This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

COMPLETE ALGEBRA

SLAUGHT AND LENNES

Educ T 129.17.795

Harvard College Library
THE GIFT OF
GINN AND COMPANY

125

3 2044 097 044 440

Roger Bacon

(1214–1294)

"All sciences rest ultimately on mathematics."

"Mathematics should be regarded as the alphabet of all philosophy."

"Divine mathematics, which alone can purge the intellect and fit the student for the acquirement of all knowledge."

COMPLETE ALGEBRA

BY

H. E. SLAUGHT, PH.D., SCD.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF CHICAGO

AND

N. J. LENNES, PH.D.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF MONTANA

ALLYN AND BACON
Boston New York Chicago

Educ T 129.17.795
✓

HARVARD COLLEGE LIBRARY
GIFT OF
GINN & CO.
JUL 24 1941

COPYRIGHT, 1916 AND 1917, BY
H. E. SLAUGHT AND N. J. LENNES.

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

THE *Complete Algebra* is planned to cover the work of a year and a half in this subject. It consists of the authors' *Elementary* and *Intermediate* texts combined in one volume. It is in no sense a revision of their First Principles of Algebra, but a new book, designed to meet the most exacting requirements of college entrance or other examination boards and the syllabi of various states. The presentation of topics, therefore, follows the traditional order.

The *Elementary* part contains numerous attractive features, all aiming to make the subject more simple and interesting and therefore more valuable to first year pupils. Among these features the most distinctive are perhaps the following four:

> The presentation of the subject is as simple as it can be made.
>
> The book is equipped with an unusually full and complete set of exercises and problems.
>
> Vital purpose is given to the study of algebra by using it to do interesting and useful things.
>
> Emphasis is given to the human interest of algebra.

To take up these four features in detail:

The simplicity of the work is especially shown (*a*) in the careful explanation of each new subject, which is fully discussed, defined, and illustrated; (*b*) in the gradation of topics, — so carefully managed as to be almost imperceptible to the pupil; and (*c*) in the gradual and natural introduction of the symbols and processes of algebra.

Each new topic is fixed by a large number of simple illustrative examples and exercises, oral and written, and finally by practical problems. These exercises are unusually numerous and thorough, and there is at the end of the book a complete set of reviews for every chapter, followed by twenty pages of miscellaneous problems.

The interest and value of the subject are emphasized by showing (*a*) how algebra grows out of arithmetic, (*b*) how much easier it is to solve problems by algebra than by arithmetic, and (*c*) by applying these algebraic solutions to countless problems of every-day experience, as well as to those taken from other studies in the school.

To emphasize the human interest of algebra, numerous historical notes have been introduced, showing how many scholars have labored through the centuries to develop this subject. Also there are portraits of well-known mathematicians, with accounts of their lives and of their contributions to mathematics, especially to algebra.

The *Intermediate* part is designed to follow the *Elementary*. Like the latter it meets the most exacting requirements of college entrance and other examination boards and of the syllabi of various states, and the presentation of topics follows the traditional order.

While recognizing the increased maturity of the pupils, the authors nevertheless maintain in the *Intermediate* text that simple and interesting form of presentation which characterizes the *Elementary*. For example, while the axioms and fundamental laws are used in the proofs of theorems, yet these are stated and the proofs are given in an informal manner that at once attracts and holds the pupils' attention and interest.

Following are some of the features which deserve particular mention:

1. The abundant *exercises*, both oral and written, throughout the book, especially in the review of the four fundamental operations, and in factoring and fractions.

2. The unusual attention given to the *applications* of each topic in the carefully graded problems, especially in those

parts of the book where the treatment is more extended than that given in the *Elementary Algebra.*

3. The clear and *simple development* of determinants of the second and third orders, and their use in solving systems of linear equations and in testing the consistency of such systems

4. The *chapter on the number system of algebra* preceding quadratics and giving a simple and clear presentation of the elementary treatment of complex numbers just where it is most needed.

5. The chapter on *graphic representation* of quadratics, following the ordinary treatment and giving the natural sequel to the graphic work in the earlier course.

6. The *unique and ample treatment of radical expressions* and the rich collection of problems in the solution of which radicals are applied.

The brief but adequate *exposition of logarithms* for the purpose of solving problems involving higher powers and roots, as, for example, the very practical problems of compound interest.

A briefer course may be made, if desired, by the omission of certain chapters or topics which have been marked with a star. These omissions will not interrupt the continuity of the work.

It is hoped that the *Complete Algebra* will be found a practical and satisfactory instrument for the study of the subject, and that it will still further extend the *concrete study of algebra,* with which the authors have been identified.

H. E. S.
N. J. L.

JUNE, 1915.

TABLE OF CONTENTS

ELEMENTARY ALGEBRA

CHAPTER PAGE
I. Introduction to Algebra 1
II. Equations and Problems 25
III. Positive and Negative Numbers 43
IV. Addition and Subtraction of Algebraic Expressions 67
V. Multiplication and Division of Algebraic Expressions 84
VI. Special Products and Quotients 108
VII. Factors of Algebraic Expressions 122
VIII. Equations Solved by Factoring 147
IX. Common Factors and Multiples 155
X. Algebraic Fractions 160
XI. Equations involving Fractions 181
XII. Ratio and Proportion 188
XIII. Literal Equations and their Uses 196
XIV. Simultaneous Equations of the First Degree . . 207
XV. *Graphic Representation 230
XVI. Square Roots and Radicals 240
XVII. Further Operations on Radicals 262
XVIII. Quadratic Equations 278
XIX. Systems of Quadratics 297
XX. *The Binomial Formula 308
XXI. *Variables and Functions 314
XXII. Review Exercises 318

INTERMEDIATE ALGEBRA

I. Definitions and Fundamental Laws 353
II. Fundamental Operations 361
III. Factoring 379
IV. Fractions 393

CHAPTER		PAGE
V.	EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN	404
VI.	FRACTIONAL EQUATIONS IN ONE UNKNOWN	415
VII.	EQUATIONS OF FIRST DEGREE IN TWO OR MORE UNKNOWNS	421
VIII.	SYSTEMS OF EQUATIONS SOLVED BY DETERMINANTS	434
IX.	GRAPHIC REPRESENTATION OF EQUATIONS	446
X.	RATIO, VARIATION, AND PROPORTION	453
XI.	POWERS AND ROOTS	463
XII.	EXPONENTS AND RADICALS	484
XIII.	THE NUMBER SYSTEM OF ALGEBRA	506
XIV.	QUADRATIC EQUATIONS	519
XV.	SYSTEMS OF QUADRATIC EQUATIONS	544
XVI.	GRAPHIC REPRESENTATION OF QUADRATICS	558
XVII.	PROGRESSIONS	572
XVIII.	BINOMIAL FORMULA	587
XIX.	LOGARITHMS	593
COLLEGE ENTRANCE EXAMINATIONS		602–616

LIST OF PRINCIPLES

NUMBER		
I.	Addition or Subtraction of Numbers having a Common Factor	9
II.	Multiplication of the Sum or Difference of Two Numbers	13
III.	Division of the Sum or Difference of Two Numbers	15
IV.	Multiplication of the Product of Several Factors	17
V.	Division of the Product of Several Factors	18
VI.	Deducing One Equation from Another	29
VII.	Addition of Signed Numbers	47
VIII.	Subtraction of Signed Numbers	51
IX.	Multiplication of Signed Numbers	57
X.	Division of Signed Numbers	59
XI.	Arrangement and Grouping of Terms	69
XII.	Removal and Insertion of Parentheses	75
XIII.	Arrangement and Grouping of Factors	84
XIV.	Multiplication of Polynomials	89
XV.	Product of Two Powers of the Same Base	94
XVI.	Quotient of Two Powers of the Same Base	97
XVII.	Multiplying or Dividing the Terms of a Fraction	161
XVIII.	Square Root of a Product	242

HISTORICAL NOTES

PAGE

Origin of the Arabic, or Hindoo Numerals 5
Origin of the Symbols of Operation 7
Parentheses and Brackets 11
Fundamental Laws of Multiplication and Division 15
Origin of the Name "Algebra" 32
Representation of Unknown Numbers 41
Development of Negative Numbers 62
Associative and Commutative Laws of Addition 69
Associative and Commutative Laws of Factors 84
Exponents 101, 490
The Pythagorean Theorem 151
Fractional Notation 161
Ratio and Proportion 188
Literal Equations; Formulas 206
Graphic Representation of Equations 239
Radicals 242
Quadratic Equations. Imaginary Roots 291
The Binomial Formula 313
Determinants 452
Number System 510
Imaginary Numbers 515
Logarithms 601

PORTRAITS

Roger Bacon . . . . *Frontispiece*

FACING PAGE

Johann Widmann 8
François Vieta 42
Sir William R. Hamilton 70
Sir Isaac Newton 102
Pythagoras 152
John Wallis 188
René Descartes 240
Karl Friedrich Gauss 292
Blaise Pascal 314
Niels Henrick Abel 394
Gottfried Wilhelm Leibnitz 452
John Napier 538

ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION TO ALGEBRA

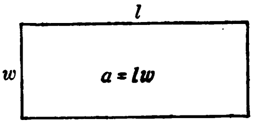
LETTERS USED TO REPRESENT NUMBERS

1. **Algebra compared with Arithmetic.** — Algebra, like arithmetic, deals with numbers. In arithmetic numbers are represented by means of the Arabic **numerals** 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. In algebra **letters**, as well as numerals, are used to represent numbers.

To illustrate this: If a blackboard is 12 feet long and 3 feet wide, its area is $12 \times 3 = 36$ square feet.

In *arithmetic* we write the rule:

$$\textit{area} = \textit{length} \times \textit{width}.$$



In *algebra* this rule is abbreviated to read:

$$a = l \times w,$$

where l and w represent the **number of units** in the length and width respectively, and a the **number of square units** in the area.

But in algebra the sign × is usually omitted between two letters, so that this rule is written:

$$a = lw.$$

Formula. The expression, $a = lw$, is called a *formula*.

A formula is a rule expressed in algebraic symbols.

Thus, $a = lw$ is a formula giving the area of a rectangle.

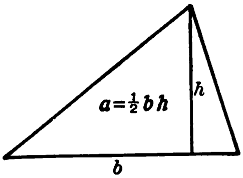
So we see that one advantage in using letters to represent numbers is to shorten the rules of arithmetic.

2. **The systematic use of letters to represent numbers** is one of the chief points of difference between algebra and arithmetic. In the following oral and written exercises note the simplicity with which the rules of arithmetic may be stated as formulas by means of letters.

ORAL EXERCISES

1. How many square inches in a slate 8 inches wide and 10 inches long? In one 6 inches wide and 9 inches long? In one w inches wide and l inches long?

2. How many square feet in the top of a table 3 feet wide and 5 feet long? In one 2 feet wide and 3 feet long? In one a feet wide and b feet long?



In arithmetic we learned that the area of a triangle is one half the product of the base and the altitude, or height. Thus, the area of a triangle is represented by the formula:

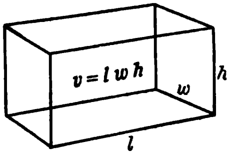
$$a = \frac{1}{2} bh.$$

3. How many square inches in a triangle whose base is 10 inches and whose altitude is 4 inches? In one whose base is 8 inches and altitude 6 inches?

In arithmetic we learned that the volume of a rectangular solid is the product of the length, width, and height; that is,

$$v = lwh.$$

4. How many cubic inches in a box 3 inches wide, 5 inches long, and 2 inches high? In one 4 inches wide, 5 inches long, and 3 inches high? In one w inches wide, l inches long, and h inches high?



5. How many cubic yards in an excavation 15 yards long, 10 yards wide, and 2 yards deep?

3. Symbols of Operation. The signs of operation, $+$, $-$, $\times$, $\div$, and the sign of equality, $=$, are used in algebra with the same meaning that they have in arithmetic. However, multiplication is often indicated by a point written above the line or by omitting the sign altogether, as in the preceding formulas.

Thus, $2 \times l \times w$ may be written $2 \cdot l \cdot w$ or $2\,lw$.
Here letters differ from numerals; for 25 means $20 + 5$, not $2 \cdot 5$.

WRITTEN EXERCISES

1. Write a formula giving the area of a rectangle. State this formula in words.

2. Using the formula $a = lw$, find the area in square rods of a field 35 rods long and 25 rods wide.

3. Using the same formula, find the area of a rectangle 17 feet wide and 20 feet long.

4. Using the formula $a = \frac{1}{2}\,bh$, find the area of a triangle whose base is 16 feet and whose altitude is 11 feet. State the process in words.

5. Using the same formula, find the area of a triangle whose base is 37 rods and whose altitude is 23 rods.

6. Given the length, width, and height of a rectangular solid, how do you find its volume? State this rule as a formula, using l, w, h, and v to represent the length, width, height, and volume, respectively. State it also in words.

7. Using the formula $v = lwh$, find the volume of a coal bin 15 feet long, 12 feet wide, and 9 feet high.

8. How many cubic feet will a freight car hold which is 40 feet long, 8 feet wide, and 7 feet high?

9. An excavation for a basement is 45 feet long, 28 feet wide, and 8 feet deep. How many cubic feet in it?

WRITTEN EXERCISES

By arithmetic 8 per cent of 90 means $.08 \times 90 = 7.2$. In this example 90 is called the **base**, .08 the **rate**, and 7.2 the **percentage**.

1. In like manner find 6 per cent of 125; 5 per cent of 350; 3 per cent of 80; and 10 per cent of 4.9. In each case how is the percentage found?

The rule for finding the percentage is stated in the following formula, in which b represents the base, r the rate, and p the percentage:

$$p = br.$$

2. Using the formula $p = br$, find the percentage if $b = 64.8$, $r = 16\%$; also if $b = 1080$, $r = 7\%$.

By a rule of arithmetic the interest on $500 for 3 years at 6 per cent is $500 \times .06 \times 3 = 90$. In this example 500 is the **principal**, .06 the **rate**, and 3 the **time**.

3. In like manner, find the interest on $900 for 7 years at 4 per cent, on $1250 for 5 years at 5 per cent, on $800 for 10 years at 6 per cent.

The rule for finding the interest is stated in the following formula, in which p represents the principal, r the rate, t the time, and i the interest:

$$i = prt.$$

4. Using the formula, $i = prt$, find the interest on $1500 for 12 years at 6 per cent. Also find i if $p = 1750$, $r = 5\%$, and $t = 4$.

5. Find the area of a rectangle 24 feet long and 18 feet wide. State the formula used.

6. Find the volume of a rectangular coal bin 14 feet long, 10 feet wide, and 6 feet deep. State the formula used.

7. Find the percentage if the base is 74 and the rate 14 per cent. State the formula used.

8. Find the interest if the principal is $840, the rate 6 per cent, and the time 3 years. State the formula used.

HISTORICAL NOTE

The Arabic or Hindu Numerals. The Arabs brought our present system of numerals to Europe when they invaded Spain from Morocco in the eighth century of our era. It is now known that these numerals, including zero, are really of Hindu origin and that they were invented between the years 200 and 650 A.D.

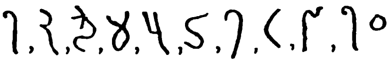
In the eleventh and twelfth centuries the Mohammedan Arabs, who had settled in Spain and Northern Africa, traded with the merchants from the great Italian cities, and in this way their system of numerals gradually spread over Europe.

In 1202 Leonardo Fibonacci of Pisa wrote a book on mathematics, the *Liber Abaci*, in which he explained the Arabic numerals for the first time to the people of Italy. He says that he published this book "in order that the Latin (Italian) race might no longer be deficient in that knowledge." The use of the new system spread but slowly, however, and it was not until the seventeenth century that it came to be used exclusively in Germany and England.

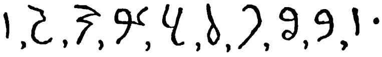
The Greeks and Romans had used letters for number symbols, but the great step in advance was taken by the Hindus when they introduced the principle of *place value* which was made possible by the symbol for zero. Thus, in $51 = 50 + 1$, the 5 means 5 tens, while in $\text{VI} = \text{V} + 1$, the V means simply five ones.

Of all mathematical inventions none has contributed more to general intelligence and material welfare than this one. To appreciate this, one needs only to try to multiply two such numbers as 589 and 642 when expressed in the Roman notation; that is, to multiply DLXXXIX by DCXLII. This difficulty alone would account for the very slow development of algebra until better symbols were invented.

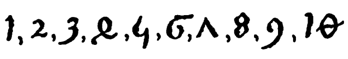
Indian Numerals about 950, A.D.



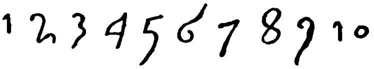
Arabic Numerals about 1100, A.D.



Of German origin about 1385, A.D.



Probably Italian about 1400, A.D.



ALGEBRAIC EXPRESSIONS

4. Algebraic Expressions. We have seen that in algebra letters as well as numerals are used to represent numbers. We have seen also that the same signs of operation are used as in arithmetic.

Any combination of numerals, letters, and signs of operation, used for the purpose of representing numbers, is called an *algebraic expression.*

E.g. 38, $18\,r$, $5\,n + 8\,n$, $3\,a - 2$ are algebraic expressions.

Expressions such as $5n + 8\,n$ and $3\,a - 2$ are said to be **written in symbols**; that is, by means of numerals, letters, and signs of operation.

The expression $3\,a - 2$ written in words would be "three times the number a diminished by 2." See how much simpler the algebraic expression is.

ORAL EXERCISES

Read the expressions in Examples 1 to 7.

1. $3 + 5$; $a + b$; $3k + 2n$; $2k + n$; $k + 2n$.
2. $7 - 3$; $a - b$; $3k - 2n$; $2k - n$; $k - 2n$.
3. 4×7; $3 \cdot 5$; ab; abc; $4mn$; $2ax$; $3cd$.
4. $10 \div 5$; $\frac{8}{3}$; $c \div d$; $m \div n$; $2k \div a$; $3n \div k$.
5. $3a$; $4abc$; $4 \cdot 5 \cdot 6$; $3mn$; $3m - 2n$; $4ab - 2c$.
6. $2k$; $2k + 1$; $2k - 1$; $4k + 1$; $4k - 1$.
7. $\frac{2}{3}h$; $\frac{3}{4}v$; $\frac{1}{5}n$; $\frac{1}{8}k$; $\frac{2}{5}abc$; $2ab$; $3cd - \frac{2}{3}mn$.

8. If one yard of wire costs 3¢, how many cents will 5 yards cost? 8 yards? n yards? State the process in words.

9. If one carfare costs 5 cents, how much will 10 carfares cost? n carfares?

10. How many days in 3 weeks? in n weeks?

11. How many feet in 4 yards? in n yards?

12. How many inches in 12 feet? in 20 feet? in k feet?

13. How many cents in \$1? in \$5? in \$$n$?

WRITTEN EXERCISES

1. If a and b are numbers, express in symbols their sum and also their product.

2. If m and n are numbers, write in symbols m divided by n; also m minus n.

3. If p and q are numbers, write the sum of 5 times p and 3 times q.

4. If a, b, and c are numbers, write in symbols that a multiplied by b equals c; also that c divided by a equals b.

If $a = 3$, $b = 5$, $c = 7$, find the value of each of the following algebraic expressions.

5. $a + b + c$.
6. $a + b - c$.
7. $2a - b + c$.
8. $5b - 2c$.
9. $16a - 6c$.
10. $4c - 3a$.
11. $bc + ac$.
12. $ab - c + 2$.
13. $4a - 2b + 3c$.
14. $2a + ab + 7c$.
15. $3b - 2c + 3$.
16. $4c - b + 3a$.
17. $abc - a - b$.
18. $ab + bc + ac$.
19. $ac + bc - ab$.

20. If m, n, and p are numbers, write 5 times the product of m and n, minus 3 times the product of m and p.

HISTORICAL NOTE

Origin of the Symbols of Operation. It was about the year 1500 A.D. that our present symbols indicating addition and subtraction first appeared in a book by a German named Johann Widmann. The sign × for multiplication was first used about 50 years later by an Englishman, William Oughtred. About the same time the sign = was first used by Robert Recorde, also an Englishman; but the sign ÷ for division does not appear until 1659, when it was used by a German, J. H. Rahn. It should not be imagined, however, that any one of these symbols came suddenly into general use. A distinguished historian of mathematics has said:

"Our present notation has arisen by almost insensible gradations as convenience suggested different marks of abbreviation to different authors; and that perfect symbolic language which addresses itself solely to the eye, and enables us to take in at a glance the most complicated relations of quantity, is the result of a long series of small improvements."

5. Factors. Many algebraic expressions, such as abc, $2mn$, $3xy$, represent products. Numbers which multiplied together form a product are called *factors of that product.*

E.g. 3 and 4 are factors of 12, as are also 2 and 6, 1 and 12. a, b, c, are factors of abc, as are also a and bc, b and ac, c and ab.

Coefficients. If an expression is the product of two factors, then either of these factors is called the *coefficient* of the other.

E.g. In $9rt$, 9 is the coefficient of rt, r is the coefficient of $9t$, and t is the coefficient of $9r$. But in such an expression the factor represented by Arabic numerals is usually regarded as the coefficient.

ADDITION AND SUBTRACTION OF NUMBERS HAVING A COMMON FACTOR

6. Addition. Two numbers which have a *common factor,* such as $5 \cdot 3$ and $8 \cdot 3$, may be added in two ways.

By the first method, *we perform the indicated multiplications and then add the products.*

Thus $$5 \cdot 3 + 8 \cdot 3 = 15 + 24 = 39.$$

By the second method, *we add the coefficients of the common factor and then multiply this sum by the common factor.*

Thus, $$5 \cdot 3 + 8 \cdot 3 = 13 \cdot 3 = 39.$$

This is evident, since 5 times a number and 8 times the *same* number make 13 times that number.

In case the common factor is represented by a letter, the second process only is available.

Thus $$5n + 8n = 13n.$$

ORAL EXERCISES

In the manner shown above add the following.

1. $2x + 4x.$
2. $4a + 3a.$
3. $13n + 4n.$
4. $3a + 2a.$
5. $5 \cdot 6 + 3 \cdot 6.$
6. $7b + 8b + b.$
7. $8t + 7t + 5t.$
8. $3r + 5r + 7r.$
9. $5 \cdot 9 + 4 \cdot 9 + 6 \cdot 9.$

Johann Widmann was born at Eger in Austria in 1460. He matriculated at Leipsic in 1480 and later became a lecturer there on algebra. In 1489 he wrote a treatise on mercantile arithmetic, in which are first found the symbols + and −, used as marks of excess and deficiency.

Widmann is chiefly interesting to us as the first German scholar whose name is associated with the subject of algebra. He has been called the originator of German algebra, though Christoff Rudolff, a pupil of Widmann's, was the first to write a textbook of algebra in the German language. Widmann was the author of a book on geometry.

7. Subtraction. Just as we add two numbers having a common factor by adding the coefficients of the common factor, so we subtract two such numbers by subtracting the coefficients.

$$\begin{array}{lr} \text{Thus, from} & 64 = 8 \cdot 8 \\ \text{subtract} & \underline{48 = 6 \cdot 8} \\ \text{Remainder} & 16 = 2 \cdot 8 \end{array} \qquad \begin{array}{lr} \text{From} & 84 = 12 \cdot 7 \\ \text{subtract} & \underline{49 = 7 \cdot 7} \\ & 35 = 5 \cdot 7 \end{array} \qquad \begin{array}{lr} \text{From} & 17\,n \\ \text{subtract} & \underline{6\,n} \\ & 11\,n \end{array}$$

ORAL EXERCISES

In this way perform the following subtractions:

1. $8 \cdot 7 - 3 \cdot 7$.
2. $6 \cdot 99 - 5 \cdot 99$.
3. $6n - 2n$.
4. $8a - 3a$.
5. $10b - 4b$.
6. $7a - 4a$.
7. $23x - 16x$.
8. $15n - 3n$.
9. $14a - 8a$.
10. $12b - 9b$.
11. $8t - 2t$.
12. $19r - 11r$.

The foregoing examples illustrate

Principle I

8. Rule. *To find the* ***sum*** *of two numbers having a common factor,* ***add*** *the coefficients of the common factor and multiply the result by the common factor.*

To find the ***difference*** *between two numbers having a common factor,* ***subtract*** *the coefficients of the common factor and multiply the result by the common factor.*

ORAL EXERCISES

Perform the following indicated operations:

1. $5x + 7x$.
2. $8x - 3x$.
3. $9a + 8a$.
4. $2m + 6m$.
5. $8y + 5y - 7y$.
6. $3a + 7a - 6a$.
7. $5ab + 7ab - 3ab$.
8. $3x + 9x - 8x$.
9. $4xy + 6xy + 4xy$.
10. $2ax + 3ax - 4ax$.
11. $6x + 14x - 7x$.
12. $20x - 8x - 3x$.
13. $5by + 7by - 3by$.
14. $4abc + 3abc + 2abc$.
15. $6ay + 9ay - 3ay$.
16. $12x + 8x + 4x$.

9. Checking Results. The substitution of numerical values for letters in order to test the correctness of an operation is called *checking the result.*

E.g. To check $2x + 7x + 4x = 13x$, we may substitute 2 for x. Thus, $2 \cdot 2 + 7 \cdot 2 + 4 \cdot 2 = 4 + 14 + 8 = 26 = 13 \cdot 2$.

WRITTEN EXERCISES

Perform the following indicated operations and check the results in the first eight by letting $t = 1$, $n = 2$, $x = 1$, $k = 1$, $s = 1$, $a = 1$.

1. $68t - 11t$.
2. $15n + 25n - 18n$.
3. $70x - 15x + 7x - 23x$.
4. $48k - 3k - 2k + 6k$.
5. $20n - 16n + 2n$.
6. $25t + 20t - 3t$.
7. $28s - 3s + 20s$.
8. $36a - 14a + 3a - 2a$.
9. $7ab - 3ab + 2ab$.
10. $34rs - 12rs - 5rs$.
11. $34xy - 18xy - 4xy$.
12. $18mn - 7mn - 3mn$.
13. $17st + 3st - 12st$.
14. $12abc - 2abc - 6abc$.
15. $42xy + 6xy - 35xy$.
16. $29rst - 18rst - 6rst$.
17. $32ac - 17ac + 2ac$.
18. $91a - 81a + 2a$.
19. $16x + 24x + 8x - 40x$.
20. $5y + 31y - 9y - 21y$.
21. $63c - 47c - 8c + 7c$.
22. $16t - 11t - 2t + 3t$.
23. $12xy - 9xy + 8xy$.
24. $39ab - 27ab - 8ab$.
25. $3rst + 9rst + 26rst - 18rst$.
26. $47abc - 14abc - 8abc + 3abc$.
27. $31xyz + 42xyz - 39xyz + 7xyz + 17xyz$.
28. $3mn + 24mn - 14mn + 8mn + 11mn$.
29. $78qr - 13qr + 8qr + 7qr - 12qr$.
30. $12lm + 7lm - 9lm + 46lm - 7lm$.
31. $145x + 17x - 125x + 280x$.
32. $40kl + 260kl - 34kl - 79kl$.
33. $7y + 14y + 28y + 56y + 112y$.
34. $93z + 47z + 82z + 235z - 406z$.

10. Equations and Identities. Two algebraic expressions representing the same number and connected by the sign = form an *equation*, as $n + 3 = 5$.

The expressions thus connected are called the **members** of the equation. In the above equation, $n + 3$ is the **left member**, and 5 is the **right member**.

In general, an equation is true only when the letters have certain special values. Thus $n+3=5$ is true only when $n=2$.

The equation $5n + 8n = 13n$ is true for all values of n. Such an equation is called an *identity*.

When it is desired to emphasize that an equation is an identity, the sign $\equiv$ is used. That is, $3a + 5a \equiv 8a$.

An equation such as $3 \times 4 = 12$, in which both members are expressed in Arabic numerals is also called an *identity*.

11. Symbols of Aggregation. **Parentheses** are used to indicate that some operation is to be extended over the whole expression inclosed by them. Thus $2(x + y)$ means that the *sum* of x and y is to be multiplied by 2, while $2x + y$ means that x alone is to be multiplied by 2.

Instead of parentheses, **brackets** [ ], **braces** { }, or the **vinculum** ——, may be used with the same meaning. All such symbols are called *symbols of aggregation*.

E.g. $2(x + y)$, $2[x + y]$, $2\{x + y\}$, $2\overline{x + y}$ all mean the same thing.

ORAL EXERCISES

Evaluate (find the value of) each of the following when

$$a = 4,\ b = 2,\ c = 1,\ x = 4,\ y = 6.$$

1. $3(a + b)$.
2. $4(a - b)$.
3. $(x + y) \div 2$.
4. $x + y \div 2$.
5. $4a - b$.
6. $3a + b$.
7. $2[x + y - 3]$.
8. $3\{a + b + c\}$.
9. $3(a + b) + c$.

Historical Note. Parentheses ( ) were first used with their present meaning by an Englishman, A. Girard, in a book on "Arithmetic," published in the year 1629. Brackets and braces are of later origin, as is also the sign $\equiv$ to denote identity.

MULTIPLICATION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

12. Multiplying a Sum. The sum of two or more arithmetical numbers may be multiplied by another number in two ways.

By the first method *we add the numbers and then multiply their sum by the other number.*

Thus
$$4(2+7)=4\cdot 9=36;$$
$$3(3+8+9)=3\cdot 20=60.$$

By the second method *we multiply each of the numbers separately and then add the products.*

Thus
$$4(2+7)=4\cdot 2+4\cdot 7=8+28=36;$$
$$3(3+8+9)=3\cdot 3+3\cdot 8+3\cdot 9=9+24+27=60.$$

But in case the numbers are represented by letters, the second process only is available.

E.g. $3(a+b)=3a+3b$ and $m(r+s)=mr+ms$.

ORAL EXERCISES

Multiply each of the following in two ways when possible:

1. $3(2+4)$.
2. $4(3+5)$.
3. $2(4+6)$.
4. $5(3+2)$.
5. $4(7+3)$.
6. $8(5+1)$.
7. $3(a+6)$.
8. $11(h+k)$.
9. $4(a+b+c)$.
10. $7(x+y)$.
11. $5(x+y+3)$.
12. $4(a+b+1)$.
13. $7(4+y+3)$.
14. $6(2+a+b)$.
15. $3(2+3+4)$.
16. $4(a+b+6)$.
17. $4(6+3)+2$.
18. $2(6+b)+3$.

13. Multiplying a Difference. The difference of two arithmetical numbers may likewise be multiplied by a given number in either of two ways.

E.g. $6(8-3)=6\cdot 5=30$,

or $6(8-3)=(6\cdot 8)-(6\cdot 3)=48-18=30$.

But in case the numbers are represented by letters, the second process only is available.

E.g. $6(r-t)=6r-6t$ and $a(c-d)=ac-ad$.

ORAL EXERCISES

Perform as many as possible of the following multiplications in two ways:

1. $7(5-2)$.
2. $12(7-4)$.
3. $5(10-8)$.
4. $7(6-3)$.
5. $9(a-2)$.
6. $8(h-4)$.
7. $5(x-1)$.
8. $3(y-2)$.
9. $a(c-d)$.
10. $m(r-s)$.
11. $x(y-z)$.
12. $t(u-v)$.
13. $x(y-8)$.
14. $a(x-7)$.
15. $b(z-4)$.

The foregoing examples illustrate

Principle II

14. Rule. *To multiply the* ***sum*** *of two numbers by a given number, multiply each of the numbers separately by the given number, and* ***add*** *the products.*

To multiply the ***difference*** *of two numbers by a given number, multiply each of the numbers separately by the given number, and* ***subtract*** *the products.*

WRITTEN EXERCISES

1. Multiply $5+7+11$ by 3 without first adding, and then check by performing the addition before multiplying.

2. Multiply $m+n$ by 4 and check for $m=5$, $n=7$.

Solution. $4(m+n)=4m+4n$.
Check. $4(5+7)=4\cdot 12=48$, also
$4\cdot 5+4\cdot 7=20+28=48$.

3. Multiply $x+y$ by r and check for $x=2$, $y=4$, $r=6$.

4. Multiply $r+s$ by k and check for $r=4$, $s=5$, $k=6$.

5. Multiply $a+b$ by m and check for $a=3$, $b=2$, $m=4$.

6. Multiply $m-n+2$ by c and check for $m=5$, $n=2$, and $c=3$.

7. Multiply $a-b-c$ by d and check for $a=10$, $b=3$, $c=4$, and $d=8$.

DIVISION OF THE SUM OR DIFFERENCE OF TWO NUMBERS

15. Dividing a Sum or a Difference. In dividing the sum or difference of two arithmetical numbers by a given number, the process may be carried out in two ways.

By the first method:

$$(12 + 8) \div 2 = 20 \div 2 = 10;$$
$$(20 - 12) \div 4 = 8 \div 4 = 2.$$

By the second method:

$$(12 + 8) \div 2 = (12 \div 2) + (8 \div 2) = 6 + 4 = 10;$$
$$(20 - 12) \div 4 = (20 \div 4) - (12 \div 4) = 5 - 3 = 2.$$

If the numbers in the dividend are represented by letters, then usually the second method only is available.

E.g. $(r + t) \div 5 = (r \div 5) + (t \div 5)$.

Division in algebra is often indicated by the fractional form.

Thus, $(r + t) \div 5 = (r \div 5) + (t \div 5)$ is commonly written

$$\frac{r+t}{5} = \frac{r}{5} + \frac{t}{5}.$$

In either case this is read: *r plus t divided by 5 equals r divided by 5 plus t divided by 5.*

ORAL EXERCISES

Perform each of the following divisions in two ways when this is possible:

1. $(4 + 12) \div 4$.
2. $(8 + 4) \div 2$.
3. $(10 + 5) \div 5$.
4. $(9 + 12) \div 3$.
5. $(16 + 8) \div 4$.
6. $(12 - 6) \div 3$.
7. $(18 - 12) \div 6$.
8. $(21 - 14) \div 7$.
9. $(x - y) \div 2$.
10. $(y - z) \div 6$.
11. $(x + y) \div 4$.
12. $(m + r) \div 9$.
13. $(m - r) \div 9$.
14. $(r - t) \div 5$.
15. $(r + t) \div 5$.
16. $(8 + 6 - 4) \div 2$.
17. $(9 + 6 - 3) \div 3$.
18. $(12 - 4 + 8) \div 4$.
19. $(a + b - c) \div 2$.
20. $(a + 4 + 6) \div 2$.
21. $(8 - b - c) \div 4$.
22. $(9 + c + d) \div 3$.
23. $(x - y + 12) \div 6$.

The foregoing examples illustrate

Principle III

16. **Rule.** *To divide the* ***sum*** *of two numbers by a given number, divide each number separately and* ***add*** *the quotients.*

To divide the ***difference*** *of two numbers by a given number, divide each number separately and* ***subtract*** *the quotients.*

WRITTEN EXERCISES

1. Divide $72 + 56$ by 8 without first adding.

2. Divide $144 - 36$ by 12 without first subtracting.

3. Divide $r + t$ by 5 and check the quotient when $r = 15$, $t = 25$; also when $r = 60$, $t = 30$.

4. Multiply $7 + 9$ by 3 without first adding 7 and 9.

5. Find the product of 12 and $a + b$, checking the result when $a = 5$, $b = 7$.

Perform the following indicated operations:

6. $3(a + b + c + d)$. Check for $a = 1$, $b = 2$, $c = 3$, $d = 4$.

7. $7(r - s + t - x)$. Check for $r = t = 5$, $s = x = 4$.

8. $(m + n + r) \div 4$. Check for $m = 64$, $n = 32$, $r = 8$.

9. $(x + y + z) \div 5$. Check for $x = 15$, $y = 10$, and $z = 5$.

10. $74\,rs - 67\,rs - 2\,rs - 3\,rs$.

11. $(63 - 35 - 14 + 21) \div 7$.

12. $a(4 - d + b + c + 3)$.

13. $(21 - x - y + 3 + c)k$.

HISTORICAL NOTE

Fundamental Laws for Multiplication and Division. The fundamental character of Principles II and III was not fully appreciated until the first part of the last century. Principle II states what is called the **Distributive Law of Multiplication** with respect to addition and subtraction. That is, the multiplier is *distributed* over the multiplicand. The name was first used by a Frenchman, F. J. Servois, in a paper published in 1814. Principle III states the **Distributive Law of Division** with respect to addition and subtraction. Compare notes on pages 69 and 84.

MISCELLANEOUS EXERCISES

1. $(48 + 36 + 24 + 12) \div 6$.
2. $(35 - 20 + 30 - 40) \div 5$.
3. $3(x + y + z) + 2(y - z + x)$.
4. $3(a + b + c) + 2(a - b + c)$.
5. $5(x + y + z) + 3(x - y + z)$.
6. $4(ab + cd + ef) + 2(ab + cd - ef)$.
7. $49\,pq + 18\,pq - 62\,pq + 3\,pq$.
8. $13\,xyz + 3\,xyz - 8\,xyz - 7\,xyz$.
9. $(q + r + s + t - a - b) \div c$.
10. $35\,lm - 33\,lm + 7\,lm - 3\,lm - 2\,lm$.
11. $k(l + m + n + r - s - t)$.
12. $a(c + d - e + f - g)$.
13. $27\,abc - 19\,abc - 4\,abc + 8\,abc$.
14. $(a + r + s - t - q) \div 3$.

15. For what values of a, b, c, d are the following equations true?

(a) $ab + ac + ad = a(b + c + d)$.

(b) $ab + ac - ad = a(b + c - d)$.

(c) $\dfrac{a + b + c}{d} = \dfrac{a}{d} + \dfrac{b}{d} + \dfrac{c}{d}$.

MULTIPLICATION OF A PRODUCT

17. **Multiplying a Product.** In multiplying a product like $3 \cdot 5$ by 2 the work may be carried out in three different ways.

Thus, $2 \cdot (3 \cdot 5) = 2 \cdot 15 = 30$;

or $2 \cdot (3 \cdot 5) = 6 \cdot 5 = 30$;

or $2 \cdot (3 \cdot 5) = 3 \cdot 10 = 30$.

In multiplying $3\,n$ by 2 we could *indicate* the work in these three ways, but we can *perform* it in the second way only; namely, $2 \cdot (3\,n) = 6\,n$.

We may evidently choose any *one* of the factors of the multiplicand and multiply this one alone by the multiplier.

ORAL EXERCISES

Find each product in several ways and verify by comparing results.

1. $2(2 \cdot 3)$.
2. $2(3 \cdot 4)$.
3. $2(4 \cdot 5)$.
4. $3(3 \cdot 4)$.
5. $2(2 \cdot 3 \cdot 4)$.
6. $3(2 \cdot 3 \cdot 4)$.
7. $2(5 \cdot 10)$.
8. $2(5 \cdot 6)$.
9. $4(5 \cdot 5)$.
10. $3(5 \cdot 4)$.
11. $8(6 \cdot 2)$.
12. $9(3 \cdot 4.)$
13. $7(2 \cdot 3)$.
14. $10(2 \cdot 4)$.
15. $2(4 \cdot 5 \cdot 2)$.

These examples illustrate

Principle IV

18. **Rule.** *To multiply the product of several factors by a given number, multiply any* ***one*** *of the factors by that number.*

Principles IV and II should be carefully contrasted.

Thus, $2(2 \cdot 3 \cdot 5) = 4 \cdot 3 \cdot 5 = 2 \cdot 6 \cdot 5 = 2 \cdot 3 \cdot 10 = 60,$
but $2(2 + 3 + 5) = 4 + 6 + 10 = 20.$

In multiplying the product of several numbers we operate upon any ***one*** *of them, but in multiplying the sum or difference of numbers we operate upon* ***each*** *of them.*

WRITTEN EXERCISES

Multiply as many as possible of the following in two or more ways.

1. $7(3 \cdot 4 \cdot 5)$.
2. $8(7 \cdot 2 \cdot 3)$.
3. $9(2 \cdot 3 \cdot 4)$.
4. $15(2ab)$.
5. $13(5xy)$.
6. $23(8xyz)$.
7. $4(19 \cdot 5)$.
8. $15(7abc)$.
9. $15(7ab)$.
10. $33(4mn)$.
11. $47(2xy)$.
12. $12(16rs)$.
13. $6(30xy)$.
14. $9(3 \cdot 4 \cdot 5)$.
15. $7(8 \cdot 5)$.
16. $8(9 \cdot 5)$.
17. $3(9 \cdot 2 \cdot 6)$.
18. $2(4 \cdot 7 \cdot 5)$.
19. $6(5 \cdot 4 \cdot 6)$.
20. $5(4 \cdot 5 \cdot x)$.
21. $4(5 \cdot 8 \cdot y)$.

DIVISION OF A PRODUCT

19. Dividing a Product. Just as in multiplying the product of several factors by a given number, so also in dividing such a product by a given number, we may operate upon any one of the factors separately.

E.g. $(4 \cdot 6 \cdot 10) \div 2 = 240 \div 2 = 120.$
Also $(4 \cdot 6 \cdot 10) \div 2 = 2 \cdot 6 \cdot 10 = 120,$
$(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 3 \cdot 10 = 120,$
and $(4 \cdot 6 \cdot 10) \div 2 = 4 \cdot 6 \cdot 5 = 120.$

Note that in each case only one factor is divided.

ORAL EXERCISES

Perform each of the following divisions in more than one way where possible:

1. $(5 \cdot 8 \cdot 3) \div 2.$
2. $20\,abc \div 4.$
3. $12\,abc \div 3.$
4. $(3 \cdot 20 \cdot 8) \div 4.$
5. $14\,xyz \div 7.$
6. $12\,abc \div c.$
7. $(10 \cdot 5 \cdot 3) \div 5.$
8. $14\,xyz \div x.$
9. $(16 \cdot 18 \cdot 13) \div 8.$
10. $5\,abc \div a.$
11. $15\,xy \div 3.$
12. $5\,(10\,x + 15\,y) \div 5.$
13. $8\,(3\,x - 4\,y) \div 4.$
14. $6\,(2\,s - 3\,t) \div 3.$

The foregoing examples illustrate

Principle V

20. Rule. *To divide the product of several factors by a given number divide any* ***one*** *of the factors by that number.*

21. Cancellation. Principle V is already known in arithmetic in the process called *cancellation.*

Thus, in the fraction $\frac{2 \cdot 6 \cdot 9}{3}$, the 3 may be cancelled out of either 6 or 9, giving $\frac{2 \cdot 6 \cdot 9}{3} = 2 \cdot 2 \cdot 9$ or $2 \cdot 6 \cdot 3$.

Contrast Principles III and V.

By Principle V, $(4 \cdot 6 \cdot 8) \div 2 = 2 \cdot 6 \cdot 8 = 4 \cdot 3 \cdot 8 = 4 \cdot 6 \cdot 4 = 96$.
By Principle III, $(4 + 6 + 8) \div 2 = 2 + 3 + 4 = 9$.

That is, *in dividing the product of several numbers we operate upon any* ***one*** *of them, but in dividing their sum or difference we operate upon* ***each*** *of them.*

WRITTEN EXERCISES

Perform the following indicated operations, remembering that a fraction is merely an indicated division:

1. $3a(7 - c + b) \div a$.
2. $5bc(d - e + 3) \div bc$.
3. $19(3a - 6b + 9c) \div 3$.
4. $13(8 - 4b + 12a) \div 4$.
5. $14(7 - 7m + 14n) \div 7$.
6. $12a(3b - 3c + 9) \div 3$.
7. $24(16b - 8c + 24d) \div 8$.
8. Divide $7a \cdot 14b \cdot 21c$ by 7 in three different ways.
9. Add $5a$, $\frac{12a}{2}$, $\frac{21a}{3}$, and $\frac{18a}{6}$, using Principles V and I.
10. From $\frac{28xy}{4}$ subtract $\frac{21xy}{7}$, using Principles V and I.
11. From $\frac{14a}{2} + \frac{10a}{5}$ subtract $\frac{6a}{3}$.
12. Find the sum of $\frac{16x}{8}$, $\frac{20x}{5}$, $\frac{16x}{4}$, $7x$, and $3x$.
13. Find the sum of $\frac{100rs}{10}$, $\frac{90rs}{9}$, and $\frac{25rs}{5}$.
14. From $25xy$ subtract $\frac{13xyz}{z}$.
15. Add $\frac{8abc}{c} + \frac{18ab}{3} + \frac{7abd}{d} + \frac{abe}{e}$.
16. Add $\frac{18a}{6} + \frac{5ab}{b} + \frac{4ax}{x} + \frac{am}{m}$.
17. Add $\frac{56x}{28} + \frac{48x}{12} + \frac{32x}{4} + \frac{24x}{3}$.

ORDER OF INDICATED OPERATIONS

22. Order of Operations. In a succession of indicated operations the final result depends in some cases upon the *order* in which the operations are performed, while in some cases it does not. This is shown in the following examples:

(1) $6 + 3 \cdot 8$ would give $9 \cdot 8 = 72$, if the addition were performed first; and would give $6 + 24 = 30$, if the multiplication were performed first.

(2) $24 \div 2 \cdot 3$ gives $12 \cdot 3 = 36$, if the division is performed first; and gives $24 \div 6 = 4$, if the multiplication is performed first.

(3) $24 \div 6 \div 2$ equals 2 if we first divide 24 by 6, and equals 8 if we first divide 6 by 2.

(4) $12 \times 6 \div 3 = 24$ no matter in what order the operations are performed.

(5) $18 + 10 - 8 = 20$, no matter in what order the operations are performed.

(6) $4 \times 3 \times 2 = 24$, no matter in what order we multiply.

23. To prevent mistakes, and to make usage uniform, the following rules have been adopted:

In an expression involving additions, subtractions, multiplications, and divisions, when no symbols of aggregation (§ 11) *are involved,*

(1) *All multiplications are performed first, and these may be taken in any order;*

(2) *All divisions are performed next, and these are taken in the order in which they occur from left to right;*

(3) *Finally, additions and subtractions are performed, and these may be taken in any order.*

These rules are illustrated by the following examples:

(1) $5 + 3 \cdot 4 - 8 \div 2 = 5 + 12 - 8 \div 2 = 5 + 12 - 4 = 13$;

(2) $8 - 2 \div 2 - 2 \times 2 + 3 = 8 - 2 \div 2 - 4 + 3 = 8 - 1 - 4 + 3 = 6$;

(3) $8 \cdot 2 \div 4 + 28 \div 2 \cdot 2 = 16 \div 4 + 28 \div 4 = 4 + 7 = 11$;

(4) $16 + 24 \div 3 \cdot 4 - 8 \cdot 3 \div 2 = 16 + 24 \div 12 - 24 \div 2$
$= 16 + 2 - 12 = 6$.

ORAL EXERCISES

1. What is the value of $4 \cdot 3 + 2 \cdot 4 - 3 \cdot 5$?
2. What is the value of $5 \cdot 4 - 6 + 2 \cdot 5$?
3. What is the value of $2 \cdot 3 \cdot 4 + 12 \div 4$?
4. What is the value of $3(4 + 2) + 6$?
5. What is the value of $2(3 + 2 \cdot 3) - 4(1 + 3)$?

If $a = 2$, $b = 3$, $c = 4$, find the value of each of the following. Remember that a fraction is an indicated division.

6. $2ab - c$.
7. $\dfrac{bc - 4a}{a}$.
8. $\dfrac{3ac + 2b}{2b}$.
9. $5a + 2b - 3c$.
10. $10c - 10b - 4a$.
11. $9a + 3c - 10b$.
12. $\dfrac{8a + c + 5b}{5}$.
13. $\dfrac{4c - 2b + 3a}{4a}$.
14. $2a + 3b + 4c$.
15. $6a - b - c$.

WRITTEN EXERCISES

If $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$, $z = 6$, find the value of each of the following expressions:

16. $(6ab \div 4) \div 3$.
17. $6ab \div (4 \div 3)$.
18. $(12cx \div 2z)b$.
19. $12cbx \div 2z$.
20. $12cx \div 2zb$.
21. $(6xyz \div 2b) \div 2c$.
22. $6xyz \div (3b \div 2c)$.
23. $5abc - 3x + 2yz$.
24. $3xyz - 8a + 5b$.
25. $3xyz - (8a + 5b.)$
26. $3axy - 4c - b$.
27. $3axy - 4(c - b)$.
28. $3axy - 4(c + b)$.
29. $2bz - (2x - c)$.
30. $2bz - 2(x - c)$.
31. $2bz - 2x - c$.
32. $30 - (x - a)$.
33. $30 - x - a$.
34. $3 + 5 \cdot 6 - 4 \cdot 3$.
35. $3 + 5(6 - 4) \cdot 3$.
36. $5(a + b + c - x)$.
37. $2cxy \div ab + 5$.
38. $4xy \div 3ab + 4$.
39. $4xy \div (4a \div b)$.
40. $2axy \div (5b \div 2a)$.
41. $(2axy \div 5b) \div 2a$.

24. Importance of the Principles. The five principles studied in this chapter, together with others which will be introduced when needed, will be found of increasing importance as we proceed. Your success in the further study of algebra will depend in no small degree upon the clearness with which you understand their real significance.

The most effective way to master these principles is by means of simple numerical examples such as were used in introducing each one. Make a list of these principles in abbreviated form for yourself and note how frequently your own errors and those of your classmates are due to direct violations of one or more of them.

WRITTEN EXERCISES

If $a = 5$, $b = 3$, $c = 2$, find the value of each of the following:

1. $ac + bc$ and $(a + b)c$.

2. $ac - bc$ and $(a - b)c$.

3. $abc, a \times (bc)$ and $b \times (ac)$.

4. $a(b + c)$ and $ab + ac$.

5. $a(b - c)$ and $ab - ac$.

6. $\frac{ab}{c}, \frac{a}{c} \times b$ and $a \times \frac{b}{c}$.

7. $\frac{a+b}{c}$ and $\frac{a}{c} + \frac{b}{c}$.

8. $\frac{a-b}{c}$ and $\frac{a}{c} - \frac{b}{c}$.

9. In each of the above compare the results obtained in the two parts.

10. $\frac{4a - 5b}{5} + \frac{b + 9}{4}$

11. $\frac{8b - 9c}{3} + \frac{3a + b}{6}$

12. $\frac{8ab - 5bc}{6} + \frac{2ab + 3c}{b}$

13. $\frac{6ac + 5bc}{2a} + \frac{ab + 3c}{b}$

14. $\frac{12a + 5b}{5} + \frac{6c + 4b}{6}$

15. $\frac{12bc + 4b}{12} + \frac{3ac + 2b}{6}$

16. $\frac{10ab - 3c}{2b} + \frac{5bc - 2a}{a}$

17. $\frac{4b + 6a - 3c}{6}$

18. $\frac{12c + 4ac - 4c}{8}$

19. $(6a + 5c) \div 2a$

REVIEW QUESTIONS

1. How would $3 \cdot 5$ and $7 \cdot 5$ be added in arithmetic? Why cannot $3n$ and $7n$ be added in the same manner? State the principle by which $3n$ and $7n$ are added. Test the identity $3n + 7n = 10n$ by substituting any convenient value for n.

2. What kind of numbers may be added by Principle I? Have the numbers ac and bc a common factor? What is it? What is the coefficient of this common factor in each? What is the sum of these coefficients? Is the equality $ac + bc = (a + b)c$ true no matter what numbers are represented by a, b, and c? When this can be said of an equality, what is it called?

3. How is $5 \cdot 9$ subtracted from $11 \cdot 9$ in arithmetic? In what different manner may this operation be performed? Why is it sometimes necessary to perform subtraction in the second way? In the identity $31x - 12x = 19x$, what number is represented by x? Test the equality by substituting any convenient number for x. Is this equality true for every value of x?

Principle I may be conveniently abbreviated as follows:

$$\begin{cases} ac + bc = (a+b)c, \\ ac - bc = (a-b)c. \end{cases}$$

4. How is $11 + 3$ multiplied by 4 in arithmetic? In what different way may this operation be performed? Why is it sometimes necessary to multiply in the second way? State in full the principle by which $a + 8$ is multiplied by 7.

Principle II may be abbreviated thus:

$$\begin{cases} c(a + b) = ca + cb, \\ c(a - b) = ca - cb. \end{cases}$$

Notice that the identities in Principle II are the same as those in Principle I read in reverse order. Thus, Principle II may be called the reverse or **converse** of Principle I.

5. How is $12 + 18$ divided by 6 in arithmetic? In what different way may this division be performed? Why is it sometimes necessary to perform division in the second way? State in full the principle used in performing the operation $(6x + 9y) \div 3$. How do you divide $(6x - 9y)$ by 3?

Principle III may be abbreviated thus:

$$\frac{a+b}{k} = \frac{a}{k} + \frac{b}{k}, \qquad \frac{a-b}{k} = \frac{a}{k} - \frac{b}{k}.$$

6. How is the product $2 \cdot 3 \cdot 5$ multiplied by 4 in arithmetic? In what different way may this multiplication be performed? Why is it ever performed in the second way? What are the factors of ab? How is the product of two numbers multiplied by another number? Should *both* factors be multiplied by the number, or only one? Is it permissible to multiply *either one* we choose?

Principle IV is abbreviated thus:

$$k \times (ab) = (ka) \times b = a \times (kb).$$

7. Divide $2 \cdot 4 \cdot 6 \cdot 20$ by 2 without first performing the multiplication indicated in $2 \cdot 4 \cdot 6 \cdot 20$. Do this in several ways and show that all the quotients obtained are equal. State in full the principle used.

Principle V is abbreviated thus:

$$(ab) \div k = \frac{a}{k} \times b = a \times \frac{b}{k}.$$

8. Contrast Principles II and IV; also III and V.

9. Why are our numerals called Arabic numerals? When and how were they brought into Europe?

10. What is the most important point in which the Arabic system differs from the Roman system?

11. What is meant by the distributive law of multiplication and the distributive law of division?

CHAPTER II

EQUATIONS AND PROBLEMS

The principles developed in the last chapter will now be used in the solution of equations and problems.

SOLUTION OF EQUATIONS

25. Unknowns. A letter whose value we seek by means of an equation is called an *unknown* in that equation.

Satisfying an Equation. A number is said to *satisfy an equation* if, when substituted for the unknown, it reduces the equation to an identity.

Thus, $3x = 18$ is said to be satisfied by $x = 6$, because this value of x reduces the equation to the identity $18 = 18$.

Equivalent Equations. If two equations which are not identities are satisfied by the same numbers, they are called *equivalent equations.*

Thus, $3x = 18$ and $6x = 36$ are equivalent equations because they are not identities and are both satisfied by $x = 6$.

Also, $n + 2 = 5$ and $2n + 4 = 10$ are equivalent equations, because they are both satisfied by $n = 3$ and by no other value of n.

ORAL EXERCISES

1. Is $x + 4 = 9$ satisfied by $x = 4$? by $x = 5$? by $x = 6$?

2. Is $2x + 9 = x + 12$ satisfied by $x = 2$? by $x = 3$?

3. Is $\frac{x}{4} + 3 = 6$ satisfied by $x = 4$? by $x = 8$? by $x = 12$?

4. Is $3(x + 2) = 3x + 6$ satisfied by $x = 1$? by $x = 2$? by $x = 3$? by $x = 4$?

26. To solve an equation is to find a value of the unknown which will satisfy the equation.

A value of the unknown which satisfies an equation is called a **root** or **solution** of the equation.

The following examples illustrate methods used in solving equations.

Example 1. Solve the equation $x - 5 = 9$. (1)

Solution. This equation states that 9 is 5 less than the number x. That is, if 9 be increased by 5, the result is x.

Hence, $x = 14$ is the solution of equation (1).

This result may be obtained by adding 5 *to each member of equation* (1), thus

$$x + 5 - 5 = 9 + 5,$$

or

$$x = 14. \qquad (2)$$

Example 2. Solve the equation $x + 7 = 12$. (1)

Solution. This equation states that 12 is 7 more than the number x. That is, if 12 be diminished by 7, the result is x.

Hence, $x = 5$ is the solution of equation (1).

This result may be obtained by subtracting 7 *from both members of* (1), thus

$$x + 7 - 7 = 12 - 7,$$

or

$$x = 5. \qquad (2)$$

Example 3. Solve the equation $\frac{1}{3}x = 7$. (1)

Solution. This equation states that one third of the number x is 7. That is, x is 3 times 7, or 21.

Hence $x = 21$ is the solution of equation (1).

This result may be obtained by multiplying both members of (1) *by* 3, thus

$$3 \times \tfrac{1}{3}x = 3 \cdot 7,$$

or

$$x = 21. \qquad (2)$$

Example 4. Solve the equation $5x = 30$. (1)

Solution. This equation states that 5 times the number x is 30. That is, x is one fifth of 30, or 6.

Hence, $x = 6$ is the solution of the equation.

This result may be obtained by dividing both members of equation (1) *by* 5, thus

$$\frac{5x}{5} = \frac{30}{5},$$

or

$$x = 6. \qquad (2)$$

ORAL EXERCISES

Solve the following equations and explain each step involved, as in the foregoing illustrative examples.

1. $2x = 16$.
2. $11x = 33$.
3. $\frac{1}{3}y = 6$.
4. $\frac{1}{5}x = 7$.
5. $\frac{1}{6}n = 20$.
6. $6w = 42$.
7. $x + 4 = 8$.
8. $x + 9 = 16$.
9. $x - 2 = 8$.
10. $x - 3 = 10$.
11. $x + 8 = 16$.
12. $x - 3 = 12$.
13. $w + 2 = 14$.
14. $w - 2 = 14$.
15. $\frac{1}{4}w = 20$.
16. $7w = 28$.
17. $3n - 1 = 5$.
18. $3n + 1 = 10$.

27. Deriving Equivalent Equations. The above examples illustrate four ways of operating upon an equation, so as to produce in each case a new equation *which is equivalent to the original equation.*

Each of these operations *changes the value of both members, but changes them both in the same way.*

The object of each step is to obtain an equation whose solution is more apparent than that of the preceding equation.

In the example below, equations (2) and (3) show two ways *of changing the form, but not the value, of one member alone,* thus producing a new equation which is equivalent to the original equation.

Example 5. Solve the equation

$$w + 2(w + 5) = 58. \tag{1}$$

By Principle II, $w + 2w + 10 = 58$. (2)

By Principle I, $3w + 10 = 58$. (3)

Subtracting 10 from both members, $3w = 48$. (4)

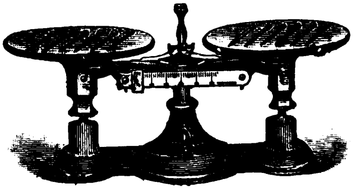
Dividing both members by 3, $w = 16$. (5)

Check. Putting $w = 16$ in (1), $16 + 2(16 + 5) = 16 + 2 \cdot 21 = 58$.

The operations involved in passing from (1) to (2) and (2) to (3) are called **form changes**.

All the operations involved in Principles I to V are *form changes* of this character. See the list at the end of Chapter I. Other form changes will be considered as need arises..

28. Illustrating the Operations on an Equation. The members of an equation may be likened to the **scale-pans of a common balance** in which are placed objects of uniform weight, say tenpenny nails. The scales balance only when the weights are the same in both pans; that is, when the number of nails is the same.



ORAL EXERCISES

1. If there are ten nails on each side, do the scales balance?

2. If now six more nails are added to one side, what must be done to the other side to keep the scales balanced? If three nails are subtracted from one side what must be done to the other side to keep the scales balanced?

3. If the number of nails on one side is doubled, what must be done to the other side to keep the balance? If the number is divided by four?

4. If the scales are balanced, and if the nails in either pan are rearranged in any way, will the scales continue to balance?

Preserving the Balance. The above examples show that if the scales are in balance, they will remain so under two kinds of changes in the weights:

(*a*) When the number of nails in the two pans is increased or diminished by the same amount; corresponding to **like changes in value of both members of an equation.**

(*b*) When the number in each pan is left unaltered but the nails are rearranged in groups or piles in any manner; corresponding to **form changes** in either member of an equation.

The equation, then, is like a balance, and its members are to be operated upon only in such ways as to preserve the balance.

WRITTEN EXERCISES

Solve the following equations as in example 5, § 27:

1. $x + 4x = 15$.
2. $12x + 3x = 30$.
3. $7x - 3x = 12$.
4. $x + 3(x - 1) = 17$.
5. $4(3x - x) = 8$.
6. $5(2x + 1) = 15$.
7. $3(4r - 2) = 18$.
8. $y + 2(y + 2) = 16$.
9. $r + 3(r + 4) = 24$.
10. $6l + 2(l - 3) = 34$.
11. $\frac{1}{3}w + 4w = 26$.
12. $\frac{1}{2}w + 2w + 3 = 13$.

The foregoing examples illustrate

Principle VI

29. **Rule.** ***An equation may be changed into an equivalent equation by means of any of the following operations:***

(1) ***Adding the same number to both members;***

(2) ***Subtracting the same number from both members;***

(3) ***Multiplying both members by any known number not zero;***

(4) ***Dividing both members by any known number not zero;***

(5) ***Changing the form of either member in any way which leaves its value unaltered.***

The operations under Principle VI are hereafter referred to in detail by means of the initial letters, *A* for *addition*, *S* for *subtraction*, *M* for *multiplication*, *D* for *division*, and *F* for *form changes* which leave the value of a member unaltered.

NOTE. — The statements (1) to (4) in Principle VI include the following so-called **Axioms** or **self-evident truths**:

(1) If equals are added to equals, the sums are equal.

(2) If equals are subtracted from equals, the remainders are equal.

(3) If equals are multiplied by equals, the products are equal.

(4) If equals are divided by equals, the quotients are equal.

Principle VI, however, includes *more than these axioms*, since it states that the new equation obtained each time is *equivalent* to the original equation.

DIRECTIONS FOR WRITTEN WORK

30. In solving an equation the successive steps may be written as in the following examples:

1. $25(n+1)+6(4n-3)=50+31n+2(3-n)-9.$ (1)

By F, using Principle II, we obtain from (1)

$$25n+25+24n-18=50+31n+6-2n-9. \qquad (2)$$

By F, using Principle I, we obtain from (2)

$$49n+7=29n+47. \qquad (3)$$

Subtracting 7 and $29n$ from each member of (3) and using Principle I, we have $20n=40.$ (4)

Dividing each member of (4) by 20,

$$n=2. \qquad (5)$$

Check. Substitute $n=2$ in equation (1).

For convenience this work can be abbreviated as follows:

$25(n+1)+6(4n-3)=50+31n+2(3-n)-9.$ (1)

By F, II, $25n+25+24n-18=50+31n+6-2n-9.$ (2)

By F, I, $49n+7=29n+47.$ (3)

By $S \mid 7, 29n$, $20n=40.$ (4)

By $D \mid 20$, $n=2.$ (5)

$S \mid 7, 29n$ means that 7 and $29n$ were subtracted from both members of the preceding equation. $D \mid 20$ means that the members of the preceding equation were divided by 20.

Similarly, in case we wish to indicate that 6 is to be added to each member of an equation, we should write $A \mid 6$, and if each member is to be multiplied by 8, we should write $M \mid 8$.

2. $17n+4(2+n)-6=5(4+n)-5+3n.$ (1)

By F, II, $17n+8+4n-6=20+5n-5+3n.$ (2)

By F, I, $21n+2=15+8n.$ (3)

By $S \mid 2, 8n$, $21n-8n=15-2.$ (4)

By F, I, $13n=13.$ (5)

By $D \mid 13$, $n=1.$ (6)

Check. Substitute $n=1$ in equation (1).

3. $\frac{x}{3}+\frac{x}{2}+x=11.$ (1)

By $M \mid 6$, $2x+3x+6x=66.$ (2)

By F, I, $11x=66.$ (3)

By $D \mid 11$, $x=6.$ (4)

31. Transposing Terms. By use of Principle VI, a term may be **transposed** from one member of an equation to the other, provided its sign is changed.

E.g. in deriving equation (4) from (3) in Example 2, page 30, $8n$ is subtracted from both sides by mentally *dropping* it on the right and *indicating* its subtraction on the left. Likewise when 2 is subtracted from both sides it *disappears* on the left and *appears* on the right with the *opposite* sign. Each of these indicated subtractions might have been performed *mentally*, thus writing equation (5) directly from (3).

After a little practice this shorter process of *transposing terms, changing the signs*, and *combining similar terms mentally* should always be used.

WRITTEN EXERCISES

Solve the following equations, putting the work in a form similar to that on page 30. Check the first ten.

1. $13x + 40 - x = 88.$
2. $3x + 15 + 2x = 18 + 4x.$
3. $5x + 3 - x = x + 18.$
4. $13y + 12 + 5y = 32 + 8y.$
5. $4m + 6m + 4 = 9m + 6.$
6. $7m + 18 + 3m = 2m + 50.$
7. $y + 42 + 45y = 76 + 12y.$
8. $42x + 56 = 20x + 122.$
9. $12m + 3 - 3m = 38 + 2m.$
10. $15m + 3 - 2m = 3m + 53.$
11. $a + 7 + 3a = 2a + 45.$
12. $5b + 30 + 6b = 3b + 150.$
13. $3c + 18 + 14c = 6c + 51.$
14. $17x + 4 + 3x = 7x + 30.$
15. $3y + 4 + 2y + 6 = y + 7 + y + 3 + 30.$
16. $5x + 3 + 2x + 3 = 2x + 5 + 3x + 3 + x.$
17. $2x + 4x + 9 - x + 6 = 20 + 2x + 5 + x.$
18. $18 + 6m + 30 + 4m = 4m + 8 + 12 + 3m + 3 - m + 29.$
19. $6x + 8 + x + 4 + 5x = 7x + 32 - x - 20.$
20. $32x + 4 + 7x = 66 + 3x + 5x.$
21. $7(m + 6) + 10m = 42 + 5m + 24.$
22. $6x + 4(4x + 2) + 3(2x + 7) = 85.$
23. $8 + 7(6 + 6n) + 2n = 2(4n + 5) + 18n + 49.$
24. $5(9x + 3) + 4(3x + 2) = 18x + 36.$

32. An **equation may be translated into a problem.** For example, the equation $21x + 2 = 8x + 15$ may be interpreted as follows: *Find a number such that 21 times the number plus 2 is 15 greater than 8 times the number.*

ORAL EXERCISES

Translate each of the following into a problem:

1. $2x + 5 = 21.$
2. $3x - 7 = 19.$
3. $6 + 5x = 16.$
4. $4x - 3 = 2x + 9.$
5. $3(x + 1) = 18.$
6. $2x + 3 + x = x + 7.$
7. $5x - 3 - x = x + 9.$
8. $8 + 3x + 2x = 4x + 19.$

HISTORICAL NOTE

Origin of the Name Algebra. The Arabs brought their first algebra into Europe in the first half of the ninth century. It bore the name Al-jebr-wàl-muqābala. The word *al-gebr*, from which the word *algebra* is derived, means "transposition" and refers to transposing terms in solving equations. The word *wàl-muqābala* means the process of simplification or *form changes*. Thus it appears that the Arabs regarded the solution of equations as the main business of algebra.

SOLUTION OF PROBLEMS

33. One great object in the study of algebra is to **simplify the solution of problems.** This is done by using letters to represent numbers, by stating problems in the form of equations, and by the systematic solutions of the equations.

Illustrative Problem. 1. The shortest railway route from Chicago to New York is 912 miles. How long does it take a train averaging 38 miles an hour to make the journey?

Solution. Let t be the number of hours required. Then $38t$ is the distance traveled. But 912 miles is the given distance traveled.

Hence $$38t = 912. \qquad (1)$$

By $D|38$ $$t = 24. \qquad (2)$$

Check. The distance traveled in 24 hours at 38 miles per hour is

$$38 \times 24 = 912.$$

Hence $t = 24$ satisfies the conditions of the problem.

Illustrative Problem. 2. For how many years must \$850 be invested at 5 % simple interest in order to yield \$255?

Solution. From arithmetic, we have

$$principal \times rate \times time = interest,$$

or

$$prt = i.$$

Hence, from the conditions of the problem,

$$850 \times .05 \times t = 255, \qquad (1)$$

or

$$42.5\,t = 255.$$

By $D \mid 42.5$,

$$t = 6. \qquad (2)$$

Check. The interest on \$850 for 6 years at 5 % is

$$6 \times .05 \times 850 = 255.$$

Hence $t = 6$ satisfies the conditions of the problem.

NOTE. — Substituting $t = 6$ in equation (1) would not fully check the solution, since that equation might be incorrect. It is necessary to see that the solution satisfies the *problem itself*.

Illustrative Problem. 3. A boy, an apprentice, and a master workman have the understanding that the apprentice shall receive twice as much as the boy and the master workman five times as much as the boy. How much does each get, if the total amount received for a piece of work is \$104?

Solution. Let n represent the number of dollars received by the boy.

Then, $2n$ is the number of dollars received by the apprentice,

and $5n$ is the number of dollars received by the master workman.

Hence, $n + 2n + 5n$ represents the total amount received.

Therefore, $$n + 2n + 5n = 104. \qquad (1)$$

By Principle I, $$8n = 104. \qquad (2)$$

By $D \mid 8$, $$n = 13. \qquad (3)$$

Hence, the amount received by the boy is 13 dollars. The apprentice receives $2n$ dollars or \$26, and the master workman receives $5n$ dollars or \$65.

Check. By the conditions of the problem the sum of the amounts obtained should be \$104; the apprentice should receive twice as much as the boy and the master workman five times as much as the boy. That is, we should have

$$13 + 26 + 65 = 104, \quad 26 = 2 \cdot 13 \quad \text{and } 65 = 5 \cdot 13.$$

Rules for Solving Problems. From the preceding illustrations we see how much more simply problems may be solved by algebra than by arithmetic.

The following rules will help in solving problems:

(1) *Represent the unknown quantity by a letter.*

(2) *Translate the words of the problems into an equation involving this letter.*

(3) *Solve this equation by use of Principle VI.*

(4) *Verify the solution by showing that the requirements of the problem are fulfilled.*

ORAL PROBLEMS

1. Three times a certain number is 24. What is the number?

If x is the number, then $3x = 24$.

2. If $\frac{1}{3}$ of a certain number is 5, find the number. First state the equation.

In each of the following, state the equation and then give the required number.

3. If 6 is added to a certain number, the sum is 22. Find the number.

4. If 8 is subtracted from a certain number, the remainder is 10. Find the number.

5. Twice a certain number added to the number itself gives 12. Find the number.

6. Three times a number less the number itself equals 18. Find the number.

7. Twice a number plus three times the number equals 25. Find the number.

8. Six times a number less 4 times the number equals 8. Find the number.

9. If $3x$ plus $4x$ equals 42, find the value of x.

10. If $2a$ is subtracted from $9a$, the remainder is 21. Find the value of a.

11. If $7x$ is subtracted from $12x$, the remainder is 15. Find the value of x.

12. If $4y$ is added to $6y$, the sum is 100. Find the value of y.

13. If $3x$, $4x$, and $8x$ are added, the sum is 30. Find the value of x.

14. Twice a number plus three times the number plus four times the number is 36. Find the number.

15. If n is a number, how do you represent 10 times that number?

16. If n is a number, how do you represent that number plus 3 times itself?

17. If n is a number, how do you represent 5 times that number plus 3 times the number plus 8 times the number?

WRITTEN PROBLEMS

Solve the following problems by means of equations, and check each result by testing whether it satisfies the conditions of the problem.

1. Five times a certain number equals 80. What is the number?

2. Twelve times a number equals 132. What is the number?

3. A tank holds 750 gallons. How long will it take a pipe discharging 15 gallons per minute to fill the tank?

4. The cost of paving a block on a certain street was $7 per front foot. How long was the block, if the total cost was $4620?

5. A city lot sold for $7500. What was the frontage, if the selling price was $225 per front foot?

6. An encyclopedia contains 18,000 pages. How many volumes are there, if they average 750 pages to the volume?

7. For how many years must \$3500 be invested at 6% simple interest to yield \$2205?

8. At what rate must \$2500 be invested for 3 years in order to yield \$412.50?

Suggestion. By the conditions of the problem, $2500 \times r \times 3 = 412.50$.

9. At what rate must \$6800 be invested for 7 years in order to yield \$2380?

10. How many dollars must be invested for 5 years at $4\frac{1}{2}$% simple interest to yield \$351?

Suggestion. By the conditions of the problem, $p \times .04\frac{1}{2} \times 5 = 351$.

11. How many dollars must be invested for 6 years at $4\frac{3}{4}$% simple interest to yield \$2422.50?

12. A cut in an embankment is 500 yards long and 4 yards deep. How wide is it if 18,760 cubic yards are removed?

Suggestion. By the conditions of the problem, $500 \times w \times 4 = 18{,}760$.

13. How deep is a rectangular cistern which holds 500 cubic feet of water, if it is 6 feet wide and 8 feet long?

14. How long is a box containing 2240 cubic inches, if its width is 14 inches and its depth 10 inches?

15. The greater of two numbers is 5 times the less, and their sum is 180. What are the numbers?

16. A number increased by twice itself, 4 times itself, and 6 times itself, becomes 429. What is the number?

17. A father is 3 times as old as his son, and the sum of their ages is 48 years. How old is each?

18. In a company there are 39 persons. The number of children is twice the number of grown people. How many are there of each?

34. Further Hints for Translating Problems into Equations. Skill in solving problems depends upon attention to the following points:

(1) *Read and understand* clearly the statement of the problem, as it is given in words.

(2) *Represent the unknown number* by some letter, say the initial letter of a word, which will keep its meaning in mind. If there are more unknown numbers than one, try to express the others in terms of the letter first selected.

(3) Form two algebraic expressions which, according to the conditions of the problem, represent the same number, and set them equal to each other, *thus forming an equation.*

ORAL EXERCISES

1. If n is a number, represent in symbols a number 7 greater than n; 5 less than n; 8 times as great as n; one third as great as n.

2. Write in symbols n increased by k; n decreased by k; n multiplied by k; n divided by k.

3. If the sum of two numbers is 10 and one of them is x, what is the other number?

4. If two numbers differ by 6 and the smaller is x, what is the other number?

5. If two numbers differ by 6 and the greater is x, what is the other number?

6. If A has m dollars, and B has 15 dollars more than A, how do you represent B's money? If C's money is twice B's, how do you represent C's money?

7. A father is 34 years older than his son. If x represents the age of the son, how do you represent the father's age? How do you represent their ages 5 years ago?

8. If n is an integer, how do you represent the next higher integer? The second higher? The next lower? The second lower?

Note. — The whole numbers or integers are obtained by ordinary counting, beginning with *one*. If any integer such as 12 is given, the next higher integers may be represented by $12 + 1$, $12 + 2$, $12 + 3$, and so on. The next lower integers may be represented by $12 - 1$, $12 - 2$, $12 - 3$, and so on.

9. What is the value of $2n$, for $n = 1, 2, 3, 4, 5, 6$, etc.? If n is any integer, the number represented by $2n$ is called an *even* integer *since it contains the factor 2.*

10. If $2n$ is any even integer, represent the next higher even integer.

11. Represent each of four consecutive even integers, the smallest of which is $2n$.

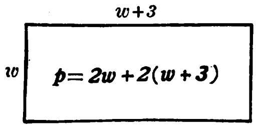
12. What is the value of $2n + 1$, for $n = 1, 2, 3, 4, 5$, etc.? If n is any integer, the number represented by $2n + 1$ is called an *odd* integer *since it does not contain the factor 2.*

13. If $2n + 1$ is any odd integer, represent the next higher odd integer.

14. Represent four consecutive odd integers, the smallest of which is $2n + 1$.

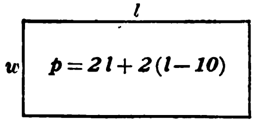
15. If x is a number, express in terms of x a number 5 less than 3 times x; also a number 5 times the remainder when 3 is subtracted from x.

16. If w and l are the width and length respectively of a rectangle, how do you represent its perimeter? (The perimeter of a rectangle means the sum of the lengths of its four sides.)



17. The length of a rectangle is 3 feet greater than its width. If w is the width, how do you represent its length? its perimeter?

18. Express the perimeter of a rectangle in terms of its length l if its width is 10 inches less than the length.



19. Express the perimeter of a rectangle in terms of its length l, if the length is 6 inches greater than the width.

WRITTEN PROBLEMS

Check each solution by finding whether the result satisfies the conditions stated in the problem:

1. Four times a certain number plus 3 times the number minus 5 times the number equals 48. What is the number?

2. One number is 4 times another, and their difference is 9. What are the numbers?

3. Find a number such that when 4 times the number is subtracted from 12 times the number, the remainder is 496.

4. Thirty-nine times a certain number, plus 19 times the number, minus 56 times the number, plus 22 times the number, equals 12. Find the number.

5. There are three numbers whose sum is 80. The second is 3 times the first, and the third twice the second. What are the numbers?

6. There are three numbers such that the second is 11 times the first and the third is 20 times the first. The difference between the second and third is 36. Find the numbers.

7. There are three numbers such that the second is 8 times the first and the third is 3 times the second. The third number less the second equals 48. Find the numbers.

8. The number of representatives and senators together in the United States Congress is 531. The number of representatives is 51 more than 4 times the number of senators. Find the number of each.

9. The area of Illinois is 6750 square miles more than 10 times that of Connecticut. The sum of their areas is 61,640 square miles. Find the area of each state.

10. Find three consecutive integers whose sum is 144.

11. Find four consecutive integers such that the last plus twice the first equals 48.

12. Find three consecutive even integers whose sum is 54.

13. Find three consecutive even integers such that 3 times the first is 12 greater than the third.

14. Find two consecutive integers such that 3 times the first plus 7 times the second equals 217.

15. Find two consecutive integers such that 7 times the first plus 4 times the second equals 664.

16. Find four consecutive odd integers such that 7 times the first equals 5 times the last.

17. Find the side of a square whose perimeter is 64 inches.

18. A rectangle is 4 inches longer than it is wide. Find its length and width if the perimeter is 40 inches.

19. A rectangle is twice as long as it is wide. Find its dimensions if the perimeter exceeds the length by 60.

20. The length of a rectangle is $1\frac{1}{2}$ times as great as its width. Find its dimensions if the perimeter exceeds the width by 40 inches.

21. The width of a rectangle is $\frac{1}{2}$ of its length and the perimeter exceeds the length by 50 inches. Find its dimensions.

22. The melting point of iron is 450 degrees centigrade higher than 5 times that of tin. Three times the number of degrees at which iron melts plus 7 times the number at which tin melts equals 6410. Find the melting point of each metal.

23. How many dollars will amount to $620 in 4 years at 6% simple interest?

Solution. From arithmetic we have

$$amount = principal + interest,$$

or

$$a = p + i = p + prt.$$

Hence, by the conditions of the problem,

$$620 = p + .06 \times 4 \times p.$$

By Principle II, $620 = p(1 + .24).$

By *D*, $p = \frac{620}{1.24} = 500.$

24. Find what principal invested for 6 years at 4½% simple interest will amount to $1270.

25. Find what principal invested for 12 years at 5½% simple interest will amount to $4150.

26. How much must be invested at 6% interest to amount to $2650 at the end of one year?

27. How much must be invested at 5% simple interest to amount to $2025 at the end of 7 years?

28. At what rate of interest per year must $800 be invested to amount to $1000 in 5 years?

HISTORICAL NOTE

Representation of Unknown Numbers. The historical beginnings of algebra are found in the attempted solution of problems. The earliest work known on algebra is by an Egyptian priest, Ahmes, who used "heap" to represent the unknown. One of his problems reads, "heap, its seventh, its whole, it makes nineteen," which means: Solve the equation $\frac{x}{7} + x = 19$.

Diophantus, a Greek (300 A.D.), used the letter *s* to stand for the unknown. The Arabians used the word "thing" and the early European algebraists often used the Latin equivalent *res*. François Vieta used capital letters such as A. The fame of Vieta was such that other writers were led to follow him. Though many writers before Vieta had, now and then, used letters to represent the unknowns, we may regard the final establishment of this custom to date from his work.

René Descartes (see page 239) fixed the custom of using the last letters of the alphabet for the unknowns.

REVIEW QUESTIONS

1. Define equation; identity. State in detail how the equation and the identity differ. Give an example of each.

2. What value of x satisfies the equation $x + 4 = 9$? What value of x will satisfy the equation obtained by adding 7 to each member of this equation? by adding 12? 24?

3. If 4 be added to the first member of the equation $x + 4 = 9$, and 6 to the second member, what value of x will satisfy the equation thus obtained?

4. If the *same* number is added to each member of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

5. If *different* numbers are added to the members of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

6. If the *same* number is subtracted from each member of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

7. If *different* numbers are subtracted from the members of an equation, is the resulting equation equivalent to the first equation? Illustrate by an example.

8. Ask and answer questions, similar to the two preceding, about multiplying and dividing both members of an equation by the *same or different* numbers. Illustrate each by examples.

9. *Each step in solving an equation consists in changing one equation into another equivalent equation whose solution is more apparent than that of the original equation.*

Hence what operations may be performed in solving an equation?

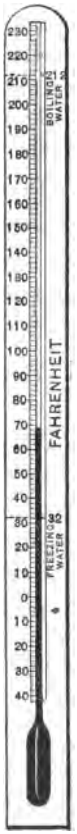
10. State Principle VI in full.

François Vieta (Viète) (1540–1603) was a French lawyer who wrote on mathematics for his own amusement. He printed and distributed his mathematical papers at his own expense. He used letters systematically to represent numbers, and from his time on this became the universal custom in algebra.

Such characters as +, −, though used as early as 1489 by Johann Widmann, did not come into general use before the time of Vieta. His keenness of mind was once shown by his discovery of the key to a Spanish cipher consisting of more than 500 characters. In their astonishment the Spaniards appealed to the Pope, accusing Vieta of using Black Art.

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS



35. A New Kind of Number. Thus far the numbers used have been precisely the same as in arithmetic, though their representation by means of letters and some of the methods used in operating upon them are peculiar to algebra.

We now proceed to the study of a **new kind of number.**

Examples. What is the highest temperature you have ever seen recorded on the thermometer? the lowest?

In answering these questions you not only give certain numbers, but you attach to each a certain **quality.** The temperature is **above zero** or **below zero**; that is, the degrees on the thermometer are measured in **opposite** directions from a starting point which is marked zero.

Many other pairs of quantities possess such **opposite qualities**; for instance, motion to the *right* and to the *left*, *gain* and *loss*, *credit* and *debit.*

It has been found useful in mathematics to extend the number system of arithmetic so as to make it apply directly to cases like these. The opposite qualities involved are designated by the words **positive** and **negative.**

It is commonly agreed to call *above zero* positive and *below zero* negative; motion to the *right* positive and to the *left* negative; *credit* positive and *debit* negative; *gain* positive and *loss* negative.

36. Positive and Negative Numbers. The signs $^+$ and $^-$ stand respectively for the words *positive* and *negative*, and numbers marked with these signs are called *positive* and *negative numbers* respectively. See § 46.

Thus, $5°$ above zero is written $+5°$, and $15°$ below zero is written $-15°$.

When no sign of quality is written, the positive sign is understood.

E.g. $+5°$ is usually written $5°$.

Positive and negative numbers are sometimes called **signed numbers**, because each such number consists of a numerical part, together with a **sign of quality** expressed or understood.

The numerical part of a signed number is called its **absolute value.**

Thus, the absolute value of $+3$ and also of -3 is 3.

37. Graphic Representation of Signed Numbers. *The integers of arithmetic* may be arranged in a series beginning at zero and extending indefinitely toward the right.

Thus, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots$

The integers of algebra may be arranged in a series beginning at zero and extending indefinitely *both to the right and the left.*

Thus, $\cdots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots.$

One of the most extensive uses of signed numbers is for marking the points on a straight line. On an unlimited straight line mark any point zero. On both sides of this point lay off equal divisions, as shown in the figure on page 45.

In order to describe the position of any one of these division points, we need not only an integer of arithmetic, to specify *how far* the given point is from the point marked zero, but also a **sign of quality** to indicate on *which side* of this point it is.

E.g. $+6$ marks the division point 6 units to the right of zero, and -5 marks the point 5 units to the left of zero. Such a diagram is called the **scale of signed numbers**.

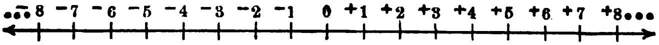
Fractions would of course be represented by points between the integral division points.

ADDITION OF SIGNED NUMBERS

38. Addition by Counting. In arithmetic two numbers may be added by starting with one number and counting *forward* as many units as there are units in the other number.

E.g. To add 3 to 5 we start with 5 and count 6, 7, 8.

Two signed numbers are added in the same manner except that the *direction*, forward or backward, in which we count, is determined by the sign ${}^{+}$ or ${}^{-}$, of the number which we are adding.



Thus, to add ${}^{+}3$ to ${}^{+}5$ begin at ${}^{+}5$ and count 3 to the *right*.
To add ${}^{-}3$ to ${}^{+}5$ begin at ${}^{+}5$ and count 3 to the *left*.
To add ${}^{-}3$ to ${}^{-}5$ begin at ${}^{-}5$ and count 3 to the *left*.
To add ${}^{+}3$ to ${}^{-}5$ begin at ${}^{-}5$ and count 3 to the *right*.
The results are as follows:
${}^{+}5 + {}^{+}3 = {}^{+}8$; ${}^{+}5 + {}^{-}3 = {}^{+}2$; ${}^{-}5 + {}^{-}3 = {}^{-}8$; ${}^{-}5 + {}^{+}3 = {}^{-}2$.
${}^{+}5 + {}^{-}3 = {}^{+}2$ is read *positive* 5 *plus negative* 3 *equals positive* 2.
${}^{-}5 + {}^{-}3 = {}^{-}8$ is read *negative* 5 *plus negative* 3 *equals negative* 8.
In like manner, read the other two.

ORAL EXERCISES

Using the number scale above perform the following additions by counting.

1. ${}^{+}3 + {}^{-}1$.
2. ${}^{+}4 + {}^{-}2$.
3. ${}^{+}5 + {}^{-}3$.
4. ${}^{+}5 + {}^{-}5$.
5. ${}^{+}5 + {}^{-}7$.
6. ${}^{+}5 + {}^{-}8$.
7. ${}^{+}8 + {}^{-}7$.
8. ${}^{+}6 + {}^{-}4$.
9. ${}^{+}7 + {}^{-}7$.
10. ${}^{+}9 + {}^{-}5$.
11. ${}^{-}2 + {}^{-}3$.
12. ${}^{-}3 + {}^{-}5$.
13. ${}^{-}4 + {}^{+}6$.
14. ${}^{-}5 + {}^{+}5$.
15. ${}^{-}7 + {}^{+}8$.
16. ${}^{-}6 + {}^{+}5$.
17. ${}^{-}6 + {}^{-}2$.
18. ${}^{-}7 + {}^{+}7$.
19. ${}^{-}8 + {}^{+}4$.
20. ${}^{-}8 + {}^{+}6$.
21. ${}^{-}4 + {}^{-}3$.
22. ${}^{-}3 + {}^{+}7$.
23. ${}^{+}8 + {}^{-}9$.
24. ${}^{+}5 + {}^{-}3$.
25. ${}^{+}7 + {}^{-}5$.
26. ${}^{+}8 + {}^{-}7$.
27. ${}^{-}3 + {}^{-}4$.
28. ${}^{-}1 + {}^{+}7$.
29. ${}^{-}2 + {}^{-}5$.
30. ${}^{-}7 + {}^{+}5$.

39. Further Illustrations of Signed Numbers. The meaning of positive and negative numbers is further explained in the following

Illustrative Problems: 1. If a man gains $ 1500 and then loses $ 800, what is the net result? *Answer,* $ 700 gain.

In this case the result is obtained by subtracting 800 from 1500. Yet this is not really a problem in subtraction but in addition. That is, we are not asking for the difference between $ 1500 gain and $ 800 loss, but for the *net result* when the gain and the loss are taken together, or the **sum** of the profit and the loss. Hence, we say $ 1500 gain + $ 800 loss = $ 700 gain, or using positive and negative signs,

$$^{+}1500 + {}^{-}800 = {}^{+}700.$$

2. The assets of a commercial house are $ 250,000, and the liabilities are $ 275,000. What is the net financial status of the house? *Answer,* $ 25,000 net liabilities.

Thus,

$ 250,000 assets + $ 275,000 liabilities = $ 25,000 net liabilities.

Or $$^{+}250{,}000 + {}^{-}275{,}000 = {}^{-}25{,}000.$$

3. The thermometer rises 18 degrees and then falls 28 degrees. What direct change in temperature would produce the same result? *Answer,* 10 degrees fall.

Thus, 18° rise + 28° fall = 10° fall.

Or $$^{+}18 + {}^{-}28 = {}^{-}10.$$

4. A man travels 700 miles east and then 400 miles west. What direct journey would bring him to the same final destination? *Answer,* 300 miles east.

Thus, 700 miles east + 400 miles west = 300 miles east.

Or $$^{+}700 + {}^{-}400 = {}^{+}300.$$

Positive and negative numbers may be written in columns and added. Thus the above examples would stand as follows:

$$\begin{array}{r} ^{+}1500 \\ ^{-}800 \\ \hline ^{+}700 \end{array} \qquad \begin{array}{r} ^{+}250{,}000 \\ ^{-}275{,}000 \\ \hline ^{-}25{,}000 \end{array} \qquad \begin{array}{r} ^{+}18 \\ ^{-}28 \\ \hline ^{-}10 \end{array} \qquad \begin{array}{r} ^{+}700 \\ ^{-}400 \\ \hline ^{+}300 \end{array}$$

The preceding exercises illustrate

Principle VII

40. Rule. *To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.*

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case the signs of two numbers are opposite and their absolute values are equal, their sum is zero.

41. Signed Numbers which Cancel each other. We have seen that in finding the sum of two numbers whose signs are *opposite,* they tend to cancel each other.

Thus, in adding ${}^{+}5$ and ${}^{-}8$, the ${}^{+}5$ cancels ${}^{-}5$ out of ${}^{-}8$ and leaves ${}^{-}3$ as the sum. In adding ${}^{-}5$ and ${}^{+}8$, the ${}^{-}5$ cancels ${}^{+}5$ and leaves ${}^{+}3$ as the sum.

ORAL EXERCISES

In this manner find the sums of the following:

1. $\begin{array}{r}{}^{+}7\\ \underline{{}^{-}4}\end{array}$

2. $\begin{array}{r}{}^{-}9\\ \underline{{}^{+}7}\end{array}$

3. $\begin{array}{r}{}^{-}8\\ \underline{{}^{-}3}\end{array}$

4. $\begin{array}{r}{}^{+}12\\ \underline{{}^{-}7}\end{array}$

5. $\begin{array}{r}{}^{-}8\\ \underline{{}^{+}12}\end{array}$

6. $\begin{array}{r}{}^{+}12\\ \underline{{}^{-}7}\end{array}$

7. $\begin{array}{r}{}^{-}8\\ \underline{{}^{-}3}\end{array}$

8. $\begin{array}{r}{}^{-}9\\ \underline{{}^{-}4}\end{array}$

9. $\begin{array}{r}{}^{-}4\\ \underline{{}^{+}8}\end{array}$

10. $\begin{array}{r}{}^{+}9\\ \underline{{}^{-}15}\end{array}$

11. $\begin{array}{r}{}^{+}13\\ \underline{{}^{-}17}\end{array}$

12. $\begin{array}{r}{}^{+}12\\ \underline{{}^{-}14}\end{array}$

13. $\begin{array}{r}{}^{+}7\\ \underline{{}^{-}19}\end{array}$

14. $\begin{array}{r}{}^{-}14\\ \underline{{}^{+}8}\end{array}$

15. $\begin{array}{r}{}^{+}30\\ \underline{{}^{-}40}\end{array}$

16. $\begin{array}{r}{}^{+}20\\ \underline{{}^{-}50}\end{array}$

17. $\begin{array}{r}{}^{+}15\\ \underline{{}^{-}25}\end{array}$

18. $\begin{array}{r}{}^{+}10\\ \underline{{}^{-}20}\end{array}$

19. $\begin{array}{r}{}^{-}25\\ \underline{{}^{+}35}\end{array}$

20. $\begin{array}{r}{}^{-}15\\ \underline{{}^{+}24}\end{array}$

21. $\begin{array}{r}{}^{-}12\\ \underline{{}^{-}18}\end{array}$

22. $\begin{array}{r}{}^{+}20\\ \underline{{}^{-}50}\end{array}$

23. $\begin{array}{r}{}^{+}14\\ \underline{{}^{-}21}\end{array}$

24. $\begin{array}{r}{}^{+}18\\ \underline{{}^{-}24}\end{array}$

25. $\begin{array}{r}{}^{-}9\\ \underline{{}^{-}12}\end{array}$

26. $\begin{array}{r}{}^{-}12\\ \underline{{}^{-}8}\end{array}$

27. $\begin{array}{r}{}^{+}12\\ \underline{{}^{-}16}\end{array}$

28. $\begin{array}{r}{}^{+}18\\ \underline{{}^{-}12}\end{array}$

29. $\begin{array}{r}{}^{+}20\\ \underline{{}^{-}35}\end{array}$

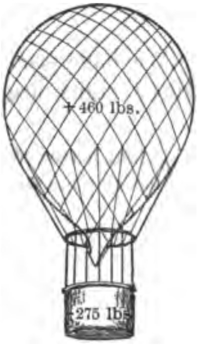
30. $\begin{array}{r}{}^{+}35\\ \underline{{}^{-}20}\end{array}$

Algebraic Sum. The sum of signed numbers obtained as in the preceding exercises is called their *algebraic sum.*

Signed numbers find application in any situation where *opposite qualities* of the kind here considered are present. Besides those already mentioned, other instances occur in the applications below.

WRITTEN EXERCISES

1. A balloon which exerts an upward pull of 460 pounds is attached to a car weighing 275 pounds. What is the net upward or downward pull? Express this as a problem in addition, using positive and negative numbers.


+460 lbs.
275 lbs.

Solution. 460 lbs. upward pull plus 275 lbs. downward pull equals 185 lbs. net upward pull. Using positive numbers to represent upward pull and negative numbers to represent downward pull, this equation becomes

$$^{+}460 + {}^{-}275 = {}^{+}185.$$

In each of the following translate the solution into the language of algebra by means of signed numbers as in Example 1.

2. A balloon which exerts an upward pull of 600 pounds has a 450-pound weight attached to it. What is the net upward or downward pull?

3. A man's property amounts to $45,000 and his debts to $52,000. What is his net debt or property?

4. The assets of a bankrupt firm amount to $245,000 and the liabilities to $325,000. What are the net assets or liabilities?

5. A man can row a boat at the rate of 6 miles per hour. How fast can he proceed against a stream flowing at the rate of $2\frac{1}{2}$ miles per hour? against one flowing 7 miles per hour?

6. A steamer which can make 12 miles per hour in still water is running against a current flowing 15 miles per hour. How fast and in what direction does the steamer move?

42. Averaging Signed Numbers. Half the sum of two numbers is called their *average*. Thus 6 is the average of 4 and 8. Similarly, the *average of three numbers* is one third of their sum, and in general *the average of n numbers* is the sum of the numbers divided by n.

Find the average of each of the following sets:

1. 10, 12, 14, 16, 18.
2. 7, 9, 11, 13, 15, 8.

The average gain or loss per year for a given number of years is the *algebraic sum* of the yearly gains and losses divided by the number of years.

Illustrative Problem. A man lost $ 400 the first year, gained $ 300 the second, and gained $ 1000 the third. What was the average loss or gain?

Solution. $({}^{-}400 + {}^{+}300 + {}^{+}1000) \div 3 = {}^{+}900 \div 3 = {}^{+}300.$
That is, the average gain is $ 300.

WRITTEN PROBLEMS

1. Find the average of $ 1800 loss, $ 3100 loss, $ 6800 gain, $ 10,800 loss, and $ 31,700 gain.

Suggestion. Add all the positive members separately and all negative members separately. Then combine the two sums.

2. Find the average of $ 180 gain, $ 360 loss, $ 480 loss, $ 100 gain, $ 700 gain, $ 400 gain, $ 1300 loss, $ 300 gain, $ 4840 gain, and $ 12,000 gain.

Find the average yearly temperatures at the following places, the monthly averages having been recorded as given below:

3. For New York City: ${}^{+}29^{\circ}, {}^{+}33^{\circ}, {}^{+}39^{\circ}, {}^{+}46^{\circ}, {}^{+}53^{\circ}, {}^{+}63^{\circ}, {}^{+}67^{\circ}, {}^{+}67^{\circ}, {}^{+}61^{\circ}, {}^{+}52^{\circ}, {}^{+}47^{\circ}, {}^{+}41^{\circ}$.

4. For St. Vincent, Minnesota: ${}^{-}5^{\circ}, 0^{\circ}, {}^{+}15^{\circ}, {}^{+}35^{\circ}, {}^{+}55^{\circ}, {}^{+}60^{\circ}, {}^{+}66^{\circ}, {}^{+}63^{\circ}, {}^{+}55^{\circ}, {}^{+}40^{\circ}, {}^{+}22^{\circ}, {}^{+}5^{\circ}$.

5. For Nerchinsk, Siberia: ${}^{-}23^{\circ}, {}^{-}13^{\circ}, {}^{-}10^{\circ}, {}^{+}35^{\circ}, {}^{+}55^{\circ}, {}^{+}70^{\circ}, {}^{+}70^{\circ}, {}^{+}64^{\circ}, {}^{+}50^{\circ}, {}^{+}30^{\circ}, {}^{+}5^{\circ}, {}^{-}15^{\circ}$.

SUBTRACTION OF SIGNED NUMBERS

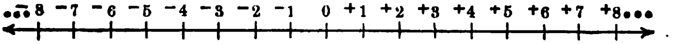
43. Subtraction is the process of finding the number which added to the subtrahend will equal the minuend.

Thus, to subtract 3 from 8 is to find the number which added to 3 will make 8. That is, $8 - 3 = 5$ because $3 + 5 = 8$.

Then the test for subtraction is

$$\textit{Remainder} + \textit{Subtrahend} = \textit{Minuend}.$$

In subtracting signed numbers we may start on the scale at the point indicated by the subtrahend and find *how far* and in *which direction* we must count in order to reach the minuend.



Thus ${}^{+}8 - {}^{+}5 = {}^{+}3$, because from ${}^{+}5$ to ${}^{+}8$ is 3 in the *positive* direction.
${}^{+}8 - {}^{-}5 = {}^{+}13$, because from ${}^{-}5$ to ${}^{+}8$ is 13 in the *positive* direction.
${}^{-}8 - {}^{-}5 = {}^{-}3$, because from ${}^{-}5$ to ${}^{-}8$ is 3 in the *negative* direction.
${}^{-}8 - {}^{+}5 = {}^{-}13$, because from ${}^{+}5$ to ${}^{-}8$ is 13 in the *negative* direction.

Similarly, perform the following subtractions:

1. ${}^{-}4 - {}^{-}2$.
2. ${}^{-}5 - {}^{+}6$.
3. ${}^{-}8 - {}^{-}4$.
4. $7 - {}^{-}6$.
5. $8 - {}^{-}8$.
6. ${}^{-}4 - {}^{-}7$.
7. ${}^{-}7 - {}^{-}4$.
8. ${}^{-}3 - {}^{-}8$.

44. A Short Rule for Subtraction. Since ${}^{+}8 - {}^{-}5 = {}^{+}13$, and since ${}^{+}8 + {}^{+}5 = {}^{+}13$, it follows that *subtracting* ${}^{-}5$ from ${}^{+}8$ gives the same result as *adding* ${}^{+}5$ to ${}^{+}8$. Similarly, ${}^{-}8 - {}^{-}5 = {}^{-}3$ and ${}^{-}8 + {}^{+}5 = {}^{-}3$.

Hence, *subtracting a negative* number is equivalent to *adding a positive* number with the same absolute value.

Since ${}^{+}8 - {}^{+}5 = {}^{+}3$, and since ${}^{+}8 + {}^{-}5 = {}^{+}3$, it follows that subtracting ${}^{+}5$ from ${}^{+}8$ gives the same result as *adding* ${}^{-}5$ to ${}^{+}8$ Similarly, ${}^{-}8 - {}^{+}5 = {}^{-}13$ and ${}^{-}8 + {}^{-}5 = {}^{-}13$.

Hence, *subtracting a positive* number is equivalent to *adding a negative* number with the same absolute value.

These statements are illustrated by such facts as: *Removing a debt* is equivalent to *adding property* and *removing property* is equivalent to *adding debt.*

Positive and negative numbers may be written in columns and subtracted.

Thus $\begin{array}{r} +8 \\ \underline{+5} \\ +3 \end{array}$ $\quad\begin{array}{r} +8 \\ \underline{-5} \\ +13 \end{array}$ $\quad\begin{array}{r} -8 \\ \underline{-5} \\ -3 \end{array}$ $\quad\begin{array}{r} -8 \\ \underline{+5} \\ -13 \end{array}$

ORAL EXERCISES

Perform the following subtractions by changing the sign of the subtrahend and adding:

1. $\begin{array}{r} ^{-}5 \\ \underline{^{-}2} \end{array}$ 3. $\begin{array}{r} ^{+}3 \\ \underline{^{-}5} \end{array}$ 5. $\begin{array}{r} ^{-}16 \\ \underline{^{-}12} \end{array}$ 7. $\begin{array}{r} ^{-}16 \\ \underline{^{+}4} \end{array}$ 9. $\begin{array}{r} ^{+}16 \\ \underline{^{-}12} \end{array}$

2. $\begin{array}{r} ^{-}4 \\ \underline{^{+}1} \end{array}$ 4. $\begin{array}{r} ^{+}57 \\ \underline{^{-}32} \end{array}$ 6. $\begin{array}{r} ^{+}12 \\ \underline{^{+}50} \end{array}$ 8. $\begin{array}{r} ^{-}19 \\ \underline{^{-}24} \end{array}$ 10. $\begin{array}{r} ^{-}48 \\ \underline{^{-}23} \end{array}$

The preceding exercises illustrate

Principle VIII

45. Rule. *To subtract one signed number from another signed number, change the sign of the subtrahend and then add it to the minuend.*

The change in the sign of the subtrahend may be made *mentally* without rewriting the problem. Check by showing that *Remainder* + *Subtrahend* = *Minuend.*

WRITTEN EXERCISES

Perform the following subtractions, changing the signs of the subtrahends *mentally:*

1. $^{-}10 - {}^{-}5$.
2. $^{-}15 - {}^{+}5$.
3. $^{+}20 - {}^{-}15$.
4. $^{+}11 - {}^{+}3$.
5. $^{-}11 - {}^{+}5$.
6. $^{-}17 - {}^{-}20$.
7. $^{+}6 - {}^{-}14$.
8. $^{+}7 - {}^{-}9$.
9. $^{-}11 - {}^{+}6$.
10. $^{-}21 - {}^{-}6$.
11. $^{+}93 - {}^{+}22$.
12. $^{+}17 - {}^{-}13$.
13. $^{-}78 - {}^{-}37$.
14. $^{+}57 - {}^{+}84$.
15. $^{-}48 - {}^{-}31$.
16. $^{-}39 - {}^{-}95$.
17. $91 - {}^{-}3$.
18. $^{-}38 - {}^{+}74$.

Subtraction always Possible in Algebra. In arithmetic subtraction is possible only when the subtrahend is less than, or equal to, the minuend.

E.g. In arithmetic we cannot subtract 5 from 2 since there is no positive number which added to 5 gives 2.

However, in algebra, by means of negative numbers we can as easily perform the subtraction, 2 minus 5, as 5 minus 2.

Thus, $2 - 5 = {}^-3$, since ${}^-3 + 5 = 2$.

Similarly, $0 - {}^+5 = {}^-5$, since ${}^-5 + {}^+5 = 0$, and ${}^-1 - {}^+6 = {}^-7$ since ${}^-7 + {}^+6 = {}^-1$.

46. Double Use of the Signs + and –. In § 36 we agreed that when no sign of quality is written, the sign $^+$ is understood. Hence we may write:

$$ {}^+8 + {}^+5 = 8 + 5. \tag{1}$$

$$ {}^+8 - {}^+5 = 8 - 5. \tag{2}$$

By Principle VII, we have

$$ {}^+8 + {}^-5 = {}^+8 - {}^+5 = 8 - 5. \tag{3}$$

By Principle VIII, we have

$$ {}^+8 - {}^-5 = {}^+8 + {}^+5 = 8 + 5. \tag{4}$$

These examples show how we may dispense with the special signs of quality $^+$, or $^-$, as follows:

1. *Positive numbers are written without any sign indicating quality except where special emphasis is desired, in which case the sign + is used.*

2. *A negative number when standing alone is preceded by the sign –. Thus ${}^-5$ is written -5.*

3. *When a negative number is combined with other numbers, its quality is indicated by the sign – with parentheses inclosing it.*

Thus, $8 + {}^-5$ is written $8 + (-5)$,
and $8 - {}^-5$ is written $8 - (-5)$.

But in such cases it is customary to apply Principles VII and VIII and write at once $8 - 5$ instead of $8 + (-5)$ and $8 + 5$ instead of $8 - (-5)$.

WRITTEN EXERCISES

Work Examples 1–16 twice, first adding, then subtracting.

1. $\begin{array}{r} -25 \\ 14 \\ \hline \end{array}$

2. $\begin{array}{r} 25 \\ -14 \\ \hline \end{array}$

3. $\begin{array}{r} 25 \\ 14 \\ \hline \end{array}$

4. $\begin{array}{r} -25 \\ -14 \\ \hline \end{array}$

5. $\begin{array}{r} -28 \\ -10 \\ \hline \end{array}$

6. $\begin{array}{r} 28 \\ -10 \\ \hline \end{array}$

7. $\begin{array}{r} -28 \\ 10 \\ \hline \end{array}$

8. $\begin{array}{r} 28 \\ 10 \\ \hline \end{array}$

9. $\begin{array}{r} 13 \\ -20 \\ \hline \end{array}$

10. $\begin{array}{r} -13 \\ -20 \\ \hline \end{array}$

11. $\begin{array}{r} -13 \\ 20 \\ \hline \end{array}$

12. $\begin{array}{r} 13 \\ 20 \\ \hline \end{array}$

13. $\begin{array}{r} 16 \\ 30 \\ \hline \end{array}$

14. $\begin{array}{r} -16 \\ 30 \\ \hline \end{array}$

15. $\begin{array}{r} -16 \\ -30 \\ \hline \end{array}$

16. $\begin{array}{r} 16 \\ -30 \\ \hline \end{array}$

Perform the operations indicated.

17. $20-(-5)$
18. $20+(-5)$
19. $20-(-30)$
20. $20x-30x$
21. $15a-20a$
22. $15a-(-20a)$
23. $16n-25n$
24. $40s-(-30s)$
25. $15b-35b$
26. $43x-71x$

EXPLANATORY NOTE

On the Double Use of Signs. It is clear from the examples in this chapter that signed numbers are needed to represent actual conditions in life, as in case of the thermometer. While from now on such numbers will be distinguished, in accordance with universal custom, by the signs $+$, $-$, it should be understood that each of these signs is thus made to represent either one of two entirely different things, namely, an *operation* or a *quality*.

After we acquire some understanding of the matter, this double use of the signs seldom leads to confusion, since we can always tell from the context which use is meant. For example, in $5-3$, the sign $-$ means *subtraction*, while in $x=-3$ it means *negative*.

But for the sake of avoiding confusion at the outset, and to make clear that a negative number is not necessarily a *subtrahend* and that a positive number is not necessarily an *addend*, we have up to this time used the special signs $^{+}$, $^{-}$, which could be readily distinguished from the signs of addition and subtraction. They will now be discontinued.

WRITTEN EXERCISES

Perform the following indicated operations:

1. Find the value of $a + b$ if (1) $a = 4, b = -5$; (2) $a = -2, b = -7$; (3) $a = -6, b = 8$; (4) $a = 6, b = -10$.

2. Find the value of $a - b$ if (1) $a = 8, b = 8$; (2) $a = -3, b = -7$; (3) $a = 4, b = -9$; (4) $a = -3, b = 6$.

3. Find the value of $a + b + c$ if (1) $a = 3, b = -4, c = -1$; (2) $a = -7, b = 3, c = 2$; (3) $a = -1, b = -8, c = 10$.

4. Find the value of $a + b - c$ if (1) $a = 6, b = -4, c = 5$; (2) $a = -2, b = -4, c = -6$; (3) $a = 7, b = -8, c = -6$.

5. Find the value of $a - b - c$ if (1) $a = 3, b = 6, c = -2$; (2) $a = -6, b = 7, c = -12$; (3) $a = 8, b = 4, c = 8$.

6. Find the value of $-a - b + c$ if (1) $a = 7, b = -6, c = 4$; (2) $a = -2, b = -6, c = 2$.

7. Find the value of $-a + b - c$ if (1) $a = -1, b = -2, c = -3$; (2) $a = -8, b = 10, c = -2$.

8. Find the value of $-a - b - c$ if (1) $a = -3, b = -2, c = -1$; (2) $a = 6, b = -3, c = -9$.

Solve the following equations:

9. $x + 8 = 4$. *Suggestion.* Subtract 8 from each member, or transpose 8 and change its sign.

10. $x + 3 = 7$.

11. $x - 9 = 1$.

12. $3 + x = 0$.

13. $x + 13 = 7$.

14. $-4 + x = -9$.

15. $-5 + x = 4$.

16. $-5 + x = 12$.

17. $-9 + x = -18$.

18. $-35 + x = 17$.

19. $17 + x = -35$.

20. $x - 14 = -18$.

21. $x - 25 = 16$.

22. $x + 4 = 1$.

23. $x - 7 = -15$.

24. $-21 + x = -17$.

25. $-16 + x = -18$.

26. $-12 + x = -20$.

ORAL EXERCISES

Add the following:

1.	18	**4.**	− 9	**7.**	81	**10.**	16	**13.**	− 9
	− 7		18		− 72		− 24		− 7
2.	− 6	**5.**	− 24	**8.**	46	**11.**	− 16	**14.**	− 8
	5		17		− 38		− 7		14
3.	− 7	**6.**	− 18	**9.**	− 36	**12.**	− 8	**15.**	− 17
	− 14		42		24		− 9		4

Subtract the following:

16.	30	**19.**	4	**22.**	7	**25.**	− 10	**28.**	10
	− 15		− 12		− 7		20		− 40
17.	− 4	**20.**	6	**23.**	− 20	**26.**	− 10	**29.**	70
	8		− 8		14		− 20		− 40
18.	− 4	**21.**	− 7	**24.**	10	**27.**	− 15	**30.**	− 16
	− 12		7		− 20		15		8

Solve each of the following, using an equation involving positive and negative numbers.

31. A dove which can fly 40 miles per hour in calm weather is flying against a hurricane blowing at the rate of 60 miles per hour. How fast and in what direction is the dove moving?

32. If of two partners, one loses $ 1400 and the other gains $ 3700, what is the net result to the firm?

33. A man's income is $ 2400 and his expenses $ 1500 per year. What is the net result for the year?

34. A man loses $ 800 and then loses $ 600 more. What is the combined loss? Indicate the result as the sum of two negative numbers.

MULTIPLICATION OF SIGNED NUMBERS

47. The multiplication of signed numbers is illustrated by the following problems.

Illustrative Problem. A balloonist, just before starting, makes the following preparations: (*a*) He adds 9000 cubic feet of gas with a lifting power of 75 pounds per thousand cubic feet. (*b*) He takes on 8 bags of sand, each weighing 15 pounds. How do these operations affect the buoyancy of the balloon?

Solution. (*a*) A lifting power of 75 pounds is indicated by $+75$, and adding such a power 9 times is indicated by $+9$. Hence, $+9(+75) = +675$, or 675 is the total *lifting* power added.

(*b*) A weight of 15 pounds is indicated by -15, and adding 8 such weights is indicated by $+8$. Since the total weight added is 120 pounds, we have $+8(-15) = -120$, which is the total *depressing* power added.

Illustrative Problem. During the course of his journey the balloonist opens the valve and allows 2000 cubic feet of gas to escape, and later throws overboard 4 bags of sand. How does each of these operations affect the buoyancy of the balloon?

Solution. (*a*) The gas, being a lifting power, is positive, but the removal of 2000 cubic feet of it is indicated by -2, and the result is a depression of the balloon by 150 pounds; that is, $-2 \cdot (+75) = -150$.

(*b*) The removal of 4 weights is indicated by -4, but the weights themselves have the negative quality of downward pull. Hence to remove 4 weights of 15 pounds each is equivalent to increasing the buoyancy of the balloon by 60 pounds; that is, $-4 \cdot (-15) = +60 = 60$.

Notion of Multiplication Extended. These examples illustrate a natural extension of multiplication in arithmetic.

E.g. Just as $3 \cdot 4 = 4 + 4 + 4 = 12$, so $3 \cdot (-4) = -4 + (-4) + (-4) = -12$. Hence we write $+3 \cdot (+4) = +12 = 12$, and $+3 \cdot -4 = -12$.

Again, just as we take the multiplicand *additively* when the multiplier is a positive integer, so we take it *subtractively* when the multiplier is a negative integer.

E.g. $-3 \cdot (+4) = -(+4) - (+4) - (+4) = -12$,

and $-3 \cdot (-4) = -(-4) - (-4) - (-4) = -(-12) = +12$.

Hence we write $-3 \cdot (+4) = -12$ and $-3 \cdot (-4) = +12 = 12$.

Two numbers may be conveniently placed in columns for multiplying.

Thus, $\begin{array}{r} +4 \\ +3 \\ \hline +12 \end{array} \qquad \begin{array}{r} +4 \\ -3 \\ \hline -12 \end{array} \qquad \begin{array}{r} -4 \\ +3 \\ \hline -12 \end{array} \qquad \begin{array}{r} -4 \\ -3 \\ \hline +12 \end{array}$

ORAL EXERCISES

Explain the following indicated multiplications and find the product in each case.

1. $-3 \cdot (-10)$.
2. $10 \cdot (-3)$.
3. $10 \cdot (-5)$.
4. $-7 \cdot (-8)$.
5. $12 \cdot (-4)$.
6. $-7 \cdot (-6)$.
7. $-5 \cdot (-12)$.
8. $-5 \cdot (-8)$.
9. $-8 \cdot 6$.

The preceding exercises illustrate

Principle IX

48. Rule. *If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.*

In applying this principle observe that the sign of the product is obtained quite independently of the absolute value of the factors.

E.g. $\frac{3}{4} \cdot (-5) = -(\frac{15}{4}) = -3\frac{3}{4}$; $-12 \cdot (-3\frac{1}{2}) = +42 = 42$.

ORAL EXERCISES

Multiply the following:

1. $\begin{array}{r} 7 \\ -4 \\ \hline \end{array}$
2. $\begin{array}{r} -5 \\ 6 \\ \hline \end{array}$
3. $\begin{array}{r} 8 \\ -5 \\ \hline \end{array}$
4. $\begin{array}{r} -12 \\ 4 \\ \hline \end{array}$
5. $\begin{array}{r} 6 \\ -8 \\ \hline \end{array}$
6. $\begin{array}{r} 9 \\ -8 \\ \hline \end{array}$
7. $\begin{array}{r} 10 \\ -8 \\ \hline \end{array}$
8. $\begin{array}{r} 20 \\ 6 \\ \hline \end{array}$
9. $\begin{array}{r} 9 \\ -10 \\ \hline \end{array}$
10. $\begin{array}{r} 8 \\ -6 \\ \hline \end{array}$
11. $\begin{array}{r} -5 \\ 4 \\ \hline \end{array}$
12. $\begin{array}{r} -6 \\ 5 \\ \hline \end{array}$

WRITTEN EXERCISES

Multiply the following pairs of numbers:

1. $\begin{array}{r} -14 \\ -\ 3 \\ \hline \end{array}$

2. $\begin{array}{r} -14 \\ 3 \\ \hline \end{array}$

3. $\begin{array}{r} 14 \\ -\ 3 \\ \hline \end{array}$

4. $\begin{array}{r} -40 \\ 5 \\ \hline \end{array}$

5. $\begin{array}{r} -35 \\ -\ 5 \\ \hline \end{array}$

6. $\begin{array}{r} 45 \\ 6 \\ \hline \end{array}$

7. $\begin{array}{r} 25 \\ 4 \\ \hline \end{array}$

8. $\begin{array}{r} -25 \\ 4 \\ \hline \end{array}$

9. $\begin{array}{r} -25 \\ -\ 4 \\ \hline \end{array}$

10. $\begin{array}{r} -4.2 \\ 6 \\ \hline \end{array}$

11. $\begin{array}{r} -5.4 \\ -\ 4 \\ \hline \end{array}$

12. $\begin{array}{r} 15a \\ -\ 4 \\ \hline \end{array}$

13. $\begin{array}{r} -22x \\ -\ 3 \\ \hline \end{array}$

14. $\begin{array}{r} -18m \\ 4 \\ \hline \end{array}$

15. $\begin{array}{r} -16ab \\ -\ 2 \\ \hline \end{array}$

49. The product of several signed numbers is found as illustrated in the following:

$-2\cdot 5\cdot(-3)\cdot(-4)\cdot 6=-10\cdot(-3)\cdot(-4)\cdot 6=30\cdot(-4)\cdot 6$ $=-120\cdot 6=-720$. That is, the first two factors are multiplied together, then their product is multiplied by the next factor, and so on, until all the factors are multiplied.

Since the product of all positive factors is positive, the final sign depends upon the number of negative factors. If this number is *even*, the product is positive; if it is *odd*, the product is negative.

E.g. If there are 3 negative factors, the product is negative; if there are 4, it is positive.

In the following exercises determine the sign of the product before finding its absolute value.

ORAL EXERCISES

1. $-4\cdot 3\cdot(-2)\cdot(-2)$.
2. $-2\cdot(-3)\cdot(-5)\cdot 3$.
3. $-5\cdot(-3)(-2)(-4)(-1)$.
4. $-5\cdot(-4)\cdot 3\cdot(-2)$.
5. $8\cdot(-9)\cdot(-1)\cdot(-2)$.
6. $-5\cdot(-2)\cdot(-3)\cdot(-4)$.

DIVISION OF SIGNED NUMBERS

50. Test for Division. In arithmetic we test the correctness of division by showing that the quotient multiplied by the divisor equals the dividend.

E.g. $27 \div 9 = 3$, because $9 \cdot 3 = 27$.

Hence **division** may be defined as the process of finding one of two factors when their product and the other factor are given.

The given product is the **dividend**, the given factor is the **divisor**, and the factor to be found is the **quotient**.

In dividing signed numbers the above test determines the *sign* of the quotient as well as its *absolute value*.

E.g. $-42 \div (+6) = -7$, because $-7 \cdot (+6) = -42$;
also $-42 \div (-6) = +7$, because $+7 \cdot (-6) = -42$.

So in every case the test of the correctness of division is:

Quotient × *Divisor* = *Dividend.*

Find the following quotients and check as above:

1. $-25 \div 5$.
2. $-ab \div a$.
3. $5xy \div (-x)$.
4. $-9rs \div (-3)$.
5. $75y \div (-15)$.
6. $-121x \div 11$.

The preceding exercises illustrate

Principle X

51. Rule. *The quotient of two signed numbers is positive if the dividend and divisor have like signs, negative if they have opposite signs.*

ORAL EXERCISES

1. $\frac{6}{-3}$
2. $\frac{6}{3}$
3. $\frac{-6}{3}$
4. $\frac{-6}{-3}$
5. $\frac{8}{-4}$
6. $\frac{-8}{-4}$
7. $\frac{-8}{4}$
8. $\frac{-15}{-5}$
9. $\frac{-15}{5}$
10. $\frac{20}{-10}$
11. $\frac{-20}{10}$
12. $\frac{-20}{-10}$
13. $\frac{48}{16}$
14. $\frac{-48}{-16}$
15. $\frac{-48}{16}$

EXERCISES

Perform the following indicated divisions, and check the first ten by multiplying quotient by divisor. Do Examples 1–21 orally.

1. $\frac{-28}{7}$.
2. $\frac{-42}{-6}$.
3. $\frac{51}{-17}$.
4. $\frac{-21}{-3}$.
5. $\frac{-75}{5}$.
6. $\frac{-16}{-1}$.
7. $\frac{-49}{7}$.
8. $\frac{64}{-4}$.
9. $\frac{4\cdot(-9)}{-3}$.
10. $\frac{-3\cdot 8}{-4}$.
11. $\frac{10\cdot(-9)}{-5}$.
12. $\frac{3\cdot(-4)\cdot(-2)\cdot 6}{-3}$.
13. $\frac{-42a}{-3}$.
14. $\frac{-42x}{3}$.
15. $\frac{42x}{-3}$.
16. $\frac{75abc}{-a}$.
17. $\frac{-100xy}{-x}$.
18. $\frac{-320y}{80}$.
19. $\frac{25xy}{-x}$.
20. $\frac{-196mn}{-14}$.
21. $\frac{-39ab}{-13a}$.

22. A man lost $300, $500, and $700 during three consecutive months. Express his average monthly loss as a quotient.

23. During five consecutive days the minimum temperature was − 5°, − 8°, − 10°, − 4°, − 6° respectively. Find the average of these temperatures.

24. A trader lost $250 in each of three months and gained $75 during each of the four succeeding months. Find the average gain or loss for the seven months.

25. On a cold day the following temperatures was observed: − 20°, − 16°, − 12°, − 4°, 2°, 8°, 4°, − 2°. Find the average of these readings.

26. Find the average of the numbers: 280, − 960, − 840, − 360, 860, − 260, − 180, 530, and − 480. See suggestions under Example 1, page 49.

52. The Principles applied to Signed Numbers. While Principles I–V were studied in connection with unsigned, or arithmetic numbers only, it is now important to note that they all apply to *signed* numbers as well.

In the statement of these principles the word *number* will from now on be understood to refer either to the ordinary numbers of arithmetic or to the signed numbers of algebra, as occasion may require. It should also be noticed that the numbers of arithmetic are used as freely in algebra as in arithmetic. It is only when we wish to distinguish them from negative numbers that they are called *positive* numbers.

ORAL EXERCISES

If $a = 6$, $b = 4$, $c = -2$, evaluate the following:

1. $a(b + c)$.
2. $a(b - c)$.
3. $a(c - b)$.
4. $c(a + b)$.
5. $c(b - a)$.
6. $c(a - b)$.

WRITTEN EXERCISES

1. Find the quotient $\dfrac{2(-3)(-4)(-6)(-8)(-2)}{4(-12)(-16)}$.

Solution. There are five negative factors in the dividend and two in the divisor. Hence the sign of the dividend is $-$ and that of the divisor is $+$. (See § 49.) Therefore the sign of the quotient is $-$. See § 51. We now cancel as if all the factors were positive and prefix the negative sign to the quotient, obtaining -3.

In like manner find the following quotients:

2. $\dfrac{a(-b)(-e)(-d)}{ab(-e)(-d)}$.
3. $\dfrac{3(-12)(-18)(-32)}{6 \cdot 8(-24)(-3)}$.
4. $\dfrac{-8 \cdot 9(-3)(-4)}{-3(-4)(-2)(-6)}$.
5. $\dfrac{ab(-c)(d)(-e)}{(-a)(-b)(-d)(e)}$.
6. $\dfrac{x(-y)(-z)(-4)}{xy(-4z)}$.
7. $\dfrac{2x(-y)(-z)(-v)}{-xv(-z)y(-3)}$.

The **number system of algebra**, so far as we have now studied it, consists of the *numbers of arithmetic* together with the *negative numbers.*

HISTORICAL NOTE

The Development of Negative Numbers. The Greeks had no conception of a negative number as distinct from a number to be subtracted. Diophantus states that in the multiplication of $(a - b)$ by $(c - d)$, *a subtraction multiplied by a subtraction gives an addition.* That is, $(-b)(-d) = bd$. But in applying the rule he takes care that a is greater than b, and c greater than d. However, the Hindus appear to have had quite clear notions of a purely "negative number" as distinct from a number to be subtracted. They recognized the difference between positive and negative numbers by attaching to one the idea of debt and to the other that of assets, or by letting them represent distances in opposite directions. The Arabs, however, failed to understand the negative numbers and did not include them in the algebra which they brought to Europe. (See page 32.) Until the beginning of the seventeenth century, mathematicians dealt almost exclusively with positive numbers.

Thomas Harriot, an Englishman (1560–1621), was the first to write a negative number all by itself. Thus an equation like $x = -3$ was never written by any of Harriot's predecessors.

The negative numbers were brought permanently into mathematics by René Descartes. (See page 239.) Trying to number all the points of a complete straight line, Descartes was compelled to start at some point and **number in both directions**. Then it became convenient to distinguish the numbers on the two sides of this starting point as positive and negative, respectively.

Sir Isaac Newton (see page 101) was the first to let a letter stand for any number, negative as well as positive. In such a formula as $a(b + c) = ab + ac$, the predecessors of Newton had restricted the letters to represent any **positive** numbers, while Newton regarded the letters as representing *any numbers whatever,* either positive or negative. This was of very great importance, since it greatly reduced the number of formulas required.

Negative numbers appeared "absurd" or "fictitious" until a **visual** or **graphical representation of them** was discovered. Cajori in his history of elementary mathematics says: "Omit all illustrations by lines, thermometers, etc., and negative numbers will be as obscure to modern students as they were to the early algebraists." From the experience of the early mathematicians it would appear that if the pupil wishes really to understand positive and negative numbers, he must study with care applications such as are given in the first part of this chapter.

INTERPRETATION AND USE OF NEGATIVE NUMBERS

53. A negative result obtained in solving a problem may have a natural interpretation, or it may indicate that the conditions of the problem are impossible.

A similar statement holds regarding fractional or zero answers in arithmetic. For example, if we say there are twice as many girls as boys in a schoolroom and 35 pupils in all, the number of boys would be $35 \div 3 = 11\frac{2}{3}$, which indicates that the conditions of the problem are impossible.

Again, if three buildings cost in all $18,500, and if the second cost $9000 more than the first, and the third $9500 more than the second, then the cost of the first building would be zero, which is impossible.

Illustrative Problem. The crews on three steamers together number 94 men. The second has 40 more than the first, and the third 20 more than the second. How many men in each crew?

Solution. Let $n =$ number of men in first crew.

Then, $n + 40 =$ number of men in second crew,

and $n + 40 + 20 =$ number of men in third crew.

Hence, $n + n + 40 + n + 40 + 20 = 94,$

and
$$3n + 100 = 94.$$
$$3n = -6.$$
$$n = -2.$$

Here the negative result indicates that the conditions of the problem are *impossible.*

Illustrative Problem. A real estate agent gained $8400 on four transactions. On the first he gained $6400, on the second he lost $2100, on the third he gained $5000. Did he lose or gain on the fourth transaction, and how much?

Solution. Since we do not know whether he gained or lost on the fourth transaction, we represent the unknown number by n, which may be positive or negative, as will be determined by the solution of the problem.

Then we have $6400 + (-2100) + 5000 + n = 8400.$ (1)

Hence, by VII, *F*, $9300 + n = 8400.$ (2)

By S, $n = 8400 - 9300.$ (3)

By VIII, $n = -900.$ (4)

In this case the negative result indicates that the conditions of the problem are possible, and that there was a *loss* on the fourth transaction.

PROBLEMS

In the following problems give the solutions in full and state all principles used.

In case a negative answer is found, state whether this answer has a natural interpretation, or whether it indicates that the conditions of the problem are impossible.

1. A man gains $2100 during one year. During the first three months he loses $125 per month, then gains $500 per month during the next five months. What is the average gain or loss per month during the remaining four months?

2. A man rowing against a swift current goes 9 miles in 5 hours. The second hour he goes two miles less than the first, the third three miles more than the second, the fourth one mile more than the third, and the fifth one mile more than the fourth. How many miles did he go during each of the five hours?

3. There are three trees the sum of whose heights is 108 feet. The second is 40 feet taller than the first, and the third is 30 feet taller than the second. How tall is each tree?

In the next three examples find the average yearly temperatures, the average monthly temperatures being as here given:

4. Port Conger, off the northwest coast of Greenland: − 37°, − 43°, − 32°, − 15°, 14°, 18°, 35°, 34°, 25°, 4°, − 17°, − 30°.

5. Franz Joseph's Land: − 20°, − 20°, − 10°, 0°, 15°, 30°, 35°, 30°, 20°, 10°, 0°, − 10°.

6. North Central Siberia: − 60°, − 50°, − 30°, 0°, 15°, 40°, 40°, 35°, 30°, 0°, − 30°, − 50°.

7. A merchant gained an average of $2800 per year for 5 years. The first year he gained $3000, the second $1500, the third $4000, and the fourth $2400. Did he gain or lose during the fifth year, and how much?

8. A certain business shows an average gain of $ 4000 per year for 6 years. During the first five years the results were: $ 8000 loss, $ 10,000 gain, $ 7000 gain, $ 3000 gain, and $ 12,000 gain. Find the loss or gain during the sixth year.

9. A commercial house averaged $ 10,000 gain for 6 years. What was the loss or gain the first year if the remaining years show: $ 8000 gain, $ 24,000 gain, $ 2000 loss, $ 20,000 gain, and $ 30,000 gain, respectively?

REVIEW QUESTIONS

1. Name several pairs of opposite qualities all of which are conveniently described by the words *positive* and *negative*. What symbols are used to replace these words when applied to numbers?

2. When loss is added to profit, is the profit increased or decreased? What algebraic symbols may be used to distinguish the numbers representing profit and loss?

3. On the number scale indicate what is meant by $+2$; by -2. Indicate what is meant by the sign + in $5+2$; by the sign − in $5-2$; by the sign − in $x=-2$.

4. Why do we call positive and negative numbers *signed* numbers? What is meant by the *absolute value* of a number?

5. State Principle VII in full.

6. How is the correctness of subtraction tested in arithmetic? Is the same test applicable to subtraction in algebra?

7. Illustrate the subtraction of positive and negative numbers by an example involving profit and loss.

8. Show by counting on the number scale that the result of subtraction gives the *distance* from subtrahend to minuend and that the sign of the remainder shows the *direction* from subtrahend toward the minuend. For example, use $8-(-5)$ and $-8-(+5)$ to illustrate this.

9. How do negative numbers make subtraction possible in cases where it is impossible in arithmetic?

10. What is a convenient rule for subtracting signed numbers? State Principle VIII.

11. Write an equation whose solution is a negative number.

12. Give an example in which positive and negative numbers are multiplied. State Principle IX.

13. Define division. How do we obtain the law of signs in division? State Principle X. What is the test of the correctness of division?

14. Explain how one set of signs + and − can be used to indicate both quality and operation.

15. By means of Principles VII, VIII, IX, and X, simplify the expressions, $a+b$, $a-b$, $a \cdot b$, $\frac{a}{b}$, after substituting in each various positive and negative values of a and b.

16. Add Principles VII, VIII, IX, and X to the list which you made in Chapters I and II. It is absolutely necessary that you remember the rules stated in these principles. Any short phrases that will assist you in this are of value. For instance, the following:

VII. *In addition, positive and negative numbers tend to cancel each other. The common sign or the sign of the numerically greater is the sign of the result.*

VIII. *In subtraction, change the sign of the subtrahend and add.*

IX. *In multiplication, two like signs give + and two unlike signs give −.*

X. *In division, like signs give + and unlike signs give −.*

CHAPTER IV

ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS

54. Building Algebraic Expressions. In the preceding chapter we have noticed that in solving problems we are led to represent numbers by means of algebraic expressions which are formed by combining several algebraic symbols.

E.g. If x is a number representing my age in years, then $x - 10$ is the number representing my age 10 years ago, and $2(x - 10)$ is double the number representing my age ten years ago.

Such expressions are now to be studied more in detail.

55. Polynomials; Terms. An algebraic expression composed of parts connected by the signs $+$ and $-$ is called a **polynomial.** Each of the parts thus connected, together with the sign preceding it, is called a **term.**

E.g. $5a - 3xy - \frac{2}{3}rt + 99$ is a polynomial whose terms are $5a$, $-3xy$, $-\frac{2}{3}rt$, and $+99$. The sign $+$ is understood before $5a$.

A polynomial of two terms is called a **binomial**; one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.** Terms which are to be added are called **addends.**

E.g. $5a - 3xy$ is a binomial; $5a - 3xy - \frac{2}{3}rt$ is a trinomial whose terms are the monomials $5a$, $-3xy$, $-\frac{2}{3}rt$.

According to the above definition $x + (b + c)$ may be called a binomial though it is equivalent to the trinomial $x + b + c$.

In this case x is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3t + 4x - 5(a + b)y$ a trinomial having the simple terms $3t$, $4x$, and the compound term $-5(a + b)y$.

56. Similar Terms. Two terms which have a factor in common are said to be **similar with respect to that factor.**

E.g. $5a$ and $-3a$ are similar with respect to a; $-3xy$ and $-7x$ are similar with respect to x; $5a$ and $-5b$ are similar with respect to 5; $7abc$ and $-\frac{2}{3}abc$ are similar with respect to abc.

Similar terms may be combined by Principle I.

E.g. $5a + 3a = (5+3)a = 8a$; $3xy - 7x = x(3y - 7)$; $5a - 5b = 5(a - b)$; $ax + bx - cx = (a + b - c)x$.

ORAL EXERCISES

Select the common factor and combine the similar terms in each of the following:

1. $5a + 3a$.
2. $4n + 5n$.
3. $2b + 10b$.
4. $9c + 8c + 7c$.
5. $14x + 7x + 4x$.
6. $2k + 4k + 8k$,
7. $8y + 3y + 5y$.
8. $16z + 2z + 3z$.
9. $12v + 4v + 6v$.
10. $7m + 4m + 8m$.
11. $8a - 3a + 2a$.
12. $3x + 7x - 6x$.
13. $5y - 2y - y$.
14. $5ab + 3ab + 2ab$.
15. $7x - 5x + 4x$.
16. $3a - 2a + 4a$.

WRITTEN EXERCISES

Combine similar terms in the following:

17.	ax	**20.**	$3ab$	**23.**	$7ax$	**26.**	$5ax$
	bx		$-2ab$		$3bx$		$3ax$
	cx		$5ab$		$12cx$		$2ax$
18.	$2ar$	**21.**	$11rs$	**24.**	$6ab$	**27.**	$4xy$
	$3br$		$-2st$		$7ac$		$-3yz$
	$-2cr$		$4as$		$-ad$		$-5wy$
19.	$12cd$	**22.**	$4xyz$	**25.**	$9abc$	**28.**	$7xyc$
	$-4cf$		$2xyz$		$3dbc$		$-5ayx$
	$-5gc$		$-3xyz$		$-4ebc$		$2bxy$

ADDITION AND SUBTRACTION OF POLYNOMIALS

Addition of Polynomials. In adding polynomials we use

Principle XI

57. **Rule.** *If several terms are to be added, they may be arranged and combined in any desired order.*

The truth of this principle may be seen from simple examples:

Thus, $2 + 3 + 5 = 3 + 2 + 5 = 2 + (3 + 5) = (3 + 2) + 5 = 10.$
Also, $8 + (-2) + 6 = -2 + 8 + 6 = -2 + (8 + 6) = 12.$

HISTORICAL NOTE

Associative and Commutative Laws of Addition. The fundamental character of Principle XI was first recognized about one hundred years ago. The principle as here given combines in one statement two laws of algebra: (1) **the associative law**, first so called by F. S. Servois (1814); (2) **the commutative law**, first so called by Sir William Hamilton.

The associative law states that addends may be *grouped* in any manner. Thus, $a + b + c = a + (b + c) = (a + b) + c.$

The commutative law states that addends may be put in any desired *order*. Thus, $a + b = b + a.$

58. **Arranging Terms in Columns.** In adding polynomials the work may be arranged conveniently by placing similar terms in the same column. This is permissible by Principle XI.

Example. Add $5x - 6y + 4z + 5at$; $-3x + 11y - 16z - 9bt$; and $-7y + 8z$.

Arranging similar terms in columns, and applying Principles I and VII, we have

$$\begin{array}{rrrl}
5x & -\ \ 6y & +\ \ 4z & +\ 5at \\
-3x & +\ 11y & -\ 16z & -\ 9bt \\
 & -\ \ 7y & +\ \ 8z & \\
\hline
2x & -\ \ 2y & -\ \ 4z & +\ (5a - 9b)t
\end{array}$$

Check. Putting $x = y = z = t = a = b = 1$, in each of the polynomials, and in their sum we have:

$$\begin{array}{rrrrrr}
5 & -\ 6 & +\ \ 4 & +\ 5 & & = \ \ 8 \\
-3 & +\ 11 & -\ 16 & -\ 9 & & = -17 \\
 & & -\ \ 7 & +\ 8 & & = \ \ 1 \\
\hline
2 & -\ 2 & -\ \ 4 & +\ 5 & -\ 9 & = -8
\end{array}$$

WRITTEN EXERCISES

Arrange similar terms in columns and add:

1. Add $7b - 3c + 2d$; $-2b + 8c - 13d$.

2. Add $6x - 3y + 4t - 7z$; $x - 5y - 3t$; $4x - 4y + 8t$.

3. Add $7a - 4x + 12z$; $8a - 3x + 2z$; $2a + 4x - 3z$; $5a - 2x - 4z$.

4. Add $5ac + 3bc - 14c + 8b$; $2b + 3c - 12bc - 3ac$; $4b + 4c + bc - ac$; $2bc + 4ac + c$; $3b - 4c$.

5. Add $16xy - 13cd$; $15ab - 2xy$; $34cd - 3xy + 2ab$; $14cd - 3xy - 2ab$.

6. Add $34ax + 4by - 3z$; $2by + 5z$; $3ax - 7by + 5z$; $7ax + 4by - 4z$.

7. Add $3ab + 4cd - 2ae$; $ab - 3cd + 3ae$; $3cd - 2ab$; $4cd - 5ae + 7ab$.

8. Add $7ax - 13by + 5$; $9ax + 8by - 4$; $3by - 12ax$; $4ax + 7by - 9$.

9. Add $5ab - 3 \cdot 67 + 5(x-1)$; $5 \cdot 67 + 3ab - 2(x-1)$; $3(x-1) - 4 \cdot 67 + 2ab$.

10. Add $11(c-9) + 3(x+y) + 21wu$; $-71wu - 5(x+y)$; $18wu + 2(x+y) - 13(c-9)$.

11. Add $5(a+b) - 3(c-d)$; $3(c-d) - 8(a+b)$; $-2(a+b)$; $13(c-d) - 4(a+b)$.

12. Add $3 + 4(c-d) - 5(a-b-c)$; $4(a-b-c) + 5(c-d)$; $3(a-b-c) - 9(c-d) + 12$.

13. Add $(a-b) - 3(c-d) + 4(a+b)$; $5(a-b) + 4(c-d)$; $7(c-d) - 9(a-b) + 3(a+b)$.

14. Add $7(x-y) - 4(x+y) + 4ab$; $9(x+y) + 3(x-y)$; $6(x-y) - 2ab - 3(x+y)$.

15. Add $3(x-5) + 4(b+c) + 3(x-y)$; $8(b+c) - 5(x-y)$; $8(x-5) - 7(b+c) - 4(x-y)$; $3(x-y) + (x-5)$.

Sir William Rowan Hamilton (1805–1865) was born in the city of Dublin, of Scotch parents. Already in early childhood he gave evidence of a brilliant mind. As a young man at college "amongst a number of competitors of more than ordinary merit he was first in every subject and at every examination."

Before taking his final examinations he was appointed in 1827 to the professorship of astronomy in the University of Dublin. He made a profound study of algebra and created a new branch of that subject which is called *quaternions*. In the opinion of some writers this will ultimately be regarded as one of the great discoveries of the nineteenth century.

16. Add $16(a+b-c)-3(x-y)+2(a-b)$; $2(x-y)-4\,xy$; $3(a-b)+(a+b-c)$; $7(a-b)+4(x-y)-8(a+b-c)$.

17. Add $6(a-b)-5(x+y)+7(x-z)-4\,abc$; $7(x-z)+5$; $9(x+y)+(a-b)+2\,abc$; $11(a-b)+10\,abc+3(x-z)+8(x+y)$.

18. Add $2(x-y+z)+7(a-x)-3(z-y)$; $3(x-y+z)-4(a-x)+5(z-y)$; $5(a-x)-2(z-y)+5(x-y+z)$; $5(x-y+z)+4(z-y)$.

19. Add $2\,ab+4(a-b)+a(b+3)+b(a+2)$; $4\,ab-2(a-b)+2\,a(b+3)-2\,b(a+2)$; $2\,b(a+2)+a(b+3)-2\,ab$.

20. Add $x(y-z)+y(x-z)+z(x-y)$; $5\,xy-2\,x(y-z)+3\,z(x-y)$; $-3\,xy+2\,x(y-z)$; $-xy-x(y-z)+2\,y(x-z)$.

59. Adding Polynomials without Rewriting them. In practice polynomials may be added without writing the similar terms in columns.

For example, to add $-3\,a+2\,b-4\,c$, $5\,a+4\,b+2\,c$, and $7\,a-3\,b-5\,c$, we first pick out the terms having a as a factor and add them *mentally*, then the terms containing b, and finally those containing c:

Thus, $-3\,a+5\,a+7\,a=9\,a$; $2\,b+4\,b-3\,b=3\,b$; $-4\,c+2\,c-5\,c=-7\,c$.

Hence the sum $9\,a+3\,b-7\,c$ may be written down at once.

ORAL EXERCISES

Pick out similar terms and add mentally:

1. $5\,x+3\,y-2\,z$; $-2\,x-7\,y+4\,z$.
2. $-3\,m+7\,n-6\,p$; $6\,m-5\,n+3\,p$.
3. $3\,k-7\,r-5\,s$; $9\,r-7\,k+8\,s$.
4. $3\,b+7\,ax+2\,d$; $7\,b-4\,ax+5\,d$.
5. $5\,ax+2\,bd-3\,c$; $-7\,ax-bd+6\,c$.
6. $3\,mn+4\,pq-7\,rs$; $-8\,pq+9\,rs-5\,mn$.
7. $3(a+b)+7(x-y)$; $-9(a+b)+11(x-y)$.
8. $3\,x-2\,y+4(a-b)$; $-8\,x+6\,y+8(a-b)$.
9. $4\,x-2\,y+3\,z$; $-5\,x+4\,y-8\,z$; $7\,x+3\,y-8\,z$.

60. Subtraction of Polynomials. Since subtraction is performed by adding the subtrahend, with its sign changed, to the minuend, we arrange the terms as in addition.

This is illustrated as follows:

From $15\,ab - 17\,xy + 11\,rt$ subtract $-5\,ab + 4\,xy - 5\,rt$.

Arranging as on page 69 and applying Principles I and VIII:

$$\begin{array}{r} 15\,ab - 17\,xy + 11\,rt \\ \underline{-\ 5\,ab + \ \ 4\,xy - \ \ 5\,rt} \\ 20\,ab - 21\,xy + 16\,rt \end{array}$$

As suggested in § 45, it is sufficient to change the signs of the subtrahend *mentally*, rather than to rewrite them before adding to the minuend.

ORAL EXERCISES

Pick out the similar terms and subtract mentally.

1. From $5\,x - 3\,y + 7\,z$ subtract $2\,x + 7\,y - 9\,z$.
2. From $7\,r - 7\,s + 5\,t$ subtract $9\,r + 8\,t - 4\,s$.
3. From $6\,m - 9\,n - p$ subtract $-4\,m + 12\,n - 8\,p$.
4. From $5\,ab - 7\,bc + 11\,ac$ subtract $12\,ab - 4\,bc - 4\,ac$.
5. From $2(x-3) + 3(y-4) + 4(z-5)$ subtract $-4(x-3) + 2(y-4) + 3(z-5)$.
6. From $5\,a - b + 2\,c - 7(x-y)$ subtract $-11\,a - 8\,b - 5\,c + 4(x-y)$.
7. From $2\,xy - 3(x-y) + 5\,yz$ subtract $-8\,xy - 7(x-y) - 6\,yz$
8. From $6\,ay - 3\,bz - 8\,(a-b)$ subtract $9(a-b) + 6\,bz - 11\,ay$.
9. From $2(a+b+c) + 2\,x - 3\,y$ subtract $6(a+b+c) - 3\,x + 5\,y$.
10. From $12(a+b-c) - 3\,ab + 2\,a - 3\,b$ subtract $-3(a+b-c) + 6\,ab - 5\,a + 2\,b$.
11. From $5\,x - 7\,a + 8\,b + 3\,c - 9\,d$ subtract $-3\,b + 6\,c - 6\,d - 3\,x + 3\,a$.

WRITTEN EXERCISES

Arrange similar terms in columns.

1. From $9x + 3y - 11z$ subtract $-5x + 8y - 3z$.
2. From $12ab - 3cd + 12xy$ subtract $3ab + 2cd - 11xy$.
3. From $9xc + 4ad - 3cz + 5y$ subtract $3y - 3ad + 5cz$.
4. From $13t + 5mx - 5cv$ subtract $2t - 4mx - 3cv$.
5. From $3v - 2w + 5mn - 4xz$ subtract $-v + 5w - 3mn$.
6. From $31b + 4xy + 16ax - 4$ subtract $8b - 5xy - 3ax$.
7. From $4 - 3a - 5xz - 3vy - x$ subtract $7a + 2xz + 4vy$.
8. From $8xy - 3x + 4y$ subtract $-2xy + 13w + 4x - 2y$.
9. From $2ab - 5 + 7v + 13abc$ subtract $3ab + v + 8abc$.
10. From $8acx - 4by - 3cy$ subtract $4acx + 2by + 4cy - 49$.
11. From $31 \cdot 45 - 7xy$ subtract $12 \cdot 45 + 9xy$.
12. From $3abc - 4 + 2(x + y) - 3xy$
    subtract $28 + 4xy - 3(x + y) + 8abc$.
13. From $21 + 9(xy - z) + 3(a + b)$
    subtract $8(xy - z) - 8(a + b) + 15$.
14. From $5ax - 3by + 4ax + 5by$ subtract $5by + 3ax + 7by$.
15. From $15 \cdot 48 + 8ab + 49x$ subtract $7 \cdot 48 - 9ab - 14x$.
16. From $19(r - 5s) + 13(5x - 4) + 7(x - y)$
    subtract $17(5x - 4) - 5(x - y) - 11(r - 5s)$.
17. From $30 + 14(x - 5yz) - 13(5y - z)$
    subtract $32 + 8(5y - z) - 7(x - 5yz)$.
18. From $a(b + c) + 4(m + n) - 16c$
    subtract $9(m + n) + 31c - d(b + c)$.
19. From $5(7x - 4) + 3(5y - 3x) + 35$
    subtract $56 - 9(7x - 4) + 8(5y - 3x)$.
20. From $(3a + 9b - 12c) \div 3$
    subtract $(6a - 12b - 18c) \div 6$
21. From $(axy + ayz - axz) \div a$ subtract $y(x + z) - 2xz$.
22. From $(abc + cxy) \div c$ subtract $2ab - 2xy$.

EXERCISES IN ADDITION AND SUBTRACTION

1. Add $5x - 3y - 7r + 8t$, $-7x + 18y - 4r$, $-7t - 20x$, $-24y + 18r - 15t$, and $13x + 15y + 11r + 6t$.

Check the sum by substituting $x = 1$, $y = 1$, $r = 1$, $t = 1$.

2. Add $17a - 9b$, $3c + 14a$, $b - 3a$, $a - 17c$, $a - 3b$, and $+4c$. Check for $a = 1$, $b = 2$, $c = 3$.

3. Add $2x + 3y - t$, $-6y + 8t$, $-x + y - t$, $-4t + 7x$, and $3y$. Check by putting each letter equal to 1.

4. Add $17r + 4s - t$, $2t + 3u$, $2r - 3s + 4t$, $5u - 6t$, $7r - 3s + 8u$, and $8r - 2t + 6u$. Check as in Example 3.

5. Add $3h + 2t + 4u$ and $h + 3t + 3u$. Check by putting $h = 100$, $t = 10$, $u = 1$; *i.e.* $324 + 133 = 457$.

6. Add $4h + 3t + u$ and $3h + 2t + 7u$. Check as in 5.

7. Write 247, 323, 647, 239, and 41, as number expressions like those in Exs. 5 and 6 and then add them.

8. Add 647, 391, 276, and 444 as in Ex. 7.

9. Add $4t - u$, $5t - u$, $6t - u$, $7t - u$. Check for $t = 10$, $u = 1$.

10. Simplify: $3xyz - 2xyz + 5xyz - 4xyz + xyz - xyz$.

11. Subtract $5a - 3b + 6c$ from $-8a + 7b - 11c$.

12. From $7xy + 8xz + 9yz$ subtract $17xy - 19xz - 20yz$.

13. From $6a - 3y$ subtract $8y - 3z$.

14. From $3p - 4q + 8r$ subtract $7p - 11r + 11q$.

15. From $2x - 3y$ subtract $5x + 7y + 2a - 3b$.

16. From the sum of $18abc - 27xyz + 13rst$ and $-11abc + 16xyz - 52rst$ subtract $67rst - 39abc$.

17. To the difference between the subtrahend $15x - 18y + 27z$ and the minuend $117x + 97y - 81z$ add $4x - 6y + 3z$.

18. Add $11(x - y) + 15(a - b)$ and $-20(x - y) - 37(a - b)$ and from the sum subtract $135(x - y) - 213(a - b)$.

ALGEBRAIC EXPRESSIONS IN PARENTHESES

61. Removing Parentheses. The sign + before parentheses means that each term within is to be added to what precedes, and the sign − means that each term within is to be subtracted from what precedes.

By Principle VII

$$a + (+b) = a + b \text{ and } a + (-b) = a - b;$$

and by Principle VIII,

$$a - (+b) = a - b \text{ and } a - (-b) = a + b.$$

Hence, we have

Principle XII

62. Rule. *Parentheses preceded by the sign + may be removed without further change.*

Parentheses preceded by the sign − may be removed by changing the sign of each term within.

Note that in each case the sign preceding the parentheses is also removed after the operation indicated by it has been carried out, and that if no sign is written before the first term in the parentheses, the sign + is understood.

Remove the parentheses and simplify the following:

Example 1. $3a + (a - b + 4) - (2a + 3b - 2)$
$= 3a + a - b + 4 - 2a - 3b + 2 = 2a - 4b + 6$

Example 2. $5(3x + y) - 4(2x - 3y + 2)$
$= 15x + 5y - 8x + 12y - 8 = 7x + 17y - 8.$

In Example 2 we multiply the terms within the first parentheses by 5 and those in the second by 4 and then remove the parentheses by Principle XII.

ORAL EXERCISES

Remove parentheses in the following:

1. $a + (x - y)$.
2. $a - (x + y)$.
3. $a - (x - y)$.
4. $-(-x - y)$.
5. $x + (a + b - c)$.
6. $x + (a - b - c)$.
7. $x - (a - b - c)$.
8. $x - (-a - b - c)$.

WRITTEN EXERCISES

Remove the symbols of aggregation and simplify:

1. $(3x - 2y) - (4x + 3y - 2)$.
2. $x - y - 2z - (3x + 2y - 7z)$.
3. $3(a + b + c) - 2(a - b + c)$.
4. $8(5x - y + 2z) - 11(3x + y - z)$.
5. $5(7x - 4y) + 9(x - y) - 3(2x + 3y)$.
6. $8(r - s) + (2r + s) - (r - 2s)$.
7. $11t + (2t - 1) - (1 - 3t)$.
8. $9(r - s) - 3(r + s) + 2(2r - s)$.
9. $3(5x - 7y) - (4x - 3y + 2) - 5y$.
10. $5x - (8 - 4x + 7y) + (5x + 3) - (5y + 3x - 99)$.
11. $-(3a + 5b - 7c) + (8a - 4c) - (9c - 4b + 4a) - 91a$.
12. $7 - (4 - 4c + 2d - 2a) + 31c - (4 - 2a - 5d) - (-8c)$.
13. $(41ab - 21c + 4) - (36c + 15 - 78ab) + (13c - 90ab - 8)$.
14. $9by - (4c - 8by - 13) - 2c - 16 - (34by - 12c + 8by)$.
15. $6mn + (-9m - 7n + 14) - 8n + (13mn - 17m) + 34mn$.
16. $34ax - (-17ax + 42) + 8x - (14a + 24ax - 7)$.
17. $19 - (2 - 7a - 4b + 11ab) - (-2b + 8ab + 4a)$.
18. $41by - (4b - 13y + 17by) - (-5b - 17by + 13y)$.
19. $39rs - 20s - 19r - (7rs + 8s - 19r) - (15r - 5s - 56)$.
20. $x - \{8x - (2y - 3x) + (2x - 4y)\}$.

Suggestion. First remove the parentheses, then the braces or brackets.

21. $a + [a - (b + c) - 2c]$.
22. $a - \{-(a - b) + (3a - 2b)\}$.
23. $2x - 3(x - 1) - [x - 2(2x - 1)]$.
24. $a - \{a + (b - c) - 2(a + b + c)\}$.
25. $-\{a - [a + (b - c) - 2(a + b + c)]\}$.
26. $-\{-[-(-\overline{a - b} + c - d)]\}$.
27. $-\{+[-(-\overline{a - b} - c + d)]\}$.

63. Removing all Symbols in Order. In such expressions as $2a \overset{\text{I}}{-} \{4(a \overset{\text{II}}{-} b) \overset{\text{III}}{-} (3a \overset{\text{IV}}{-} 2b)\}$, the symbols of aggregation may be removed *in order as we read*, by noting that a term affected by an *even* number of minus signs is *plus*, while one affected by an *odd* number is *minus*.

Thus, in the above expression, $4a$ is affected by one minus sign (I), $4b$ by two (I and II), $3a$ by two (I and III), and $2b$ by three (I, III, and IV). Hence the expression equals $2a - 4a + 4b + 3a - 2b = a + 2b$.

ORAL EXERCISES

Remove all symbols of aggregation *as you read*, then write the result, if necessary, and simplify.

1. $a-(2a+3b)$.
2. $x-[2x-(x+y)]$.
3. $m-[3(m-n)-4m]$.
4. $r+(3a-2b)-(a-b)$.
5. $a-\{2a-(3a-b)\}$.
6. $a+\{2a+(3a-b)\}$.
7. $a+\{2a-(3a-b)\}$.
8. $a-\{2a+(3a-b)\}$.
9. $a-\{2a-(3a+b)\}$.
10. $a-\{2a-[3a-(a+b)]\}$.

11. Read Examples 20 to 27 in the preceding exercises in this manner.

64. Inserting Expressions in Parentheses. By the converse of Principle XII terms may be inclosed in parentheses with or without change of sign, according as the sign $-$ or $+$ precedes.

E.g. $a+b-c=a+(b-c)$ and $a-b+c=a-(b-c)$.

ORAL EXERCISES

In each of the following, place the last two terms in parentheses.

1. $x-y+2$.
2. $5a+3b-c$.
3. $m-n+p$.
4. $5a-3b+2c$.
5. $7m-4n-3p$.
6. $8+4b-3c$.
7. $5xy-5x+3y$.
8. $9ax+3by-4cd$.
9. $a-3b+d-5c$.
10. $13-7a-3b$.
11. $19x-3c+4e$.
12. $21ax-13bx+6dx$.

WRITTEN EXERCISES

In the following, use either method: (1) remove one symbol at a time, beginning with the innermost; (2) remove them in order as you write, beginning at the left.

1. $-[a + \{a - (a - x) - (a - \overline{x - a})\} - a]$.
2. $8x - \{6y - [3x - (2y - x) - 8y] + x\}$.
3. $-[x - \{z + (x - z) - (z - x) - z\} - x]$.
4. $2a - [2a - \{2a - (2a - 2\overline{a - a})\}]$.
5. $-[a - \{a - (x - \overline{a - x - a}) - a\} - 2a]$.
6. $-[5x - \{4y - (5z + 2y) - (2x - 5z)\}]$.
7. $16 - x - -[7x - \{8x - (9x - \overline{3x - 6x})\}]$.
8. $2x - [3y - \{4x - (5y - \overline{6x - 7y})\}]$.
9. $2a - [3b + (2b - c) - 4c + \{2a - (3b - \overline{c - 2b})\}]$.
10. $a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a - (a - \overline{2b + c})\}]$.
11. $2(3b - 5a) - 7[a - 6\{2 - 5(a - b)\}]$.
12. $-2\{a - 6[a - (b - c)]\} + 6\{b - (c + a)\}$.
13. $-3\{-2[-4(-a)]\} + 5\{-2[-2(-a)]\}$.
14. $-2\{-[-(x - y)]\} + \{-2[-(x - y)]\}$.
15. $a - (b - c) - [a - b - c - 2\{b + c - 3(c - a) - d\}]$.
16. $2x - (3y - 4z) - \{2x - [3y + 4z - 3y - (4z + 2x)]\}$.
17. $-2(a - d) - 2[b + c + d - 3\{c + d - 4(d - a)\}]$.
18. $-4(a + d) + 4(b - c) - 2[c + d + a - 3\{d + a - 4(b + c)\}]$.
19. $a - 2b - [4a - 6b - \{3a - c + (5a - 2b - \overline{3a - c + 2b})\}]$.
20. $a - [-b\{-c(-d + \overline{e - f}) + 2a - c\} + c + d]$.

In each of the following insert the last three terms in parentheses preceded by the minus sign:

21. $5 + 2a - 3 + b$.
22. $x - 4 - 2a + c$.
23. $7xy - 3x - 2y + z$.
24. $a - 2b - 4c + d$.
25. $6 - a - b - c$.
26. $3b - 4 - 2x + y$.
27. $9y - 3c + 8 - 2a$.
28. $12 + 3a - 8b + c$.
29. $c - 8b - 3d + 12$.
30. $3a - 4c + 5d - 2b$.

FORMING ALGEBRAIC EXPRESSIONS

ORAL EXERCISES

1. The sum of two numbers is 20. If x is one of them, how may the other be represented?

2. The sum of two numbers is 16. What is a convenient representation of each number?

3. How do you represent six times the number x?

4. How do you represent 6 less than twice the number n?

5. How do you represent 8 more than 4 times the number a?

6. If x is a number, how do you represent a number which is 8 more than $\frac{1}{3}$ of this number?

7. If a is a number, how do you represent a number 10 less than $\frac{3}{4}$ of this number?

8. A number is represented by x, another number is 15 less than twice this number. How do you represent the sum of these numbers?

9. A number is represented by n. Another number is 6 more than three times this number. How do you represent twice the second number?

10. A number is represented by x. Another number is 24 less than four times this number. How do you represent $\frac{1}{3}$ of the second number?

11. A number is represented by a. By how much does this number exceed 50?

12. The difference between two numbers is 10. How may we represent the numbers?

13. If a number exceeds another by 50, represent twice the smaller number plus three times the greater.

14. If n represents the number of years in my age now, how old was I five years ago? How old will I be 10 years hence?

PROBLEMS ON THE RELATIONS OF NUMBERS

1. The sum of two numbers is 20 and one is 4 greater than the other. Find the numbers.

2. If 10 is added to 4 times a number, the result is 74. Find the number.

3. Six less than twice a certain number equals 18. Find the number.

4. Eight more than 4 times a number equals 48. Find the number.

5. Ten more than $\frac{1}{2}$ of a certain number equals 18. Find the number.

6. One number is 12 greater than another. Their sum is 24. Find the numbers.

7. One number is 10 less than twice another number. The sum of the numbers is 50. Find the numbers.

8. One number is 18 more than 3 times another number. The sum of the numbers is 66. Find the numbers.

9. Find two numbers such that one is 6 more than twice the other, and the smaller plus twice the greater equals 42.

10. One number exceeds another by 50. Twice the smaller number plus the larger number equals 95. Find the numbers.

11. The difference between two numbers is 10. The smaller number plus twice the larger equals 35. Find the numbers.

12. One number exceeds another by 40. Twice the smaller number plus 3 times the greater equals 295. Find the numbers.

13. One number is 16 less than 6 times another. The sum of the numbers is 40. Find the numbers.

14. The sum of my ages 10 years hence and 5 years ago is 55. How old am I?

PROBLEMS ON THE ARRANGEMENT AND VALUE OF DIGITS

If we speak of the number whose 3 digits, in order from left to right, are 5, 3, and 8, we mean $538 = 500 + 30 + 8$. Likewise, the number whose three digits are h, t, and u is written $100\,h + 10\,t + u$. Note that htu would mean $h \times t \times u$.

Hence, when letters stand for the digits of a number written in the decimal notation, care must be taken to multiply each letter by 10, 100, 1000, etc., according to the position it occupies.

ORAL EXERCISES

1. If the sum of two digits is 9 and if x represents one of them, how do you represent the other?

2. If the difference between two digits is 3, and if x represents one of the digits, how do you represent the other?

3. Give an expression representing a number if the digit in tens' place is x and the digit in units' place is y. Also give an expression representing the number if the order of the digits is reversed.

4. Give an expression representing a number if the digit in tens' place is x and in units' place $6 - x$. Also give an expression representing the number with the order of the digits reversed.

5. Give an expression representing a number if the digit in tens' place is t and that in units' place $t + 6$. Also give an expression representing the number obtained by reversing the order of the digits.

6. If the digit in tens' place is x and in units' place $12 - x$, give an expression representing 7 times the sum of the digits.

7. If the digit in tens' place is x and the digit in units' place is 12 less than 5 times x, write an expression representing the number. Also express it with the digits reversed.

Illustrative Problem. A number is composed of two digits whose sum is 6. If the order of the digits is reversed, we obtain a number which is 18 greater than the first number. What is the number?

Solution. Let $x =$ the digit in tens' place.
Then, $6 - x =$ the digit in units' place.

Hence, the number is $10x + 6 - x$. Reversing the order of the digits, we have as the new number $10(6 - x) + x$.

Hence, $10(6 - x) + x = 18 + 10x + 6 - x.$
Solving, $x = 2$, the digit in ten's place.
and $6 - x = 4$, the digit in unit's place.

Hence, the required number is 24.

WRITTEN EXERCISES

In each of the examples 1 to 8 below the number considered is composed of two digits.

1. The digit in units' place is 2 greater than the digit in tens' place. If 4 is added to the number, the result is then equal to 5 times the sum of the digits. What is the number?

2. The digit in tens' place is 3 greater than the digit in units' place. The number is 1 more than 8 times the sum of the digits. What is the number?

3. The sum of the digits is 9. If the order of the digits is reversed, we obtain a number which is equal to 12 times the remainder when the units' digit is taken from the tens' digit. What is the number?

4. The sum of the digits is 12. If the order of digits is reversed, the number is increased by 18. Find the number.

5. The tens' digit is 2 less than its units' digit. The number is 1 less than 5 times the sum of its digits. What is the number?

6. The digit in units' place is 4 less than that in tens' place. If the order of the digits is reversed, we obtain a number which is 3 less than 4 times the sum of the digits. What is the number?

7. The digit in units' place is 2 less than twice the digit in tens' place. If the order of the digits is reversed, the number is unchanged. What is the number?

8. The digit in tens' place is 12 less than 5 times the digit in units' place. If the order of the digits is reversed, the number is equal to 4 times the sum of the digits. What is the number?

9. A number is composed of three digits. The digit in units' place is 3 greater than the digit in tens' place, which in turn is 2 greater than the digit in hundreds' place. The number is equal to 96 plus 4 times the sum of the digits. What is the number?

REVIEW QUESTIONS

1. What is a *polynomial?* a *term?* How are polynomials classified? What are *similar terms?* By what principle are similar terms added? By what principle are they subtracted?

2. In adding or subtracting polynomials how may the terms be arranged for convenience? State the principle on which this is based.

3. What is the rule for removing parentheses when preceded by the sign +? By the sign −? How may Principle XII be used for inclosing terms within parentheses?

4. Tell how to remove all the symbols of aggregation in order as you read $5x - \{x - (3x + y) - (2x - 3y)\}$. How many minus signs affect the term $3x$? How many affect $3y$? What are the final signs of these terms?

5. Add Principles XI and XII to your list expressed in symbols:

XI. $a + b + c = a + c + b = c + a + b$, etc.

XII. $a + (b - c - d) = a + b - c - d.$

$a - (b - c - d) = a - b + c + d$

CHAPTER V

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

MULTIPLICATION OF POLYNOMIALS

In **multiplying one monomial by another** we use

Principle XIII

65. Rule. *To obtain the product of two or more factors, these may be arranged and multiplied in any order.*

The truth of this principle is clear from examples such as:

$$2 \cdot 3 \cdot 5 = 2 \cdot 5 \cdot 3 = 5 \cdot 3 \cdot 2 = 5 \cdot (3 \cdot 2) = 2 \cdot (3 \cdot 5) = 30.$$

HISTORICAL NOTE

Associative and Commutative Laws of Factors. Principle XIII, like Principle XI, states two fundamental laws of algebra: (1) **the associative law of factors,** first so called by F. S. Servois; (2) the **commutative law of factors,** first so called by Sir William Hamilton.

The associative law of factors states that factors may be grouped in any combination.

Thus, $abc = a(bc) = (ab)c.$

The commutative law of factors states that factors may be multiplied together in any order.

Thus, $abc = acb = cba$, etc.

66. Repeated Factors. In multiplying algebraic expressions, the same factor may occur more than once in the same term.

Thus we may have $5 \cdot 5$ or $x \cdot x$. These are written 5^2 and x^2 respectively, and are read *5 square* and *x square.*

This is a convenient way of abbreviating written expressions.

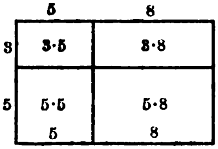
E.g. $5a \cdot 3a = (5 \cdot 3) \cdot (a \cdot a) = 15a^2$; $ay \cdot ay = aayy = a^2y^2$.

67. The product of two binomials such as $5+8$ and $5+3$ may be obtained in two ways:

(1) $(5+3)(5+8)=8\cdot 13=104$.
(2) $(5+3)(5+8)=5(5+8)+3(5+8)=5^2+5\cdot 8+3\cdot 5+3\cdot 8=104$.

The second method is illustrated by the accompanying figure in which $5+8$ is the length of a rectangle and $5+3$ is its width. The total area is the product $(5+3)\cdot(5+8)$ and it is composed of the four small areas, 5^2, $5\cdot 8$, $3\cdot 8$, and $3\cdot 5$.

The second method here used for multiplying $(5+3)(5+8)$ is the only one available when the terms of the binomials cannot be combined.

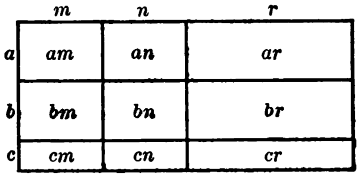


Thus, $(x+4)(x+6)=x(x+6)+4(x+6)=x^2+6x+4x+4\cdot 6$
$=x^2+10x+24,$

and $(a+b)(m+n)=a(m+n)+b(m+n)=am+an+bm+bn.$

Hence, **to multiply two binomials,** *multiply each term of one by every term of the other and add the products.*

68. Product of Two Trinomials. In a manner similar to that just illustrated we may multiply two trinomials.



From the figure we see at once that

$(a+b+c)(m+n+r)=am+bm+cm+an+bn+cn+ar+br+cr,$

in which *each term of one trinomial is multiplied by every term of the other and the products are added.*

Evidently the same process is applicable to the product of two such polynomials each containing any number of terms.

WRITTEN EXERCISES

Find the following products:

1. $(x+1)(x+2)$.
2. $(x+3)(x+5)$.
3. $(u+7)(u+4)$.
4. $(a+8)(a+3)$.
5. $(t+3)(t+7)$.
6. $(y+9)(y+2)$.
7. $(x+1)(x+7)$.
8. $(s+5)(s+3)$.
9. $(a+b)(c+d)$.
10. $(x+4)(x+3)$.
11. $(x+y)(a+b)$.
12. $(2x+3)(x+2)$.
13. $(5+x)(6+x)$.
14. $(5s+1)(s+5)$.
15. $(x+7)(3x+4)$.
16. $(a+4)(3a+1)$.
17. $(3+x)(2+5x)$.
18. $(a+b)(3a+7b)$.
19. $(x+y)(2x+3y)$.
20. $(7x+4)(x+8)$.
21. $(x+3)(2x+3)$.
22. $(2a+5)(a+7)$.
23. $(8b+3)(2b+3)$.
24. $(5a+4)(2a+3)$.
25. $(9+2x)(5+x)$.
26. $(6+y)(3+4y)$.

Example. Solve the equation

$$(x+1)(x+2)=x(1+x)+12. \qquad (1)$$

Solution. Performing the indicated multiplication,

$$x^2+3x+2=x+x^2+12 \qquad (2)$$

Subtracting x^2 from both sides,

$$3x+2=x+12. \qquad (3)$$

By $S \mid x, 2$, $\quad 2x=10. \qquad (4)$

By $D \mid 2$, $\quad x=5. \qquad (5)$

Equation (2) differs from those solved heretofore in that it contains the term x^2 in each member.

We may, however, subtract x^2 from each member, giving equation (3), which is a form already studied.

Solve the following equations:

27. $(x+3)(x+2)=(x+1)(x+2)+6$.
28. $(x+2)(x+4)=(x+3)(x+1)+9$.
29. $(x+3)(x+5)-13=(x+2)(x+4)$.
30. $(x+1)(x+7)-2=(x+2)(x+5)$.

WRITTEN PROBLEMS

1. The length of a rectangle is 6 feet greater than its width w. Express the area of the rectangle in terms of w.

2. The width w of a rectangle is 10 feet less than its length. Express the area of the rectangle in terms of w.

3. The length of a rectangle is 4 greater than its width w. Express the dimensions of this rectangle in terms of w after the width is increased by 2 and the length by 3.

4. The width w of a rectangle is 8 less than its length. Express the dimensions of this rectangle in terms of w after its length is increased by 2 and its width is increased by 4. Also express the area of the new rectangle in terms of w.

5. A field is 10 rods longer than it is wide. If its length is increased by 10 rods and its width increased by 5 rods, the area is increased by 640 square rods. What are the dimensions of the field?

Suggestions. Let $w =$ the width of the field.
Then, $w + 10 =$ its length,
and $w(w + 10) =$ its area.
The area of the increased field is $(w + 5)(w + 20)$.
By the conditions of the problem,

$$(w + 5)(w + 20) = w(w + 10) + 640.$$

6. A rectangle is 7 feet longer than it is wide. If its length is increased by 3 feet and its width is increased by 2 feet, its area is increased by 60 square feet. What are its dimensions?

7. A farmer has a plan for a granary which is to be 12 feet longer than wide. He finds that if the length is increased 8 feet and the width is increased 2 feet, the floor space will be increased by 160 square feet. What are the dimensions?

8. If the length of a rectangular flower bed is increased 3 feet and its width is increased 1 foot, its area will be increased by 19 square feet. What are its present dimensions, if its length is 4 feet greater than its width?

69. Multiplying Polynomials with Negative Terms. In §§ 67 and 68 we studied the multiplication of polynomials whose terms were all positive. The same method may be applied to polynomials having negative terms.

Example. Find the product of $(5-2)$ and $(4-3)$. This product, written out term by term as a product of two sums, would give

$$\begin{aligned}(5-2)(4-3) &= [5+(-2)][4+(-3)]\\ &= 5\cdot 4+5(-3)+4(-2)+(-3)(-2)\\ &= 20-15-8+6=3.\end{aligned}$$

Also $(5-2)(4-3)=3\cdot 1=3$.

Similarly, $(x+5)(x-2)=x^2+5x-2x-10=x^2+3x-10$,

and $(x-3)(x-5)=x^2-3x-5x+15=x^2-8x+15$.

WRITTEN EXERCISES

Perform the following indicated operations:

1. $(x-5)(x-3)$.
2. $(x-3)(x+4)$.
3. $(a-6)(a-1)$.
4. $(u+5)(u-3)$.
5. $(b+2)(b-7)$.
6. $(3-b)(4+b)$.
7. $(3+x)(7-3x)$.
8. $(n-4)(3-n)$.
9. $(a-b)(c+d)$.
10. $(a-b)(c-d)$.
11. $(x-4)(x-5)$.
12. $(a+b-c)(m-n)$.
13. $(a-b)(7a+3b)$.
14. $(5-y)(5x+3y)$.
15. $(2a-3b+c)(m+n)$.
16. $(v-t)(7v-5t)$.
17. $(3a-2)(2b-3)$.
18. $(7x-3y-4)(a-4b)$.
19. $(6+3a-b)(4c-9d-1)$.
20. $(a+m+n)(x-y+z)$.
21. $(a+b-c)(d-e+f)$.
22. $(v+t+u)(v-t-u)$.
23. $(3x-5)(2x+7)$.
24. $(2x-y-1)(x+y)$.
25. $(8a-3b-1)(2a-b)$.
26. $(2x-1+y)(3x-2y)$.
27. $(4a-b)(a+2b)$.
28. $(6x+5y)(2x-y)$.
29. $(7m-3n)(2m+5n)$.
30. $(8a-3b)(2a+3b)$.
31. $(5c+d)(2c-3d)$.
32. $(3a-4b)(3a+4b)$.
33. $(9x-4y)(4x+9y)$.
34. $(12a-5b)(3a-b)$.

The preceding exercises illustrate

Principle XIV

70. Rule. *The product of two polynomials is found by multiplying each term of one by every term of the other, and adding the products.*

71. It should be observed that Principle XIV involves a repeated application of Principle II. Thus

$$(a+b)(c+d)=(a+b)c+(a+b)d=ac+bc+ad+bd.$$

Arranging the Terms of the Product. In multiplying polynomials, the work is usually arranged so that similar terms in the product are written in columns and then added.

Example 1. Multiply $3x-2$ by $2x-5$.

Solution.

$$\begin{array}{rrr}
3x & -2 & \\
2x & -5 & \\ \hline
6x^2 & -4x & \\
 & -15x & +10 \\ \hline
6x^2 & -19x & +10
\end{array}$$

Example 2. Multiply $3x-2y+2$ by $4x-3y-2$.

Solution.

$$\begin{array}{rrrrrr}
3x & -2y & +2 & & & \\
4x & -3y & -2 & & & \\ \hline
12x^2 & -8xy & +8x & & & \\
 & -9xy & & +6y^2 & -6y & \\
 & & -6x & & +4y & -4 \\ \hline
12x^2 & -17xy & +2x & +6y^2 & -2y & -4
\end{array}$$

WRITTEN EXERCISES

In each case simplify the expressions within the parentheses as much as possible before multiplying:

1. $(x-7)(3x+4)$.
2. $(x-2)(9x+4)$.
3. $(a-x)(9x+4a)$.
4. $(11b+3a)(2b-3b+5)$.
5. $(x-2+y)(4y-3x)$.
6. $(x-5x+4)(8y-3-y)$.
7. $(7+y-x)(2y+x-1)$.
8. $(5x+3y-1)(x-2)$.
9. $(x-y+3)(5x-3y+5)$.
10. $(a-13n)(a-n+8)$.

WRITTEN EXERCISES AND PROBLEMS

Multiply the following:

1. $(13a - b - 12a)(2b - 3a)$.
2. $(6 - 4x + 3x)(7x + y - 3x + 1)$.
3. $(5x + 3y - 4x - 2y)(6y + 3x - 2y + y)$.
4. $(11b - a - 10b)(6a - 3b - 2a)$.
5. $(-7a - 1 + 8a)(5a - 8 - 3a)$.

Solve the following equations and check the results:

6. $(x + 2)(x + 3) = (x - 3)(x + 10) + 10$.
7. $(5x - 4)(6 - x) - 97 = (x - 1)(6 - 5x)$.
8. $(3n - 1)(18 - n) = (n + 6)(16 - 3n)$.
9. $(7 - a)(9a - 8) = 31 + (36 - 9a)(a + 2)$.
10. $(4a + 4)(a - 3) = (4a + 1)(a + 7) - 13a + 221$.
11. $(n + 6)(3n - 4) - 14 = (n + 8)(3n - 3)$.
12. $(8n + 6)(10 - n) + 150 = (1 - n)(8n + 3)$.
13. $(a - 1)(13 - 6a) = (6a - 3)(8 - a) - 21$.
14. $(7x - 13)(6 - x) - (x + 4)(3 - 7x) = 70$.

In the following make sure that the solutions are correct by doing the work with care and looking it over a second time.

15. $(2x - 3)(5x + 2) - 7x = (2 - 5x)(7 - 2x) + 1$.
16. $(2a - 1)(3a + 2) + a = (4 - 3a)(1 - 2a) + 7$.
17. $(8x - 1)(x + 7) + 4x = (2x + 8)(4x + 6) + 5$.
18. $(5b + 2)(3 - b) = (3 - 5b)(4 + b) + 9$.
19. $(3a - 7)(2 + a) = (5 + 3a)(2 + a)$.
20. $(x - 3)(2x + 5) - (3 - x)(5 - 2x) = 0$.
21. $(3y - 5)(5 - 6y) - (9y - 6)(3 - 2y) = 5$.
22. $(4 - a)(3 - a) + (a - 2)(a - 5) + (1 - 2a)(2 + a) = 7$.

23. Find two numbers whose difference is 6 and whose product is 180 greater than the square of the smaller.

24. There are four consecutive *even* integers such that the product of the first and second is 40 less than the product of the third and fourth. What are the numbers?

25. There are four consecutive integers such that the product of the first and third is 223 less than the product of the second and fourth. What are the numbers?

26. Find four numbers such that the second is 5 greater than the first, the third 5 greater than the second, and the fourth 5 greater than the third. The product of the first and second is 250 less than the product of the third and fourth.

27. A club makes an equal assessment on its members each year to raise a certain fixed sum. One year each member pays a number of dollars equal to the number of members of the club less 175. The following year, when the club has 50 more members, each member pays $5 less than the preceding year. What was the membership of the club the first year and how much did each pay?

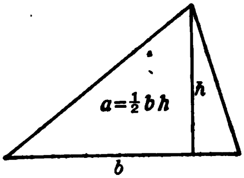
PROBLEMS ON RECTANGLES AND TRIANGLES

28. A rectangle is 10 inches longer than wide. Express its area in terms of the width w. If the width is increased by 4 and the length by 6 inches, express the area in terms of w.

29. A rectangle is 8 inches longer than wide. Express its area in terms of the width w, after the width is increased 4 inches and the length decreased 10 inches.

30. A rectangle is 5 feet longer than it is wide. If it were 3 feet wider and 2 feet shorter, it would contain 15 square feet more. Find the dimensions of the rectangle.

31. A rectangle is 6 feet longer and 4 feet narrower than a square of equal area. Find the side of the square and the sides of the rectangle.



If b is the base of a triangle, h its height or altitude, and a its area, then *area* $= \frac{1}{2}$ (*base* × *altitude*);

i.e. $$a = \frac{1}{2}bh.$$

32. The base of a triangle is 2 inches less than its altitude a. Express the area in terms of a.

33. The altitude of a triangle is 7 greater than its base b. If the altitude is increased by 8 and the base by 6, express its area in terms of b.

34. The altitude of a triangle is 16 inches less than the base. If the altitude is increased by 3 inches and the base by 2 inches, the area is increased by 52 square inches. Find the base and altitude of the triangle.

PRODUCTS OF POWERS OF THE SAME BASE

72. **Exponents; Powers.** Any number written over and to the right of a number expression is called an **index** or **exponent**. If an exponent is a *positive integer*, it shows how many times the expression is to be taken as a factor.

A product consisting entirely of equal factors is called a **power** of the repeated factor. The repeated factor is called the **base** of the power. See § 66.

E.g. x^3 means $x \cdot x \cdot x$ and is read *the third power of* x or x *cube*; x^5 means $x \cdot x \cdot x \cdot x \cdot x$, and is read *the fifth power of* x or briefly x *fifth*. In both these cases the base is x. $(x-y)^3 = (x-y)(x-y)(x-y)$ and is read $x-y$ *cubed* or *the cube of the binomial* $x-y$.

The first power of x is written without an exponent. Thus x means x^1; 2 means 2^1, etc.

Difference between a Coefficient and an Exponent. A coefficient *is a factor*, while an exponent, if a positive integer, shows how many times some number is *used as a factor*.

E.g. $5a = 5 \cdot a$ which means $a + a + a + a + a$, while $a^5 = a \cdot a \cdot a \cdot a \cdot a$.

ORAL EXERCISES

Find the following powers:

1. 2^3, 2^4, 2^5.
2. 3^2, 3^3.
3. 4^2, 4^3.
4. 5^2, 5^3.
5. 6^2, 6^3.
6. 7^2, 8^2.
7. 9^2, 10^2.
8. 11^2, 12^2.
9. 13^2, 14^2.
10. 15^2, 16^2.
11. 20^2, 30^2.
12. 40^2, 50^2.
13. 60^2, 70^2.
14. 80^2, 90^2.
15. 100^2, 1000^2.
16. 200^2, 300^2.
17. 400^2, 500^2.
18. 600^2, 700^2.

WRITTEN EXERCISES

Find the following powers:

1. 17^2, 18^2.
2. 19^2, 21^2.
3. 22^2, 23^2.
4. 24^2, 25^2.
5. 26^2, 27^2.
6. 28^2, 29^2.
7. $(a+b)^2$.
8. $(c-d)^2$.
9. $(a+b+c)^2$.
10. $(a+b-c)^2$.
11. $(3-a)^2$.
12. $(3-b-c)^2$.
13. $(a-b-c)^2$.
14. $(3a-2)^2$.
15. $(x-y+z)^2$.
16. $(2x-3y)^2$.
17. $(5a-2b)^2$.
18. $(4x+3y)^2$.

73. In the case of factors expressed in Arabic numerals multiplications like the following may be carried out in either of two ways.

E.g. $3^2 \cdot 3^4 = 9 \cdot 81 = 729,$

or $3^2 \cdot 3^4 = (3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3) = 3^{2+4} = 3^6 = 729.$

But with literal factors the second process only is possible.

E.g. $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^{2+4} = a^6.$

WRITTEN EXERCISES

In the following exercises carry out each indicated multiplication in two ways in case this is possible:

1. $5 \cdot 5^2$.
2. $5^2 \cdot 5^4$.
3. $3^2 \cdot 3^2$.
4. $7 \cdot 7^3$.
5. $a^2 \cdot a^3$.
6. $x^3 \cdot x^2$.
7. $x^7 \cdot x^4$.
8. $t^3 \cdot t^4$.
9. $t^2 \cdot t^3 \cdot t^4$.
10. $2^3 \cdot 2^2 \cdot 2^4$.
11. $3 \cdot 3^2 \cdot 3^3$.
12. $2^2 \cdot 2^3 \cdot 2^2 \cdot 2$.

Illustrative Problem. To multiply 2^k by 2^n, k and n being any two positive integers.

Solution. 2^k means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to k factors,

and 2^n means $2 \cdot 2 \cdot 2 \cdot 2$, etc., to n factors.

Hence, $2^k \cdot 2^n = (2 \cdot 2 \cdot 2 \cdots \text{ to } k \text{ factors}) \; (2 \cdot 2 \cdots \text{ to } n \text{ factors})$
$= 2 \cdot 2 \cdot 2 \cdot 2 \cdots$ to $k + n$ factors in all.

That is, $2^k \cdot 2^n = 2^{k+n}$.

The preceding examples illustrate

Principle XV

74. **Rule.** *The product of two powers of the same base is found by adding the exponents of the factors and making this sum the exponent of the common base.*

But exponents are added in multiplication only when they apply to the *same* base. Thus $2^2 \cdot 2^3 = 2^{2+3} = 2^5 = 32$; $a^2 \cdot a^3 = a^5$.

E.g. The product of $2^3 \cdot 3^2$ cannot be found by adding the exponents. It must be done as follows: $2^3 \cdot 3^2 = 8 \cdot 9 = 72$.

ORAL EXERCISES

Perform the following indicated multiplications by means of Principle XV.

1. $2^3 \cdot 2^7$.
2. $a^3 \cdot a^7$.
3. $3^4 \cdot 3^5$.
4. $x^4 \cdot x^5$.
5. $3^k \cdot 3^n$.
6. $x^k \cdot x^n$.
7. $4^a \cdot 4^b$.
8. $3^{2a} \cdot 3^{2b}$.
9. $5^{2+n} \cdot 5^{2-n}$.
10. $a^m \cdot a^n$.
11. $c^x \cdot c^{2-x}$.
12. $x^{ac} \cdot x^{bc}$.
13. $x^2 \cdot 3x^3$.
14. $x^4 \cdot 4x^5$.
15. $x^3 \cdot 5x^5$.
16. $a^k \cdot 2a^n$.
17. $a^4 \cdot 2a^n$.
18. $a^{2n} \cdot 3a^{3n}$.

Perform the following multiplications by means of Principles II and XV.

19. $x(x^2 + x + 1)$.
20. $x^2(x + 1)$.
21. $a^2(a^2 - a + 1)$.
22. $a^3(a^4 - 4a^2 + a + 1)$.
23. $y(3y^2 + 4y^4 - y^3)$.
24. $x^3(7x^4 - 5x^3 - 2x)$.
25. $a^3(3a - 2a^2b + 4ab^2)$.
26. $a^{2b}(4a^{2b} - a^{3c} + a^{2d})$.

75. Products of Monomials. In multiplying together monomials like $3a^2bc$ and $2ab^2cd$ it is convenient to arrange the factors in the product so that the same letters are associated together and likewise the numerical coefficients. This we are permitted to do by Principle XIII.

Thus, $3a^2bc \times 2ab^2cd = (3 \cdot 2)(a^2 \cdot a)(b \cdot b^2)(c \cdot c)d = 6a^3b^3c^2d.$

Notice that in the product *the exponent of each letter is the sum of the exponents of this letter in the factors, and the numerical coefficient is the product of the numerical coefficients of the factors.*

E.g. $(2a^2b^2)(5a^4b^3c) = 10a^{4+2}b^{2+3}c = 10a^6b^5c.$

This is a convenient rule for finding the product of two or more monomials.

ORAL EXERCISES

Multiply:

1. $3ab$ by $5a^2b^3$.
2. $4x^3$ by $3xy^2$.
3. $2xyz$ by $3x^2yz$.
4. $6k^2$ by $7ak^3$.
5. $3x^4y^2$ by $4xy^3$.
6. $5a^2b^3c$ by ab^4c.
7. $2b^4c^2x^3$ by $5b^4cx$.
8. $3a^3b^4c$ by abd^4.
9. $-2x^2y^2$ by $5xy^3$.
10. $-4a^2bc^3$ by $-3ab^3c^2$.
11. $-3t^5u^6$ by $-5tu^4$.
12. a^2x^2 by $-3ax^5$.
13. $5abc$ by $-abc$.
14. $-4m^2n$ by $-6mn^2$.
15. $-7a^2my$ by $-2m^2y$.
16. $-3x^3y^2$ by $2xy^4$.
17. $10^2x^2y^2$ by -10^3x^3y.
18. $5^3 \cdot 3^2 \cdot 2^2$ by $5 \cdot 3 \cdot 2^4$.
19. $2a^3bc^3$ by $3ab^4c$.
20. $6m^2n$ by $3mn^3$.
21. $2ma^5$ by $3a^2$.
22. $2b^4$ by $7b^4$.
23. $4x^my^2$ by $3x^ny^5$.
24. $3my^2$ by $4m^2y^3$.
25. $7ax^3$ by $4a^3x^2$.
26. $5a^4y$ by a^2y^3.
27. $4m^2n^3$ by m^4n^2.
28. $-3a^3b^4c$ by $5a^2b^4c^7$.
29. $-8a^3c^5$ by $-4a^7c^2x$.
30. $-4x^4y^2$ by $3x^2y^4$.

WRITTEN EXERCISES

Perform the following indicated multiplications:

1. $(a+b)(a^2+b^2)$.
2. $(a-b)(a^2+b^2)$.
3. $(a-b)(a^2-b^2)$.
4. $(x^2+y^2)(x^2-y^2)$.
5. $(x-y^2)(x^2-y)$.
6. $(3a^2x-y^2)(3a^2x+y^2)$.
7. $(3x^2-2y^2)(3x^2+2y^2)$.
8. $(2ax^2-3by^2)(2ax^2+3by^2)$.
9. $(7x^2+2y)(3x^2-2y)$.
10. $(2a^2b-c^2)(3ab^2+c^2)$.
11. $(3xy^2-5x^2y)(x-y)$.
12. $(a+b)^2(a-b)$.
13. $(a-b)^2(a+b)$.
14. $(a+b)^2(a-b)^2$.
15. $(a^3+a^2b+ab^2+b^3)(a-b)$.
16. $(a+b-c)(a+b+c)$.
17. $(3x-2y-1)(2x+y)$.
18. $(1+a+a^2)(1-a)$.
19. $(1-a+a^2-a^3)(1+a)$.
20. $(a+b)(a-b)(a^2+b^2)$.
21. $(a+b)(a^2-ab+b^2)$.
22. $(a-b)(a^2+ab+b^2)$.
23. $(x^2+x+1)(x^2-x+1)$.
24. $(x+y)(x^3-x^2y+xy^2-y^3)$.
25. $(x-y)(x^3+x^2y+xy^2+y^3)$.
26. $(x^2-xy+y^2)(x^2+xy+y^2)$.
27. $(x^2+2xy+y^2)(x-y)^2$.
28. $(x^2-y^2)(x^4+x^2y^2+y^4)$.
29. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
30. $(x^2+y^2)(x+y)(x-y)$.
31. $(x-y)(x^2+xy+y^2)(x^3+y^3)$.
32. $(x+y)(x^2-xy+y^2)(x^3-y^3)$.
33. $(x^3-y^3)(x^3+y^3)$.
34. $(x^2+2x+1)(x^2-2x+1)$.
35. $(x+y)^2(x+y)^2$.
36. $(x+y)^3(x+y)^2$.
37. $(a^2+2ab+b^2)(a-b)^2$.
38. $(3ax^2-7)(3ax^2+7)$.
39. $(3ax^2-y)(3ax^2-y)$.
40. $(x^4-x^2y^2)(x^4+x^2y^2)$.
41. $(9a^2+3ab+b^2)(3a-b)$.
42. $(x+y+z)(x-y-z)$.
43. $(a+b-c+d)^2$.
44. $(x-y-z-v)^2$.

QUOTIENT OF TWO POWERS OF THE SAME BASE

76. Illustrative Problem. To divide x^6 by x^4.

Since by § 50 the quotient times the divisor equals the dividend, we seek an expression which multiplied by x^4 equals x^6.

Since by Principle XV two powers of the same base are multiplied by adding their exponents, the expression sought must be that power of x whose exponent added to 4 equals 6. Hence the exponent of the quotient is $6-4=2$. That is, $x^6 \div x^4 = x^{6-4} = x^2$.

ORAL EXERCISES

Perform the following indicated divisions:

1. $2^4 \div 2^2$.
2. $2^3 \div 2^2$.
3. $2^4 \div 2$.
4. $3^3 \div 3^2$.
5. $3^4 \div 3$.
6. $3^4 \div 3^2$.
7. $9^{11} \div 9^{10}$.
8. $5^{13} \div 5^{12}$.
9. $7^{24} \div 7^{22}$.
10. $8^3 \div 8$.
11. $6^4 \div 6^2$.
12. $a^3 \div a^2$.
13. $a^4 \div a^3$.
14. $m^4 \div m^2$.
15. $x^4 \div x^2$.
16. $t^{14} \div t^4$.
17. $m^3 \div m$.
18. $n^6 \div n^2$.
19. $(20)^4 \div (20)$.
20. $(101)^{14} \div (101)^{13}$.
21. $(41)^7 \div (41)^6$.

The preceding exercises illustrate

Principle XVI

77. **Rule.** *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

For the present only those cases are considered in which the exponent of the dividend is greater than or equal to that of the divisor.

Notice that Principle XVI does not apply to powers of *different* bases.

E.g. The result of $3^4 \div 2^2$ does not equal any integral base to the power, $4 - 2$. This division can be performed only by first multiplying out both dividend and divisor. Thus, $3^4 \div 2^2 = 81 \div 4 = 20\frac{1}{4}$.

ORAL EXERCISES

Perform the following indicated divisions by means of Principle XVI:

1. $2^7 \div 2^3$.
2. $a^7 \div a^3$.
3. $3^4 \div 3^2$.
4. $x^4 \div x^2$.
5. $3^{3n} \div 3^{2n}$.
6. $x^{4n} \div x^{2n}$.
7. $3^{2a-1} \div 3^{a-2}$.
8. $5^{n+5} \div 5^{n+2}$.
9. $x^{a+4} \div x^{a+2}$.
10. $t^{4a} \div t^a$.
11. $x^{2a+b} \div x^{a+b}$.
12. $w^{2x} \div w^x$.
13. $(17)^{14} \div (17)^{13}$.
14. $5x^6 \div 2x^2$.
15. $(12)^4 \div (12)^3$.

WRITTEN EXERCISES

In the following, use Principles III, V, and XVI:

1. $(3 \cdot 2^4 + 5 \cdot 2^3) \div 2^2$.
2. $(3 \cdot 4^5 - 5 \cdot 4^4) \div 4^4$.
3. $(a^3b - a^4b^2) \div a^2$.
4. $(4\,x^3 + 3\,x^4) \div x^2$.
5. $(a^2m^4 - b^2m^3) \div m^3$.
6. $(4\,x^2 - 5\,x^3 + x^4) \div x^2$.
7. $(3\,a^3 + 9\,a^4 - 2\,a^5) \div a^3$.
8. $(12\,x^3y - 11\,x^2y^2 + 5\,x^4) \div x^2$.
9. $(8\,a^3 + 12\,a^4) \div 4\,a^3$.
10. $(9\,x^5 + 15\,x^6) \div 3\,x^2$.
11. $(8\,x^4 - 16\,x^8) \div 8\,x^4$.
12. $(15\,x^7 - 10\,x^3) \div 5\,x^2$.
13. $(4\,ax^2 - 8\,a^2x) \div 4\,x$.
14. $(8\,a^2x^3 - 12\,ax^2) \div 2\,x^2$.
15. $(12\,a^3x^2 - 8\,x^3a^2) \div 4\,a^2$.
16. $(16\,a^3x^3 - 8\,x^2a^3) \div 2\,x^2$.
17. $(24\,x^2y^3 + 16\,x^3y^2) \div 8\,x^2$.
18. $(24\,x^3y^2 - 16\,x^2y^3) \div 8\,y^2$.
19. $(12\,x^3y - 16\,x^2y^2) \div 4\,y$.
20. $(x^a + x^{3a}) \div x^a$.
21. $(x^m + x^{4m}) \div x^m$.
22. $(x^{3m} - x^{2m}) \div x^m$.
23. $(y^{3a} - y^{2a}) \div y^a$.
24. $(a^{2m} - a^{3m}) \div a^m$.
25. $(4\,x^8 - 8\,x^7 + 16\,x^6 - 20\,x^5 + 12\,x^4) \div 4\,x^4$.
26. $(4\,x^{8n} - 8\,x^{7n} + 16\,x^{6n} - 20\,x^{5n} + 12\,x^{4n}) \div 4\,x^{4n}$.
27. $(3\,xy^4 - 6\,x^2y^3 + 12\,x^3y^2 - 24\,y^5) \div 3\,y^2$.
28. $(3\,xy^{4a} - 6\,x^2y^{3a} + 12\,x^3y^{2a} - 24\,y^{5a}) \div 3\,y^{2a}$.
29. $(a^7b^2 - a^6b^3 + a^5b^2 - 4\,a^4) \div a^4$.
30. $(a^{7k}b^2 - a^{6k}b^3 + a^{5k}b^2 - 4\,a^{4k}) \div a^{4k}$.
31. $(x^{10}y^5 - x^9y^6 + x^8y^7 - x^7y^8 + x^6y^9) \div x^6$.
32. $(x^{10a}y^5 - x^{9a}y^6 + x^{8a}y^7 - x^{7a}y^8 + x^{6a}y^9) \div x^{6a}$.

78. **Division by Monomials.** In dealing with the quotient of two monomials the indicated division may be written in the form of a fraction and the factors common to dividend and divisor may be cancelled, that is, divided out of both numerator and denominator, just as in arithmetic.

Example 1. $15\,a^3b^2c \div 3\,a^2bx^2y = \dfrac{15\,a^3b^2c}{3\,a^2bx^2y} = \dfrac{5\,abc}{x^2y}$.

Example 2. $12\,a^2x \div 3\,ax = \dfrac{12\,a^2x}{3\,ax} = \dfrac{4\,a}{1} = 4\,a$.

ORAL EXERCISES

Give the following quotients in their simplest forms:

1. $\dfrac{2a^2b^3}{ab^2}$.
2. $\dfrac{4x^3y^4}{2xy^3}$.
3. $\dfrac{6m^2n^3}{3mn}$.
4. $\dfrac{8 \cdot 15 \cdot 14}{4 \cdot 5 \cdot 7}$.
5. $\dfrac{4x^6y^4}{x^3y^2}$.
6. $\dfrac{5x^2y^3}{2x^2y^3}$.
7. $\dfrac{8a^4b^4}{4a^2b^3}$.
8. $\dfrac{9x^7y^4}{3x^6y^2}$.
9. $\dfrac{a^4b^6c^2}{a^2b^2c^2}$.
10. $\dfrac{6 \cdot 8 \cdot 10}{3 \cdot 4 \cdot 2}$.
11. $\dfrac{12x^2y^3}{6x^2y}$.
12. $\dfrac{10a^7b^7c^2}{5a^7b^6c}$.
13. $\dfrac{48x^4y^{n+4}}{12x^3y^{n+3}}$.
14. $\dfrac{24a^{2n}b^{3n}}{6a^nb^{2n}}$.
15. $\dfrac{84x^{2m+3}y^{n+4}}{21x^{2m+1}y^{n+1}}$.
16. $\dfrac{16a^{2n}b^{3n}}{4a^nb^{2n}}$.
17. $\dfrac{14x^{3a}y^{3b}}{7x^ay^b}$.
18. $\dfrac{21a^{4n}b^{2m}}{7a^{2n}b^m}$.

WRITTEN EXERCISES

Write each in the fractional form and cancel.
Divide:

1. $4 \cdot 7 \cdot 9x^5$ by $2 \cdot 3x^2$.
2. $12 \cdot 8 \cdot 20x^8$ by $2 \cdot 4 \cdot 5x^5$.
3. $6x^3y^2z$ by $2xyz$.
4. $6^4 \cdot 5^4 \cdot x^3$ by $6^2 \cdot 5^2 \cdot x^2$.
5. $12x^{12}y^{13}$ by $4xy^3z$.
6. $5a^4b^{11}c$ by ab^4c^2.
7. $10x^4b^{14}c^3$ by $2xb^4c$.
8. $36x^4y^3$ by $6x^3y^5$.
9. $4x^2y^3 - 3x^3y^2$ by x^2y^2.
10. $18x^4y^4 - 12x^3y^3 + 6x^2y^2$ by $6x^2y^2$.
11. $49a^4 + 21a^3 - 7a$ by $7a$.
12. $12ax^4y^3 - 16a^2x^3y^2 + 8a^3xy$ by $4axy$.
13. $2x^{3a} + 4x^{4a} - 8x^{2a}$ by $2x^a$.
14. $6x^{2n+1} + 12x^{3n+1} - 10x^{n+1}$ by $2x^{n+1}$.
15. $4x^{13} - 6x^{11}b - 10x^4c$ by $2x^4$.
16. $10a^3b^2 - 20a^2b^3 + 15a^4b^4$ by $5a^2b^2$.

79. Negative Exponents. The process of division by subtracting exponents leads in certain cases to interesting results.

Thus, $x^4 \div x^4 = x^{4-4} = x^0$, which seems to have no meaning, since an exponent has been defined only when it is a *positive integer.* The exponent zero cannot indicate, as in the case of a positive integer, how many times the base is used as a factor.

We know, however, that $x^4 \div x^4 = 1$, since any number divided by itself equals unity. Thus, any number with the exponent zero is equal to unity. Hence if we use the symbol x^0, it must be interpreted to mean 1, *no matter what number is represented by x.*

Again by this process, $x^2 \div x^4 = x^{2-4} = x^{-2}$, which seems to have no meaning, since negative exponents have not been defined. But we know that if we divide numerator and denominator by x^2, as in arithmetic, $x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}$.

Hence if we use the symbol x^{-2}, it must be interpreted to mean $\frac{1}{x^2}$. Similarly $x^{-3} = \frac{1}{x^3}$, $x^{-6} = \frac{1}{x^6}$, etc. Negative exponents and also fractional exponents are the subjects of more advanced work and are considered in detail in the Intermediate Course.

ORAL EXERCISES

Express the following quotients by means of negative exponents after first cancelling any common factors, as in § 78.

1. $\frac{1}{x^2}$.
2. $\frac{1}{x^3}$.
3. $\frac{1}{x^4}$.
4. $\frac{1}{a^5}$.
5. $\frac{x}{x^3}$.
6. $\frac{a}{a^4}$.
7. $\frac{b}{b^3}$.
8. $\frac{c}{c^4}$.
9. $\frac{ab}{a^3b^3}$
10. $\frac{a^2x}{a^3x^2}$.
11. $\frac{a^3x^2}{a^5x^3}$.
12. $\frac{a^4y^4}{a^5y^6}$.

ORAL EXERCISES

Express the following in the fractional form:

1. a^{-2}.
2. b^{-3}.
3. x^{-1}.
4. a^{-4}.
5. x^{-6}.
6. a^{-10}.
7. $a^{-2}y^{-3}$.
8. $x^{-1}y^{-2}$.
9. xy^{-4}.
10. $a^{-3}b^{2}$.
11. $a^{-4}b^{-3}$.
12. $x^{-2}y^{-3}$.

WRITTEN EXERCISES

By means of negative exponents write each of the following without using the fractional form:

1. $\dfrac{5\,ax^3}{x^2y^3}$.
2. $\dfrac{a^2b^3}{x^5y^4}$.
3. $\dfrac{4\,xy^3}{x^3y^6}$.
4. $\dfrac{2\,ab^2}{x^3y^9}$.
5. $\dfrac{4}{x^my^n}$.
6. $\dfrac{4\,a}{x^{2m}y^{n+4}}$.
7. $\dfrac{4\,b^2c}{b^4c^6x^2}$.
8. $\dfrac{3\,xy}{x^ay^b}$.
9. $\dfrac{2}{x^{m+4}y^{5a+3}}$.

Express the following without using negative exponents:

10. $ax^{-3}y^{-2}$.
11. $a^{-3}yx^{-2}$.
12. $a^{-2}y^{-3}x^{2}$.
13. $xy^{-a}z^{-1}$.
14. $x^{-2}yz^{-2a}$.
15. $x^{-a}y^{-b}z$.
16. $x^{-3a}y^{-2b}z^{-1}$.
17. $a^{-1}b^{-2}c^{-a}$.
18. $a^{-m}b^{-2n-3}c$.

HISTORICAL NOTE

Exponents. The expression "power" is used by Alkarismi to denote the *square* of a number. Up to the time of Vieta it was customary to use different letters (if letters were used at all) to express the square, the cube, etc. of a number. Thus, if R represented a number, Q might represent the square of it and C the cube of it. Vieta wrote A, A quad, A cube, etc., for A, A^2, A^3, etc. Harriot wrote aa for a^2, aaa for a^3, etc. Descartes (1637) established the usage of the forms a^2, a^3, etc.

John Wallis (1616–1703) explained the meaning of negative and of fractional exponents. Thus he wrote x^{-1} for $\frac{1}{x}$, x^{-2} for $\frac{1}{x^2}$, etc.

Sir Isaac Newton (1642–1727) used exponents of any magnitude. Thus he used not only x^3, x^{-3}, $x^{\frac{1}{3}}$, but also such expressions as $x^{\sqrt{2}}$. These latter need not be discussed here.

DIVISION BY A POLYNOMIAL

80. Illustrative Example. Consider the product

$$(x^2 + 2\,xy + y^2)(x + y) = x^2(x + y) + 2\,xy(x + y) + y^2(x + y).$$

The products $x^2(x + y)$, $2\,xy(x + y)$, and $y^2(x + y)$ are called **partial products,** and their sum, $x^3 + 3\,x^2y + 3\,xy^2 + y^3$, the **complete product.**

In dividing $x^3 + 3\,x^2y + 3\,xy^2 + y^3$ by $x + y$, the quotient must be a polynomial such that when its terms are multiplied by $x + y$ the results are these partial products.

The work may be arranged as follows:

$$\begin{array}{lr|l}
\text{Dividend or product} = & x^3 + 3\,x^2y + 3\,xy^2 + y^3 & x + y, \text{ divisor.} \\
\quad\text{1st product, } x^2(x+y) = & \underline{x^3 + x^2y \phantom{{}+ 3\,xy^2 + y^3}} & x^2 + 2\,xy + y^2, \\
\text{Dividend minus 1st product} = & 2\,x^2y + 3\,xy^2 + y^3 & \text{[quotient.} \\
\quad\text{2d product, } 2\,xy(x + y) = & \underline{2\,x^2y + 2\,xy^2 \phantom{{}+ y^3}} & \\
\text{Dividend minus 1st and 2d products} = & xy^2 + y^3 & \\
\quad\text{3d product, } y^2(x + y) = & \underline{xy^2 + y^3} & \\
\text{Dividend minus 1st, 2d, and 3d products} = & 0 &
\end{array}$$

Explanation. Dividing the first term, x^3, of the dividend by the first term, x, of the divisor the quotient is x^2. Multiplying this term of the quotient by the divisor, we obtain the first partial product, $x^3 + x^2y$.

Subtracting the first partial product from the whole product, $x^3 + 3\,x^2y + 3\,xy^2 + y^3$, the remainder is $2\,x^2y + 3\,xy^2 + y^3$. Dividing the first term, $2\,x^2y$, of this remainder by x the quotient is $2\,xy$. The product of $2\,xy$ and $x + y$ is the second partial product.

In like manner the third partial product is $xy^2 + y^3$.

After subtracting the third partial product the remainder is zero.

Checking Problems in Division. Problems in division may be checked by substituting any convenient values for the letters. For instance, in the above example, if $x=1$, $y = 1$, we have:

Divisor $= x + y = 1 + 1 = 2$.
Quotient $= x^2 + 2\,xy + y^2 = 1 + 2 + 1 = 4$.
Dividend $= x^3 + 3\,x^2y + 3\,xy^2 + y^3 = 1 + 3 + 3 + 1 = 8$.

We know that we should have

Dividend = Divisor × Quotient.

But $8 = 2 \times 4$. Hence the correctness of the division is shown.

But since division by zero is impossible, care must be taken in checking not to select such values for the letters as will reduce the divisor to zero.

Sir Isaac Newton (1642–1727), probably the greatest mathematician of all time, was born near Grantham in Lincolnshire, England, in the same year that Galileo died.

He entered Cambridge in 1661 and began at once a career of unequalled productive study. The binomial theorem was one of his early discoveries. Later Newton laid the foundation of the calculus.

His chief work, the *Principia*, published in 1687, aimed "to apply mathematics to the phenomena of nature." This work placed him in the very front rank for all time among mathematicians, physicists, and astronomers.

Example. Divide $x^3 - 7x^2 + 2x^4 - 1 + 5x$ by $2x - 1 + x^2$.

Solution. We first arrange both dividend and divisor according to the descending powers of x. [divisor.

Dividend or product = $2x^4 + x^3 - 7x^2 + 5x - 1$ | x^2+2x-1,
1st product, $2x^2(x^2+2x-1) = 2x^4 + 4x^3 - 2x^2$ | $2x^2-3x+1$,
Dividend minus 1st product = $-3x^3 - 5x^2 + 5x - 1$ [quotient.
2d product, $-3x(x^2+2x-1) = -3x^3 - 6x^2 + 3x$
Dividend minus 1st and 2d products = $+x^2 + 2x - 1$
3d product, $1 \cdot (x^2 + 2x - 1) = x^2 + 2x - 1$
Dividend minus 1st, 2d, and 3d products = 0

Check. Substituting $x = 2$, we get $21 \div 7 = 3$.

81. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. *Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter. As the division proceeds, arrange each remainder in the same way.*

2. *Divide the first term of the dividend by the first term of the divisor. This quotient is the first term of the quotient.*

3. *Multiply the first term of the quotient by the divisor and subtract the product from the dividend.*

4. *Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.*

5. *Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor.*

ORAL EXERCISES

Arrange each of the following in descending powers of the letter involved:

1. $16x - 26x^2 + 3 + 11x^3 + x^4$.
2. $8a - 5a^2 + 8 - 4a^5 + 2a^4 - a^3$.
3. $3y^3 + 4 + 6y^4 - 2y - y^2$.
4. $4z^2 - 18 + 6z^4 + 18z$.

WRITTEN EXERCISES

Divide the following. Check the results in Examples 1 to 13, being careful to substitute such numbers for the letters as do not make the divisor zero:

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 - 2ab + b^2$ by $a - b$.
3. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
4. $2x^3 + 2x^2y - 4x^2 - x - 4xy - y$ by $x + y$.
5. $x^3 - x^2y + xy^2 - y^3$ by $x - y$.
6. $x^3 + 4x^2 + x - 6$ by $x + 3$.
7. $x^3 + x + 4x^2 - 6$ by $x - 1$.
8. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.
9. $x^3 + 3xy^2 + 3x^2y + y^3$ by $x^2 + y^2 + 2xy$.
10. $x^3 - 8x^2 + 75$ by $x - 5$.
11. $2a^3 + 19a^2b + 9ab^2$ by $2a + b$.
12. $x^3 + y^3$ by $x + y$.
13. $x^3 - y^3$ by $x - y$.

Divide the following:

14. $x^4 + x^3y + xy^3 + y^4$ by $x + y$.
15. $x^4 + y^4 + x^2y^2$ by $x^2 - xy + y^2$.
16. $x^4 - y^4$ by $x - y$.
17. $x^4 + x^3 - 12x^2 + 14x - 4$ by $x^2 - 3x + 2$.
18. $2x^4 + 11x^3 - 26x^2 + 16x - 3$ by $x^2 + 7x - 3$.
19. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
20. $x^5 + 10x^2 - x^4 - 27x^3 - 30x - 200$ by $x^2 - 4x - 10$.
21. $3x^2 - 4xy + 8xz - 4y^2 + 8yz - 3z^2$ by $x - 2y + 3z$.
22. $9r^2s^2 - 4r^2t^2 + 4rst^2 - s^2t^2$ by $3rs - 2rt + st$.
23. $9a^2b^2 + 16x^2 - 4a^2 - 36b^2x^2$ by $3ab + 6bx - 2a - 4x$.
24. $x^3 + x^2y + x^2z - xyz - y^2z - yz^2$ by $x^2 - yz$.
25. $a^5 - a^3b^2 + a^4b + a^2b + ab^2 - b^3$ by $a^2 + ab - b^2$.

82. Division not Exact. In case the division is not exact, the remainder may be placed over the divisor in the fractional form as in arithmetic.

Example. Divide $20a^2 - 4 + 18a^4 + 18a - 19a^3$ by $2a^2 - 3a + 4$.

Solution. Arranging dividend and divisor according to the descending powers of a, we have

$$\begin{array}{lr|l}
 & & \text{[divisor.} \\
\text{Dividend or product:} & 18a^4 - 19a^3 + 20a^2 + 18a - 4 & \underline{2a^2 - 3a + 4,} \\
\text{1st product:} & \underline{18a^4 - 27a^3 + 36a^2 \phantom{{}+ 18a - 4}} & 9a^2 + 4a - 2, \\
\text{Dividend minus 1st product:} & 8a^3 - 16a^2 + 18a - 4 & \text{[quotient.} \\
\text{2d product:} & \underline{8a^3 - 12a^2 + 16a \phantom{{}- 4}} & \\
\text{Dividend minus 1st and 2d products:} & -4a^2 + 2a - 4 & \\
\text{3d product:} & \underline{-4a^2 + 6a - 8} & \\
\text{Dividend minus all products:} & -4a + 4 &
\end{array}$$

Since $2a^2$ is not contained in $-4a$, the division ends and $-4a + 4$ is the remainder. As in arithmetic we write this as the numerator of a fraction whose denominator is the divisor. Hence, the complete result is $9a^2 + 4a - 2 + \dfrac{4 - 4a}{2a^2 - 3a + 4}$.

WRITTEN EXERCISES

In the following express the remainders in the fractional form:

Divide:

1. $4x^4 - 2x^2 + 6x^3 + 4x - 7$ by $2x^2 - x + 2$.
2. $8a^5 - 4 + 7a^4 + 2a^3 - 2a^2$ by $a^2 - 3a + 1$.
3. $3 - 4x + 7x^2 - 3x^3 + x^4$ by $x - 4$.
4. $2x^3 - 6 + 2x^2 - 6x + 6x^4$ by $x^2 + 5$.
5. $13x - 7x^2 + 8x^4 + 3x^3 + 14$ by $x^2 + 3$.
6. $21 - 7x - 9x^3 + 8x^2 + x^4$ by $x^3 + 7$.
7. $12x^3 - 4x^2 + 8x + 16x^4$ by $x^2 + x + 4$.
8. $35a - 14 + 21a^2 + 7a^3$ by $a - 5$.
9. $16x^7 - 14x^5 + 2x^3 - x + 2$ by $2x^4 - 4x^2 + 8$.
10. $2x^4 - 11x^3 + 26x^2 + 18x + 3$ by $x^2 - 7x + 3$.
11. $6x^5 - 2x^4 + 2x^3 - x^2 + x - 6$ by $3x^2 - 4x + 2$.

83. Division by Detached Coefficients. When both dividend and divisor are arranged in descending powers of the same letter, the work of dividing by a polynomial may be shortened by omitting the letters and writing only the coefficients.

Example. Divide $2x^4+x^3-7x^2+5x-1$ by x^2+2x-1.

Writing coefficients only,

$$\begin{array}{l|l}
2+1-7+5-1 & 1+2-1 \\ \cline{2-2}
\underline{2+4-2} & 2-3+1 \\
-3-5+5-1 & \\
\underline{-3-6+3} & \\
+1+2-1 & \\
\underline{+1+2-1} &
\end{array}$$

Since dividend and divisor are arranged in descending powers of x, we know that the quotient will be so arranged also. Since the first term of the quotient contains $x^4 \div x^2$, the quotient starts with x^2 and is then

$$2x^2-3x+1.$$

This example is worked without detached coefficients on page 103.

If any terms are lacking in the dividend or divisor, zero must be written as the coefficient of each such term, as in the following example.

Example. Divide x^4+x^2+1 by x^2-x+1.

Since the third and first powers are lacking in the dividend, we write zero in place of each.

$$\begin{array}{l|l}
1+0+1+0+1 & 1-1+1 \\ \cline{2-2}
\underline{1-1+1} & 1+1+1 \\
+1+0+0+1 & \\
\underline{+1-1+1} & \\
+1-1+1 & \\
\underline{+1-1+1} &
\end{array}$$

Hence, the quotient starts with $x^4 \div x^2 = x^2$, and is, then, x^2+x+1.

WRITTEN EXERCISES

Divide by detached coefficients Examples 3, 6, 7, 8, 10, 12, 13, 16, and 20, page 104.

NOTE. — Division by detached coefficients is much used in certain topics in higher algebra where the work is still further abbreviated by a process called *synthetic division*.

REVIEW QUESTIONS

1. Make a diagram to show how to multiply $(7+4)$ by $(11+8)$ without first uniting the terms of the binomials. Multiply $(a+b)$ by $(c+d)$ in the same manner.

Multiply $(12-3)$ by $(9-7)$ in two ways and compare results. State the principle by which two polynomials are multiplied.

2. Describe a convenient manner of arranging the work in multiplying polynomials. What kind of terms in the product are placed in the same column? Find the product of $7x-3y+1$ and $2x-8y-3$, arranging your work this way.

3. State Principle XIII. How do you arrange the factors in multiplying $3a^2b$, $2ab^2$, and $5a^4b^3$?

4. Define a positive integral exponent. Explain the difference between an exponent and a coefficient.

5. Under what circumstances are exponents added in multiplication? State Principle XV. Use this principle to show that $(a^2)^3=a^6$, $(a^3)^4=a^{12}$.

6. Under what circumstances are exponents subtracted in division? State Principle XVI.

7. What is meant by x^{-2}? How may $\frac{a}{b^3}$ be written with a negative exponent?

8. How may division by a polynomial be shortened by use of detached coefficients?

Add Principles XIII, XIV, XV to your list expressed in symbols:

$$\text{XIII}\begin{cases} a\cdot b\cdot c = a\cdot c\cdot b = b\cdot c\cdot a, \text{ etc.} \\ a\cdot b\cdot c = ab\cdot c = a\cdot bc \end{cases}$$

$$\text{XIV}\quad a^m\cdot a^n = a^{m+n}$$

$$\text{XV}\quad a^m \div a^n = a^{m-n}$$

CHAPTER VI

SPECIAL PRODUCTS AND QUOTIENTS

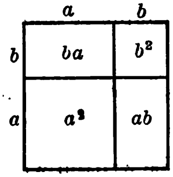
There are certain products and quotients the formulas for which should be memorized. Some of these are collected in this chapter.

THE SQUARE OF A BINOMIAL

84. The square of the sum of two numbers is found by ordinary multiplication as in § 67.

E.g. $(a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$

Hence, $\boldsymbol{(a+b)^2 = a^2 + 2ab + b^2.}$



This product is illustrated in the accompanying figure.

Translated into words, this identity is:

The square of the sum of two numbers is equal to the square of the first, plus twice the product of the two numbers, plus the square of the second.

ORAL EXERCISES

Read at sight the squares indicated by the following:

1. $(x+y)^2$.
2. $(m+n)^2$.
3. $(x+1)^2$.
4. $(x+2)^2$.
5. $(y+3)^2$.
6. $(a+4)^2$.
7. $(2a+1)^2$.
8. $(2x+1)^2$.
9. $(3+a)^2$.
10. $(3a+1)^2$.
11. $(4+x)^2$.
12. $(1+3x)^2$.
13. $(2+3a)^2$.
14. $(2a+3)^2$.
15. $(3a+4)^2$.

By the above formula we may square any binomial sum.

E.g. $(3x+2y)^2 = (3x)^2 + 2\cdot(3x)(2y) + (2y)^2 = 9x^2 + 12xy + 4y^2.$

WRITTEN EXERCISES

Write the squares indicated by the following:

1. $(2a+3b)^2$.
2. $(3y+8)^2$.
3. $(5x+4)^2$.
4. $(3a+7c)^2$.
5. $(2a+9b)^2$.
6. $(4c+5a)^2$.
7. $(8a+2b)^2$.
8. $(4m+3n)^2$.
9. $(6x+5y)^2$.
10. $(x+12)^2$.
11. $(12+7a)^2$.
12. $(10+3x)^2$.
13. $(16a+b)^2$.
14. $(8x+3y)^2$.
15. $(12a+7b)^2$.

85. Similarly, we obtain the **square of the difference of two numbers**:

$$(a-b)^2 = a^2 - 2ab + b^2.$$

That is, *the square of the difference of two numbers is equal to the square of the first, minus twice the product of the two numbers, plus the square of the second.*

Example. By means of this formula, find the square of $a-3b$.

Solution. $(a-3b)^2 = a^2 - 2\cdot a(3b) + (3b)^2 = a^2 - 6ab + 9b^2$.

ORAL EXERCISES

Read at sight the squares indicated by the following:

1. $(x-y)^2$.
2. $(m-n)^2$.
3. $(x-1)^2$.
4. $(x-2)^2$.
5. $(x-3)^2$.
6. $(a-4)^2$.
7. $(a-5)^2$.
8. $(2a-1)^2$.
9. $(2-x)^2$.
10. $(3-x)^2$.
11. $(4-x)^2$.
12. $(1-2a)^2$.

WRITTEN EXERCISES

Write the squares indicated by the following:

1. $(4x-3y)^2$.
2. $(5y-3x)^2$.
3. $(2a-3b)^2$.
4. $(5x-3)^2$.
5. $(4-3x)^2$.
6. $(4c-5a)^2$.
7. $(8a-3)^2$.
8. $(7x-4y)^2$.
9. $(3x-7y)^2$.
10. $(ab-4c)^2$.
11. $(12-7a)^2$.
12. $(7x-9y)^2$.
13. $(8a-3b)^2$.
14. $(12a-13)^2$.
15. $(9a-7b)^2$.

86. Squaring Polynomials. By means of the formulas in §§ 84 and 85, we may square any polynomial.

1. $(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2$
$= a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2.$

2. $(a+r-s+t)^2 = [(a+r)-(s-t)]^2$
$= (a+r)^2 - 2(a+r)(s-t) + (s-t)^2$
$= a^2 + 2\,ar + r^2 - 2\,(as - at + rs - rt) + s^2 - 2\,st + t^2$
$= a^2 + 2\,ar + r^2 - 2\,as + 2\,at - 2\,rs + 2\,rt + s^2 - 2\,st + t^2.$

WRITTEN EXERCISES

1. $(a+b-c)^2$.
2. $(a-b+c)^2$.
3. $(a-b-c)^2$.
4. $(a-b+3)^2$.
5. $(3\,a-2\,b+5)^2$.
6. $[7\,x-(4\,r-s)]^2$.
7. $[(a-3)-2(b+c)]^2$.
8. $[(m+3)-(n+a)]^2$.
9. $(a-b+c-d)^2$.
10. $(x-y+z-3)^2$.
11. $(a-x+b-c)^2$.
12. $(2-x+y+z)^2$.

PRODUCT OF THE SUM AND DIFFERENCE OF TWO NUMBERS

87. Examples. Find the products:

$$(x+5)(x-5) \text{ and } (x+a)(x-a).$$

Solutions

$$\begin{array}{lll}
x+5 & & \\
\underline{x-5} & & \\
x^2+5x & & \\
\underline{\quad\;\, -5x-25} & & \\
x^2 \qquad\;\; -25 & &
\end{array}
\qquad\qquad
\begin{array}{lll}
x+a & & \\
\underline{x-a} & & \\
x^2+ax & & \\
\underline{\quad\;\, -ax-a^2} & & \\
x^2 \qquad\;\; -a^2 & &
\end{array}$$

In each of these examples one factor is the sum of two numbers and the other factor is the difference of the same numbers. In each case the product is the difference of the squares of the numbers. This is expressed by the formula

$$(x+a)(x-a) = x^2 - a^2.$$

That is, *the product of the sum and difference of two numbers is equal to the difference of their squares.*

ORAL EXERCISES

Read the following products:

1. $(a+1)(a-1)$.
2. $(a+3)(a-3)$.
3. $(k-b)(k+b)$.
4. $(3-x)(3+x)$.
5. $(2a+3b)(2a-3b)$.
6. $(a+2b)(a-2b)$.
7. $(2b-1)(2b+1)$.
8. $(1+3x)(1-3x)$.
9. $(1-7y)(1+7y)$.
10. $(a-4b)(a+4b)$.
11. $(6a-3b)(6a+3b)$.
12. $(7-9a)(7+9a)$.
13. $(2c+1)(2c-1)$.
14. $(3a+b)(3a-b)$.
15. $(5k+3h)(5k-3h)$.
16. $(9m+3n)(9m-3n)$.

By means of the above formula a product may be written at once whenever the factors can be expressed as the sum and difference of the same two number expressions.

E.g.
$$\begin{aligned}(x+y-z)(x+y+z) &= [(x+y)-z][(x+y)+z]\\ &= (x+y)^2 - z^2\\ &= x^2+2xy+y^2-z^2.\end{aligned}$$

WRITTEN EXERCISES

In this manner form the following products. Verify the first five by formal multiplication.

1. $(4a+5b)(4a-5b)$.
2. $(5-6t^2)(5+6t^2)$.
3. $(3x-2y)(3x+2y)$.
4. $(x^3-y^3)(x^3+y^3)$.
5. $(3x^4-5y^4)(3x^4+5y^4)$.
6. $(16a^2b^3-3c)(16a^2b^3+3c)$.
7. $(24x+12y)(24x-12y)$.
8. $[x+(y-z)][x-(y-z)]$.
9. $(x^m+y^n)(x^m-y^n)$.
10. $[c-(a-b)][c+(a-b)]$.
11. $[x-(y+z)][x+(y+z)]$.
12. $(a+b+c)(a-b-c)$.
13. $(a+b-c)(a-b+c)$.
14. $(a-b+c)(a-b-c)$.
15. $(r-y-z)(r-y+z)$.
16. $(a+b+c)(a+b-c)$.
17. $(x+2y+z)(x+2y-z)$.
18. $(x-2y+z)(x+2y-z)$.
19. $(x-2y-z)(x+2y+z)$.
20. $(a+2b-c)(a-2b+c)$.
21. $(a-2b-c)(a-2b+c)$.
22. $(2a+3b-4c)(2a-3b+4c)$.

THE SQUARE OF A TRINOMIAL

89. Example. By multiplication find the square of $a+b+c$, and reduce the result to simplest form.

Solution. By means of the formula in § 84 we can perform this multiplication by a short method; namely,

$$\begin{aligned}(a+b+c)^2 &= [(a+b)+c]^2\\ &= (a+b)^2+2(a+b)c+c^2\\ &= a^2+2ab+b^2+2ac+2bc+c^2.\end{aligned}$$

How many terms are there in the product? How many are squares? How many are of the type $2ab$?

From this we get the following rule:

The square of a trinomial consists of the sum of the squares of its terms plus twice the product of each term by each succeeding term.

In symbols this is

$$\boldsymbol{(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.}$$

The above rule may be used to find the square of $a-b+c$ as follows:

$$\begin{aligned}a-b+c=a+(-b)+c &= a^2+(-b)^2+c^2+2a(-b)+2ac+2(-b)c\\ &= a^2+b^2+c^2-2ab+2ac-2bc.\end{aligned}$$

Similarly, $(2a+b-3c)^2$

$$\begin{aligned}&=(2a)^2+b^2+(-3c)^2+2(2a)b+2(2a)(-3c)+2b(-3c)\\ &= 4a^2+b^2+9c^2+4ab-12ac-6bc.\end{aligned}$$

Hence, *In the square of a trinomial, the squared terms are all positive and the double products are negative when one factor is negative, otherwise they are positive.*

ORAL EXERCISES

Give the squares of the following:

1. $x+y+z$.
2. $x+y-z$.
3. $x-y-z$.
4. $x-y+z$.
5. $a+b+2$.
6. $a+b-2$.
7. $a-b-2$.
8. $a-b+2$.
9. $a-2b+3$.
10. $a+2b-3$.
11. $a-2b-3$.
12. $a+2b+3$.

ORAL EXERCISES

Give the squares of the following:

1. $3-y-z$.
2. $3-x+y$.
3. $3+a+b$.
4. $a-3+b$.
5. $a+3-b$.
6. $a-3-b$.
7. $-x+y+z$.
8. $-x-y+z$.
9. $-x-y-z$.
10. $-a+b+2$.
11. $-a-b+2$.
12. $-a+b-2$.
13. $x+y+1$.
14. $x+y-1$.
15. $x-y-1$.
16. $x-y+1$.
17. $-x-y+1$.
18. $-x+y-1$.

WRITTEN EXERCISES

Find the squares of the following:

1. $3a-2b+c$.
2. $3a+2b-c$.
3. $4a-3b+2c$.
4. $4a-3b-2c$.
5. $2a+4b-6c$.
6. $x-4y+3z$.
7. $3x^2+2xy+3y^2$.
8. $a^2-b^2+c^2$.
9. $2a^2+3b^2+c^2$.
10. $ab+ac+bc$.
11. $2a^2-3b^2+4c^2$.
12. $a^2b^2+b^2c^2+c^2a^2$.
13. $2a^2b+ab^2+ab$.
14. $ab+a^2b^2+a^3b^3$.
15. $2x^2+x+1$.
16. $3x^3+2x^2+x$.
17. $ax+by+cz$.
18. ax^3+bx^2+cx.
19. $a^2b^2-b^2c^2-a^2c^2$.
20. $a^2x^2-x^2y^2+y^2z^2$.
21. $a^2x+ay+x^2y$.
22. $3a-4x+3b$.
23. $5a-6b+7c$.
24. $3x-5y-9z$.

25. Find the square of $a+b+c+d$ by writing it in the form $[(a+b+c)+d]$. Study this product and make a rule for squaring a polynomial of four terms.

Find the squares of the following:

26. $x+y+z+w$.
27. $a-b+c-d$.
28. $a-2b-c-2d$.
29. $m+2n+3r+s$.
30. $2a-3b+c-d$.
31. $3x+2y-z+w$.

CUBE OF A BINOMIAL

90. Example. Find the cube of $a+b$ by first finding the square of $a+b$ and then multiplying this result by $a+b$. How many terms are there in the product?

From this we get the following formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

In words this is:

The cube of a binomial is equal to the cube of the first term, plus three times the square of the first term multiplied by the second, plus three times the first term multiplied by the square of the second, plus the cube of the second term.

The above rule may be applied to find the cube of $a-b$, thus

$$(a-b)^3 = [a+(-b)]^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3.$$

Which terms in the product are negative and why?

Translate into words the formula:

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

EXERCISES

Find the following cubes. Read the first six at sight.

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(c+d)^3$.
4. $(c-d)^3$.
5. $(m+n)^3$.
6. $(m-n)^3$.
7. $(2x+3y)^3 = (2x)^3 + 3(2x)^2 \cdot 3y + 3 \cdot 2x(3y)^2 + (3y)^3$.
8. $(2x-3y)^3$.
9. $(x+2y)^3$.
10. $(x-3y)^3$.
11. $(3a-b)^3$.
12. $(3a+2b)^3$.
13. $(3a-2b)^3$.
14. $(x-1)^3$.
15. $(x+1)^3$.
16. $(2x+a)^3$.
17. $(2x-a)^3$.
18. $(b+6)^3$.
19. $(2a-5b)^3$.

Historical Note. The Hindus and Arabs had the formulas for $(a+b)^2$ and $(a+b)^3$. Of course their formulas were not stated in terms of our symbols. Vieta used the formula for $(a+b)^4$.

QUOTIENTS DERIVED FROM SPECIAL PRODUCTS

91. From the special products given in §§ 84–88, we may at once derive certain special quotients by observing that if either factor be used as a divisor of the product, the other factor is the quotient.

Thus, from

$$(a+b)(a+b)=a^2+2ab+b^2, \qquad (1)$$

$$(a-b)(a-b)=a^2-2ab+b^2, \qquad (2)$$

$$(a+b)(a-b)=a^2-b^2, \qquad (3)$$

$$(x+a)(x+b)=x^2+(a+b)x+ab, \qquad (4)$$

we derive

$$(a^2+2ab+b^2)\div(a+b)=a+b, \qquad (1)$$

$$(a^2-2ab+b^2)\div(a-b)=a-b, \qquad (2)$$

$$\left.\begin{array}{l}(a^2-b^2)\div(a-b)=a+b\\(a^2-b^2)\div(a+b)=a-b\end{array}\right\}, \qquad (3)$$

$$\left.\begin{array}{l}[x^2+(a+b)x+ab]\div(x+a)=x+b\\ [x^2+(a+b)x+ab]\div(x+b)=x+a\end{array}\right\}. \qquad (4)$$

EXERCISES

Give the following quotients orally if possible:

Divide:

1. $x^2+2xy+y^2$ by $x+y$.
2. $x^2+4xy+4y^2$ by $x+2y$.
3. $x^2-6xy+9y^2$ by $x-3y$.
4. x^2-y^2 by $x-y$.
5. x^4-y^4 by x^2+y^2.
6. $25a^2-16b^2$ by $5a-4b$.
7. $1-9c^2$ by $1-3c$.
8. $9c^2-1$ by $3c+1$.
9. $(a+b)^2-c^2$ by $a+b-c$.
10. $(a+b)^2-c^2$ by $a+b+c$.
11. $x^2-(y+z)^2$ by $x-y-z$.
12. $x^2-(y+z)^2$ by $x+y+z$.
13. x^2+5x+6 by $x+3$.
14. x^2-5x+6 by $x-3$.
15. $a^2+7a+12$ by $a+4$.
16. $a^2+7a+12$ by $a+3$.
17. $a^2-7a+12$ by $a-3$.
18. c^2+c-12 by $c+4$.
19. c^2+c-12 by $c-3$.
20. $x^2-2x-15$ by $x-5$.

DIVIDING THE SUM OR DIFFERENCE OF TWO CUBES

92. Examples.

$$\begin{array}{r|l|l} a+b & a^3+b^3 & a^2-ab+b^2 \\ \hline & \underline{a^3+a^2b} & \\ & -a^2b & \\ & \underline{-a^2b-ab^2} & \\ & \quad ab^2+b^3 & \\ & \quad \underline{ab^2+b^3} & \end{array}
\qquad
\begin{array}{r|l|l} a-b & a^3-b^3 & a^2+ab+b^2 \\ \hline & \underline{a^3-a^2b} & \\ & a^2b & \\ & \underline{a^2b-ab^2} & \\ & \quad ab^2-b^3 & \\ & \quad \underline{ab^2-b^3} & \end{array}$$

From these examples we have the two formulas:

$$(a^3+b^3)\div(a+b)=a^2-ab+b^2$$
$$(a^3-b^3)\div(a-b)=a^2+ab+b^2$$

That is,

The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers **minus** *their product.*

The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers **plus** *their product.*

WRITTEN EXERCISES

After performing the division as indicated, review these examples, giving the quotients orally.

Divide:

1. x^3+y^3 by $x+y$.
2. x^3-y^3 by $x-y$.
3. 2^3+3^3 by $2+3$.
4. r^3+s^3 by $r+s$.
5. $1+a^3$ by $1+a$.
6. $1-b^3$ by $1-b$.
7. a^3+1 by $a+1$.
8. a^3-1 by $a-1$.
9. a^3+8b^3 by $a+2b$.
10. a^3-8b^3 by $a-2b$.
11. x^3-27y^3 by $x-3y$.
12. $1+x^6$ by $1+x^2$.
13. $8+a^9$ by $2+a^3$.
14. $1+125x^3$ by $1+5x$.
15. 2^3x^3-1 by $2x-1$.
16. $64a^3-27b^3$ by $4a-3b$.
17. $1-8m^3$ by $1-2m$.
18. $1-2^3m^6$ by $1-2m^2$.

USING SPECIAL PRODUCTS IN EQUATIONS

Ability to form rapidly the special products given by the formulas in §§ 84–88 is important in solving certain kinds of equations. See page 86.

Example. Solve $(x+1)^2+(x-2)^2=2(x+3)(x-5)+31.$ (1)

Solution. Forming the special products, we have

$$x^2+2x+1+x^2-4x+4=2x^2-4x-30+31. \qquad (2)$$

Transposing and collecting terms,

$$2x=-4, \qquad (3)$$

$$x=-2. \qquad (4)$$

WRITTEN EXERCISES

Solve the following equations:

1. $x^2=(x-3)(x+6)-12.$
2. $(1-x)(1-3x)-(1+x)^2=2x^2-18.$
3. $(a+1)(a+2)+a(a+3)=2a(a+5)+2.$
4. $(b+3)(b-2)-(b+1)^2=-2(b+4)-1.$
5. $(z-5)^2+(z-4)(z-1)=2z(z-3)-25.$
6. $(a+4)^2+(a-1)(2a+5)=(a+4)(3a+2).$
7. $(a-1)(3a-1)-(a+1)^2=2a^2-18.$
8. $(6-a)^2+(a-3)(2a-5)=(3a+1)(a-3)+84.$
9. $(7a-18)(a+4)-(a-1)^2=6(a+2)^2-79.$
10. $(2b-30)(b-1)-5b^2=6b-3(b+5)^2+65.$
11. $(5-c)^2+(7-c)^2+(9-c)^2=(c-1)(3c-58)-93.$
12. $(5c-3)(2+c)-4(c-1)^2=(c+1)^2+54.$
13. $(8-4c)(5-c)=(c+1)^2+(c+3)(3c-8)+218.$
14. $(y-1)^2+4(y+1)^2+(1-y)(5y+6)=15y-29.$
15. $x(x+3)+(x+1)(x+2)=2x(x+5)+2.$
16. $x^2=(x-3)(x+7)-19.$
17. $(5+5x)(3-x)+2(x+1)^2+3(x+1)(x-7)=17(x+1).$
18. $(8+3x)(4-x)+(x-1)(x-2)+2(x+5)^2=105.$

PROBLEMS

1. Find two consecutive integers whose squares differ by 51.

2. Find two consecutive integers whose squares differ by 97.

3. Find two consecutive integers whose squares differ by a. Show from the form of the equation obtained that a must be an *odd* integer.

4. There is a square field such that if each of its dimensions is increased by 5 rods, its area is increased 625 square rods. How large is the field?

Suggestion. If a side of the original field is w, then its area is w^2, and the area of the enlarged field is $(w+5)^2$.

5. A rectangle is 9 feet longer than it is wide. A square whose side is 3 feet longer than the width of the rectangle is equal to the rectangle in area. What are the dimensions of the rectangle?

6. A rectangle is 16 feet longer than it is wide. If its length is increased by 4 feet and its width is decreased by 3 feet, its area is decreased by 50 square feet. What are its dimensions?

7. A field is 20 rods longer than it is wide. If its length is increased by 8 rods and its width is decreased by 5 rods, the area is decreased by 50 square rods. What are the dimensions of the field?

8. A farmer has a plan for a granary which is to be 15 feet longer than wide. He finds that if the length is decreased 3 feet and the width is increased 2 feet, the floor space will be increased by 16 square feet. What are the dimensions?

9. If the length of a rectangular flower bed is decreased 4 feet and its width is decreased 1 foot, its area will be decreased by 39 square feet. What are its present dimensions, if its length is 8 feet greater than its width?

MISCELLANEOUS EXERCISES

Perform the indicated operations:

1. $(2a-3b+c)^2$.
2. $(5x-y-4z)^2$.
3. $(3m-4n+2p)^2$.
4. $(2a-3b)^3$.
5. $(ab-cd)^3$.
6. $(4x-5y)^3$.
7. $[(a^2+3)-(b^2-2)]^2$.
8. $[(x^2-2)+(y^2-3)]^2$.
9. $(x^2-7x+6)\div(x-6)$.
10. $(144m^4-64n^6)\div(12m^2-8n^3)$.
11. $[(x-y)^2-z^2]\div(x-y+z)$.
12. $[(x+y)^2-z^2]\div(x+y-z)$.
13. $(125x^3-27y^3)\div(5x-3y)$.

REVIEW QUESTIONS

State in words each of the following formulas:

$$(a+b)^2=a^2+2ab+b^2.$$

$$(a-b)^2=a^2-2ab+b^2.$$

$$(a+b)(a-b)=a^2-b^2.$$

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

$$(a^2-b^2)\div(a+b)=a-b.$$

$$(a^2-b^2)\div(a-b)=a+b.$$

$$(a^3+b^3)\div(a+b)=a^2-ab+b^2.$$

$$(a^3-b^3)\div(a-b)=a^2+ab+b^2.$$

$$\left.\begin{aligned}[x^2+(a+b)x+ab]\div(x+a)&=x+b\\ [x^2+(a+b)x+ab]\div(x+b)&=x+a\end{aligned}\right\}.$$

CHAPTER VII

FACTORS OF ALGEBRAIC EXPRESSIONS

93. **Factors in Arithmetic and Algebra.** Factors are of great importance in arithmetic. Thus from the multiplication table, we know the factors of such numbers as 25, 42, 49, 54, 63, etc. Likewise in algebra the factors of certain algebraic expressions are so important that they must be known at sight.

94. **Prime Expressions.** An algebraic expression is said to be **prime** if it is divisible only by itself and 1. Factors of an expression, when they cannot themselves be further factored, are called *prime factors*.

Thus, 2, 3, x, $x+2$, a^2+b^2, are prime expressions.

Case I: MONOMIAL FACTORS

$$ax + ay + az = a(x + y + z).$$

95. **Common Monomial Factors.** If the terms of a polynomial contain a common monomial factor, the polynomial may be divided by the monomial, and the quotient and the divisor are factors of the polynomial.

E.g.

$$a^2 - ab = a(a - b);$$
$$5xy - 3x^2y + 4x^3y = xy(5 - 3x + 4x^2),$$
$$x^4 + 3x^3y + 3x^2y^2 + xy^3 = x(x^3 + 3x^2y + 3xy^2 + y^3).$$

Observe that factoring the various parts of a polynomial does not factor the polynomial.

E.g. $a^2 + ax + ab + by$ is not factored by writing it

$$a(a + x) + b(a + y).$$

Removing a common factor from the terms of a polynomial is nothing more than the application of Principle I.

ORAL EXERCISES

Read the following and give the factors of each:

1. $x^3 + x^2 + x$.
2. $4a^2 + 3a + a^3$.
3. $a^3b^2 + a^2b^3$.
4. $3xy - 4x^2y^2$.
5. $3ab^2c - 2a^2bc^2$.
6. $5x^3y^4 - 10x^4y^3$.
7. $2m^3n - 3mn^3$.
8. $4ax^2y - 6a^2xy^2$.
9. $4xy^2 - 2xy$.
10. $6xy - 3x^2y$.
11. $4x^3y^4 + 8x^7y^6$.
12. $8x^7y^6 - 12x^5y^8$.
13. $x^2y + xy^2$.
14. $xy^3 - x^2y^5$.
15. $x^5 + 4x^4 + 5x^3 + 3x^2$.
16. $8a^3b^2 - 12a^2b^3$.
17. $3a^2b^3c^2 - 9a^4b^3c^2$.
18. $5xyz + 25x^2y^2z^2$.
19. $4ab - 12a^2b$.
20. $3a^2b^2 - 9a^3b^2$.
21. $5ab^2 - 10ab^4$.
22. $6a^3b^3 - 3ab$.
23. $7a^2b^3 + 14ab$.
24. $8a^3b + 24a^4b$.
25. $9x^2y^3 - 6x^4y$.
26. $10x^5y^2 - 5x^2y^5$.

WRITTEN EXERCISES

Factor the following polynomials:

1. $7a^4b^3 + 14a^2b + 21ab^3$.
2. $13a^4b - 16a^3b^4 - 2a^3b^2$.
3. $15xy^4 - 20x^3y + x^2y^2$.
4. $9v^3w^4 + 21v^4w^3 - 18v^2w^2$.
5. $12a^4b^3 - 8a^3b^4 - 6a^2b^2$.
6. $11a^4x^2 - 44a^3x^4 + 33ax$.
7. $72ab^3x^4 - 36a^2b^2x^4 - 48a^4b^3x^3$.
8. $84b^7x^9y^4 + 18b^5x^3y^4 + 12b^3x^2y^2$.
9. $17a^4b^3c^2 + 51a^3b^4c^3 - 34a^2b^2c^4$.
10. $38a^{12}b^{14}c^4 - 76a^{11}b^{12}c^3 - 76a^{13}b^{11}c^2$.
11. $4x^{2a}y^{3b} + 6x^{3a}y^{2b} - 8x^{5a}y^{4b}$.
12. $3a^{4n}b^{3n} + 6a^{6n}b^{3n} - 12a^{5n}b^{5n}$.
13. $2x^ay^b + 4x^{3a}y^{4b} - 6x^{4a}y^{3b}$.
14. $3x^{2a}y^{2b} + 6x^{3a}y^{2b} - 9x^{4a}y^{4b}$.
15. $4x^{2a}y^b + 6x^{3a}y^{2b} - 9x^{2a}y^{3b}$.
16. $5x^{2a}y^b + 15x^{5a}y^{2b} - 20x^{6a}y^{3b}$.

Case II: TRINOMIAL SQUARES

$$\begin{cases} a^2 + 2ab + b^2 = (a + b)(a + b), & (1) \\ a^2 - 2ab + b^2 = (a - b)(a - b). & (2) \end{cases}$$

96. **Square Root.** If an expression is the product of two equal factors, either of these factors is called its *square root.*

Thus, a is the square root of a^2, since $a^2 = a \cdot a$.

In §§ 84 and 85 we found by multiplication:

$$(a + b)(a + b) = a^2 + 2ab + b^2,$$

and

$$(a - b)(a - b) = a^2 - 2ab + b^2.$$

Hence, $a + b$ is the square root of $a^2 + 2ab + b^2$ and $a - b$ is the square root of $a^2 - 2ab + b^2$.

97. **Trinomial Squares.** A trinomial which is the square of a binomial is called a *trinomial square.*

Thus $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are trinomial squares.

98. From a study of the two trinomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$, we learn to distinguish whether *any* given trinomial is a perfect square, as in the following examples :

1. $x^2 + 4x + 4$ is in the form of (1), since x^2 and 4 are squares each with the sign +, and $4x$ is twice the product of the square roots of x^2 and 4. Hence

$$x^2 + 4x + 4 = x^2 + 2(2x) + 2^2 = (x + 2)(x + 2) = (x + 2)^2.$$

2. $x^2 - 4x + 4$ is in the form of (2), since it differs from (1) only in the sign of the middle term. Thus

$$x^2 - 4x + 4 = x^2 - 2(2x) + 2^2 = (x - 2)(x - 2) = (x - 2)^2.$$

99. The foregoing examples lead to the following

Rule. *A trinomial is a perfect square if it contains two terms which are squares, each with the sign +, while the third term, whose sign is either + or −, is twice the product of the square roots of the other two.*

The square root of such a trinomial is the sum or the difference of these square roots according as the sign of the third term is + or −.

ORAL EXERCISES

Determine whether the following are trinomial squares, and if so indicate the two equal factors.

1. $a^2 + 2ad + d^2$.
2. $x^2 + 2xy + y^2$.
3. $x^2 - 2xy + y^2$.
4. $x^4 + 2x^2y^2 + y^4$.
5. $x^4 - 2x^2y^2 + y^4$.
6. $m^2 + n^2 - 2mn$.
7. $r^2 + s^2 + 2rs$.
8. $4x^2 - 8xy + 4y^2$.
9. $a^6 + b^6 + 2a^3b^3$.
10. $a^8 + b^8 - 2a^4b^4$.
11. $64 + t^2 - 16t$.
12. $16 + x^2 - 8x$.
13. $9 - 6y + y^2$.
14. $25x^2 + 16y^2 + 40xy$.
15. $4m^2 + n^2 + 2mn$.
16. $100 + s^2 + 20s$.
17. $64 + 49 + 112$.
18. $16a^2 + 25b^2 - 50ab$.

WRITTEN EXERCISES

Decide which of the following are trinomial squares. Find the square roots of all such:

1. $9 + 2 \cdot 3 \cdot 4 + 16$.
2. $x^2 + 4y^2 + 4xy$.
3. $9x^2 + 18xy + 9y^2$.
4. $4x^2 + 4xy + y^2$.
5. $4x^2 + 8xy + 4y^2$.
6. $25x^2 + 12xy + 4y^2$.
7. $16x^2 + 16xy + 4y^4$.
8. $9r^2 + 36rs + 25s^2$.
9. $16x^6 + 8x^3y + y^2$.
10. $4x^8 + 12a^3x^4 + 9a^6$.
11. $a^{10} + 6a^5b + 9b^2$.
12. $(a + 1)^2 + 2(a + 1)b + b^2$.
13. $(x+3)^2 + 4(x+3)y + 4y^2$.
14. $x^6 + 12x^3 + 36$.
15. $a^4 + 18a^2 + 12$.
16. $121 + 4x^8 - 44x^4$.
17. $16x^4 + 64y^4 - 64x^2y^2$.
18. $81a^2 - 216a + 144$.
19. $4a^2 + 8ab^2 + 4b^2$.
20. $9b^4 + 18b^2c^4 + 9c^8$.
21. $4x^2 + 4y^2 - 8xy$.
22. $9a^2 - 16ab + 4b^2$.
23. $9a^4 - 24a^2b + 16b^2$.
24. $25 + 49x^2 - 70x$.
25. $-30ab^2 + 9a^2 + 25b^4$.
26. $16a^2 - 24ab + 9b^2$.
27. $36x^2 - 84x + 49$.
28. $25 - 90 + 81$.
29. $64x^2 - 32x + 9$.
30. $(3 + a)^2 + b^2 - 2b(3 + a)$.

Case III: THE DIFFERENCE OF TWO SQUARES

$$a^2 - b^2 = (a + b)(a - b).$$

100. In § 87, we found by multiplication,

$$(a + b)(a - b) = a^2 - b^2.$$

Hence we have the formula

$$a^2 - b^2 = (a + b)(a - b)$$

From this formula we obtain the following

Rule. *Every binomial which is the difference between two perfect squares is the product of two binomial factors; namely, the sum and the difference of the square roots of these squares.*

E.g. $16x^2 - 9y^2$ is the difference of the two squares, $(4x)^2$ and $(3y)^2$. Hence we have

$$16x^2 - 9y^2 = (4x)^2 - (3y)^2 = (4x + 3y)(4x - 3y).$$

ORAL EXERCISES

In each of the following expressions determine whether it is the difference of two squares, and if so, find the factors.

1. $x^2 - 4y^2$.
2. $9x^2 - 36y^2$.
3. $x^4 - b^2$.
4. $4x^2 - 9b^8$.
5. $16a^4 - 9b^4$.
6. $64 - b^2$.
7. $1 - b^2$.
8. $a^2 - 1$.
9. $1 - 9x^4$.
10. $4 - 36a^2$.
11. $1 - 64a^8$.
12. $144x^2b^4 - 1$.
13. $36a^4b^6 - c^2$.
14. $1 - (x + y)^2$.
15. $4 - (x + 2y)^2$.
16. $16a^2 - 25b^2$.
17. $49x^2 - 4y^2$.
18. $225 - 64x^2y^4$.
19. $81a^2 - 144y^2$.
20. $5^8 - 3^8$.
21. $x^4 - 81y^2$.
22. $a^2 - (x + y)^2$.
23. $(x + y)^2 - a^2$.
24. $(x - y)^2 - a^2$.
25. $a^2 - (x - y)^2$.
26. $(a + 3)^2 - 16$.
27. $(x + 3)^2 - 25$.
28. $25 - (a + b)^2$.
29. $36 - 4(a + b)^2$.
30. $9(a - b)^2 - 4$.
31. $16(a - b)^2 - 4c^2$.
32. $16(a - b)^2 - 9d^2$.
33. $4 - 9(a + b)^2$.
34. $9 - 16(a + b)^2$.
35. $9a^2 - 4(x + y)^2$.
36. $25(x - y)^2 - 4z^2$.
37. $36(x + y)^2 - 25z^2$.
38. $64x^2 - 49(a + b)^2$.
39. $81a^4 - 36(b + c)^2$.

101. Following is another example of an expression which may be written as the difference of two squares:

E.g. $a^2+b^2+2ab-c^2=(a+b)^2-c^2=(a+b+c)(a+b-c)$.

WRITTEN EXERCISES

Factor each of the following:

1. $x^2-(y-z)^2$.
2. $(x-y)^2-z^2$.
3. $a^2+b^2-2ab-4$.
4. $x^2+y^2-2xy-z^2$.
5. $4a^2b^2-(a^2+b^2-c^2)^2$.
6. $a^2-(b^2+c^2+2bc)$.
7. $(2a-5)^2-(3a+1)^2$.
8. $(3x^2-y)^2-(x+y)^2$.
9. $(3a-2b)^2-(8a+5b)^2$.
10. $(3m-4)^2-(2m+3)^2$.
11. $(2r+s)^2-(3r-s)^2$.
12. $81-(a+b+c)^2$.
13. $x^4+2x^2+1-4a^2$.
14. $a^2-(x+2y)^2$.
15. $9x^2-(a-b)^2$.
16. $25m^2-(3r+2s)^2$.

102. Expressions Reducible to the Difference of Two Squares.*

Example. The trinomial $a^4+a^2b^2+b^4$ would be the difference of two squares if its middle term were $2a^2b^2$ instead of a^2b^2. Hence, if we add a^2b^2 to this term and subtract a^2b^2 from the whole expression, we shall have the difference of two squares.

Thus, $a^4+a^2b^2+b^4=a^4+2a^2b^2+b^4-a^2b^2=(a^2+b^2)^2-a^2b^2$.

WRITTEN EXERCISES

Factor the following:

1. $x^4+x^2y^2+y^4$.
2. $x^8+x^4y^4+y^8$.
3. $a^4+4b^4=(a^4+4a^2b^2+4b^4)-4a^2b^2$.
4. m^8+4n^8.
5. a^4+a^2+1.
6. y^8+y^4+1.
7. $4x^4+11x^2y^2+9y^4$.
8. $x^4-14x^2y^2+25y^4$.
9. $4a^4-29a^2b^2+25b^4$.
10. $16a^4+20a^2b^2+9b^4$.
11. $a^{4n}+a^{2n}b^{2n}+b^{4n}$.
12. $x^4-12x^2y^2+4y^4$.
13. $a^4-17a^2b^2+16b^4$.

---

* This article may be omitted without destroying the continuity.

Case IV: THE SUM OF TWO CUBES

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

103. In § 92 we found

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

Since Dividend = Divisor × Quotient, we have

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

From this formula we obtain the following

Rule. *The sum of the cubes of two numbers is the product of two factors, one of which is the sum of the numbers, and the other is the sum of their squares minus their product.*

E.g. (1) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(2) $8a^3 + 27b^3 = (2a)^3 + (3b)^3$.
$= (2a + 3b)(4a^2 - 2a \cdot 3b + 9b^2)$.

(3) $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

Notice the difference between the trinomial $x^2 - xy + y^2$ and the trinomial square $x^2 - 2xy + y^2$.

EXERCISES

Determine whether each of the following is the sum of two cubes, and if so find the factors. Read 1–6 at sight.

1. $x^3 + y^3$.
2. $a^3 + 8b^3$.
3. $27a^3 + b^3$.
4. $8a^3 + 1$.
5. $1 + 64x^3$.
6. $2^3 + 3^3$.
7. $125 + 729$.
8. $1 + 125x^6$.
9. $27x^6 + 1$.
10. $8a^3 + 27b^3$.
11. $8a^3 + 64b^3$.
12. $w^3x^6 + x^9a^3$.
13. $1 + 8a^3b^3$.
14. $64x^3 + 343$.
15. $1 + a^3$.
16. $a^3 + 9b^3$.
17. $125x^3 + y^6$.
18. $1 + x^8$.
19. $64x^2 + 27y^3$.
20. $8^3 + 10^3$.
21. $1 + 729x^6$.
22. $x^6 + y^{12}$.
23. $a^9 + b^3$.
24. $27r^3 + 125s^3$.
25. $x^9 + 27y^6$.
26. $64 + a^9$.
27. $a^3b^6 + x^3y^9$.
28. Find whether $x^3 + y^3$ is exactly divisible by $x - y$.

Case V: THE DIFFERENCE OF TWO CUBES

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

104. In § 92 we found

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

Since Dividend = Divisor × Quotient, we have

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

From this formula we obtain the following

Rule. *The difference of the cubes of two numbers is the product of two factors, one of which is the difference of the numbers, and the other is the sum of their squares plus their product.*

E.g. (1) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(2) $8a^3 - 64b^3 = (2a)^3 - (4b)^3$

$= (2a - 4b)(4a^2 + 2a \cdot 4b + 16b^2)$.

(3) $a^9 - b^9 = (a^3)^3 - (b^3)^3 = (a^3 - b^3)(a^6 + a^3b^3 + b^6)$.

Notice the difference between the factor $x^2 + xy + y^2$, and the trinomial square $x^2 + 2xy + y^2$.

EXERCISES.

Determine whether each of the following is the difference of two cubes, and if so, find the factors. Read 1–6 at sight.

1. $r^3 - s^3$.
2. $x^3 - 1$.
3. $1 - x^3$.
4. $x^6 - y^6$.
5. $1 - y^6$.
6. $y^6 - 1$.
7. $1 - a^{10}$.
8. $1 - 8a^3$.
9. $64x^3 - y^3$.
10. $27 - 125a^3$.
11. $x^3 - y^6$.
12. $x^9 - 8$.
13. $1 - 125x^3$.
14. $8 - 27x^3$.
15. $27x^3 - 64$.
16. $2^3x^3 - 1$.
17. $8x^4 - y^3$.
18. $64a^3 - 27b^3$.
19. $1 - 729x^6$.
20. $x^6 - y^{12}$.
21. $27r^3 - 125s^3$.

22. Also factor Examples 4, 5, 6, 19, and 20 as the difference of two squares; and then resolve these factors still further.

Case VI: TRINOMIALS OF THE FORM x^2+px+q

105. In § 88 were found such products as

$$(1)\ (x+5)(x+2)=x^2+7x+10.$$
$$(2)\ (x-5)(x-2)=x^2-7x+10.$$
$$(3)\ (x+5)(x-2)=x^2+3x-10.$$
$$(4)\ (x-5)(x+2)=x^2-3x-10.$$

All these are included in the form

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

in which the *coefficient of x is the algebraic sum of a and b and the last term is their product.*

106. It is possible to determine at sight whether a trinomial is of the form just considered, and if it is, to find the factors by inspection.

Illustrative Examples. Determine whether the following trinomials can be factored by inspection:

1. $x^2+7x+12$. The question is whether two numbers can be found such that their sum is $+7$ and their product 12. 3 and 4 are such numbers. Hence, $x^2+7x+12=(x+3)(x+4)$.

2. $x^2-5x-14$. Since the product of the numbers sought is -14, one number must have the sign $-$ and the other $+$; and since their sum is -5, the one having the greater absolute value must have the sign $-$. The numbers are -7 and $+2$, and we have $x^2-5x-14=(x-7)(x+2)$.

3. $x^2-7x+12=(x-3)(x-4)$. Since $(-3)(-4)=+12$ and $(-3)+(-4)=-7$.

4. $x^2+4x-12=(x+6)(x-2)$. Since $(+6)(-2)=-12$ and $(+6)+(-2)=+4$.

These examples lead to the following

Rule. *To factor a trinomial such as x^2+px+q, we try to find two numbers, a and b, whose product is q and whose algebraic sum is p. If two such numbers can be found, then the factors are $x+a$ and $x+b$.*

ORAL EXERCISES

In each of the following state what is the product of the two numbers to be found and what is their sum. Then give the factors.

1. x^2+3x+2.
2. x^2-3x+2.
3. x^2-x-2.
4. x^2+x-2.
5. x^2+4x+3.
6. x^2-2x-3.
7. x^2+2x-3.
8. x^2-4x+3.
9. x^2+5x+4.
10. x^2-5x+4.
11. x^2-3x-4.
12. x^2+3x-4.
13. x^2+6x+5.
14. x^2-4x-5.
15. x^2+4x-5.
16. x^2-6x+5.
17. x^2+7x+6.
18. x^2-5x-6.
19. x^2+5x-6.
20. x^2-7x+6.
21. x^2+5x+6.
22. x^2-x-6.
23. x^2+x-6.
24. x^2-5x+6.
25. $x^2+7x+12$.
26. x^2-x-12.

Give orally or in writing the factors of the following:

27. x^2+x-12.
28. $x^2-7x+12$.
29. x^2+6x+8.
30. x^2-2x-8.
31. x^2+2x-8.
32. x^2-6x+8.
33. x^2+8x+7.
34. x^2-8x+7.
35. x^2-6x-7.
36. x^2+6x-7.
37. x^2+6x+9.
38. x^2-6x+9.
39. $x^2+10x+9$.
40. $x^2-10x+9$.
41. x^2+8x-9.
42. x^2-8x-9.
43. $x^2+7x+10$.
44. $x^2-7x+10$.
45. $x^2+3x-10$.
46. $x^2-3x-10$.
47. $x^2+8x+12$.
48. $x^2-8x+12$.
49. $x^2-4x-12$.
50. $x^2+4x-12$.
51. $x^2+9x+20$.
52. $x^2-9x+20$.

107. It is not always possible to factor by inspection expressions of the form $x^2 + px + q$; for it may be that there are no integers whose product is q and whose algebraic sum is p.

E.g. Given $x^2 + 5x + 3$. It is easily seen that there are no two integers such that their sum is $+5$ and their product $+3$.

WRITTEN EXERCISES

Determine whether each of the following trinomials can be factored by inspection, and if so, find the factors.

1. $x^2 + 11x + 24$.
2. $x^2 + 2x - 35$.
3. $x^2 - 3x - 40$.
4. $x^2 - 2x - 24$.
5. $x^2 + x - 30$.
6. $x^2 - 3x - 8$.
7. $x^2 + 2x - 24$.
8. $a^2 - 4a - 32$.
9. $a^2 + 4a - 32$.
10. $b^2 + 15b + 56$.
11. $b^2 + 8b + 15$.
12. $b^2 - b - 56$.
13. $b^2 + b - 56$.
14. $c^2 - 3c - 15$.
15. $x^2 - 15x + 56$.
16. $x^2 + 15x - 54$.
17. $x^2 - 14x - 95$.
18. $y^2 + 21y + 98$.
19. $y^2 - 7y - 98$.
20. $x^2 - 19x + 78$.
21. $x^4 + 18x^2 + 77$.
22. $x^4 - 5x^2 - 104$.
23. $a^2 + 32a + 240$.
24. $a^4 - 11a^2 + 28$.
25. $a^4 - 11a^2 - 60$.
26. $a^2 - 14a - 51$.
27. $a^2 - 3a - 54$.
28. $x^4 - 8x^2 - 32$.
29. $a^6 - 3a^3 - 154$.
30. $x^2 - 10x + 25$.
31. $a^2b^6 - 13ab^3 - 30$.
32. $x^2 - 17xyz + 72y^2z^2$.
33. $r^8 + 6r^4s - 91s^2$.
34. $a^4c^4 + 9a^2c^2 - 162$.
35. $a^2 + 11a - 210$.
36. $m^4 + 4m^2n + 4n^2$.
37. $s^2t^2 - 15st - 54$.
38. $a^2b^2 - 27ab + 26$.
39. $l^2 + 13l + 42$.
40. $x^2y^2 - 11xy - 180$.
41. $9a^2 + 24a + 16$.
42. $81a^2 - 99a + 30$.
43. $g^2 + 26g + 133$.
44. $x^2 + 5xy - 84y^2$.
45. $r^2 + 3r - 154$.
46. $u^2 + 38uv + 165v^2$.

Case VII: TRINOMIALS OF THE FORM ax^2+bx+c.

108. Examples.

(1)
$$\begin{array}{r} 2x+5 \\ \underline{3x+2} \\ 6x^2+15x \\ \underline{4x+10} \\ 6x^2+19x+10 \end{array}$$

(2)
$$\begin{array}{r} 2x+5 \\ \underline{3x-2} \\ 6x^2+15x \\ \underline{-4x-10} \\ 6x^2+11x-10 \end{array}$$

In Example (1), the products $3x\cdot 2x=6x^2$ and $2\cdot 5=10$ are called *end products* and $2\cdot 2x=4x$ and $5\cdot 3x=15x$ are called *cross products.* Likewise, in Example (2), $6x^2$ and -10 are *end products* and $-4x$ and $15x$ are *cross products.*

In each case we see that the *final result is a trinomial, two of whose terms are the end products while the third term is the algebraic sum of the cross products.*

Likewise, examine the following:

(3)
$$\begin{array}{r} 2x-5 \\ \underline{3x+2} \\ 6x^2-15x \\ \underline{4x-10} \\ 6x^2-11x-10 \end{array}$$

(4)
$$\begin{array}{r} 2x-5 \\ \underline{3x-2} \\ 6x^2-15x \\ \underline{-4x+10} \\ 6x^2-19x+10 \end{array}$$

WRITTEN EXERCISES

In this manner obtain the following products:

1. $(2a+3)(a+3)$.
2. $(4a-1)(3a+2)$.
3. $(2x+5)(x-7)$.
4. $(7r+8)(3r-6)$.
5. $(2x+8)(9x-4)$.
6. $(3m-1)(4m+3)$.
7. $(5s-7)(2s-4)$.
8. $(2x-1)(7x+4)$.
9. $(4n-9)(5n-7)$.
10. $(8y-1)(5y+11)$.
11. $(t-5)(t+4)$.
12. $(5x-y)(2x-3y)$.
13. $(3x-2y)(x+3y)$.
14. $(4a-3y)(a+y)$.
15. $(3r-2s)(2r+s)$.
16. $(5m-n)(2m+n)$.
17. $(5a+3x)(3a-4x)$.
18. $(4a-5b)(a+3b)$.
19. $(3a+5b)(a-b)$.
20. $(3c-7d)(2c+3d)$.
21. $(2a-3b)(3a+2b)$.
22. $(6x-5y)(2x+3y)$.

109. Factoring $ax^2 + bx + c$ by Inspection. Trinomials in the form $ax^2 + bx + c$ may sometimes be factored by inspection.

Example 1. Factor $5x^2 + 16x + 3$.

If this is the product of two binomials they must be such that the end products are $5x^2$ and 3 and the sum of the cross products $16x$.

One pair of binomials having the required end products is $5x + 3$ and $x + 1$. Others are $5x + 1$ and $x + 3$; $5x - 1$ and $x - 3$; and $5x - 3$ and $x - 1$.

It is convenient to write down these possible pairs of factors as follows, as if arranged for multiplication:

$$\begin{array}{cccc} 5x + 3 & 5x - 3 & 5x + 1 & 5x - 1 \\ x + 1 & x - 1 & x + 3 & x - 3 \end{array}$$

The sum of the cross products in the first pair is $8x$, in the second pair $-8x$, in the third pair $16x$, and in the fourth $-16x$. Since $16x$ is the middle term required, the factors are $5x + 1$ and $x + 3$.

Example 2. Factor $6x^2 - 19x + 10$.

Pairs of binomials which give the right end products are

$$\begin{array}{cccc} 3x + 5 & 3x - 5 & 2x + 5 & 2x - 5 \\ 2x + 2 & 2x - 2 & 3x + 2 & 3x - 2 \end{array}$$

Of these, the ones which give the right cross products are $2x - 5$ and $3x - 2$.

Hence $$6x^2 - 19x + 10 = (2x - 5)(3x - 2).$$

WRITTEN EXERCISES

Factor the following:

1. $2x^2 + 5x + 2$.
2. $2x^2 + 3x - 2$.
3. $2x^2 - 3x - 2$.
4. $2x^2 - 5x + 2$.
5. $2x^2 + 7x + 3$.
6. $2x^2 - 7x + 3$.
7. $2x^2 + 5x - 3$.
8. $2x^2 - 5x - 3$.

From these examples we deduce the following

Rule. *To factor a trinomial of the form* $ax^2 + bx + c$, *write down the possible pairs of binomials which give the proper end products. Select that pair whose cross products give the proper algebraic sum.*

WRITTEN EXERCISES

In this manner factor the following:

1. $3x^2 + 5x + 2$.
2. $3x^2 + x - 2$.
3. $3x^2 - 5x + 2$.
4. $3x^2 - x - 2$.
5. $3x^2 + 7x + 2$.
6. $3x^2 - 7x + 2$.
7. $3x^2 + 5x - 2$.
8. $3x^2 - 5x - 2$.
9. $3x^2 + 17x + 10$.
10. $3x^2 - 17x + 10$.
11. $3x^2 + 13x - 10$.
12. $3x^2 - 13x - 10$.
13. $5x^2 + 6x + 1$.
14. $5x^2 - 6x + 1$.
15. $5x^2 + 4x - 1$.
16. $5x^2 - 4x - 1$.
17. $6x^2 + 7x + 1$.
18. $6x^2 - 7x + 1$.
19. $3x^2 + 4x + 1$.
20. $3x^2 - 4x + 1$.
21. $5x^2 - 17x - 12$.
22. $5x^2 + 17x - 12$.
23. $9a^2 + 9a + 2$.
24. $2x^2 + 11x + 12$.
25. $9x^2 + 36x + 32$.
26. $2x^2 - x - 28$.

In the following try to find the factors without writing all the pairs which give proper end products.

27. $12s^2 + 11s + 2$.
28. $6t^2 + 7t - 3$.
29. $6x^2 - x - 2$.
30. $5r^2 + 18r - 8$.
31. $14a^2 - 39a + 10$.
32. $5x^2 + 26x - 24$.
33. $2x^2 - 5x + 2$.
34. $2m^2 - m - 3$.
35. $7c^2 - 3c - 4$.
36. $5x^4 + 9x^2 - 18$.
37. $7a^4 + 123a^2 - 54$.
38. $6c^2 - 19c + 15$.
39. $3a^2 - 21a + 30$.
40. $6d^2 + 4d - 2$.
41. $20a^2 - a - 99$.
42. $12c^2 + 25c + 12$.
43. $8 + 6a - 5a^2$.
44. $15 - 5x - 10x^2$.
45. $6 - 5x - 4x^2$.
46. $3h^2 - 13h + 14$.
47. $15r^2 - r - 2$.
48. $2t^2 + 11t + 5$.
49. $10 - 5x - 15x^2$.
50. $5x^2 - 33x + 18$.

Case VIII: FACTORS FOUND BY GROUPING

$$ax + ay + bx + by = (a + b)(x + y).$$

110. Another method of general application will now be applied to polynomials of four terms.

Example 1. Find the factors of $ax + ay + bx + by$.

By Principle I, the first two terms may be added and also the last two.

Thus, $ax + ay + bx + by = a(x + y) + b(x + y)$.

These two compound terms have a *common factor*, $(x + y)$, and may be added with respect to this factor by Principle I.

Thus, $a(x + y) + b(x + y) = (a + b)(x + y)$.

Hence, $ax + ay + bx + by = (a + b)(x + y)$.

Example 2. Factor $ax - ay - bx + by$.

Combining the first two terms with respect to a and the second two with respect to $-b$, we have,

$$ax - ay - bx + by = a(x - y) - b(x - y).$$

Again combining with respect to the factor $x - y$,

$$ax - ay - bx + by = (a - b)(x - y).$$

The success of this method depends upon the possibility of so grouping and combining the terms as to reveal a *common binomial factor*.

WRITTEN EXERCISES

Factor the following:

1. $ab^2 + ac^2 - db^2 - dc^2$.
2. $6ms - 15nt + 9ns - 10mt$.
3. $8ax - 10ay + 4bx - 5by$.
4. $2a^2 + 3ak - 14an - 21nk$.
5. $ac + bc + ad + bd$.
6. $ax^2 - bx^2 - ay^2 + by^2$.
7. $8ac - 20ad - 6bc + 15bd$.
8. $2ax - 6bx + 3by - ay$.
9. $5 + 4a - 15c - 12ac$.
10. $15b - 6 - 20bc + 8c$.
11. $2n^2 - cn + 2nd - cd$.
12. $5ax - 15ay - 3bx + 9by$.
13. $3xa - 12xc - a + 4c$.
14. $3xy - 4mn + 6my - 2xn$.
15. $7mn + 7mr - 2n^2 - 2nr$.
16. $a - 1 + a^3 - a^2$.
17. $3s + 2 + 6s^4 + 4s^3$.
18. $as^2 - 3bst - ast + 3bt^2$.
19. $3mn + 6m^2 - 2am - an$.
20. $2ar + 2as + 2br + 2bs$.

Case IX: THE SQUARE OF A TRINOMIAL

$$\boldsymbol{a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc = (a + b + c)^2.}$$

111. In § 89 we found

(1) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$

(2) $(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc.$

(3) $(a - b + c)^2 = a^2 + b^2 + c^2 - 2\,ab + 2\,ac - 2\,bc.$

(4) $(a - b - c)^2 = a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc.$

A study of these forms enables us to determine whether a polynomial of six terms is a perfect square; namely,

(1) *Three of the terms must be squares each with the sign* $+$.

(2) *Each of the other three terms must be twice the product of the square roots of two of the square terms.*

(3) *The signs of these products must all be* $+$, *or else two of them must be* $-$ *and one* $+$.

Example. Find whether the following is a perfect square:

$$4\,x^2 - 12\,xy - 16\,xz + 9\,y^2 + 24\,yz + 16\,z^2$$

Solution. The terms $4\,x^2$, $9\,y^2$, and $16\,z^2$ are all squares, each with the sign $+$. The square roots of these are $2\,x$ or $-2\,x$, $3\,y$ or $-3\,y$, $4\,z$ or $-4\,z$. By trial we find that $2(2\,x)(-3\,y) = -12\,xy$; $2(2\,x)(-4\,z) = -16\,xz$; and $2(-3\,y)(-4\,z) = 24\,yz$.

Hence the given polynomial is equal to $(2\,x - 3\,y - 4\,z)^2$.

WRITTEN EXERCISES

In each of the following determine whether the polynomial is a perfect square, and if so indicate its square root.

1. $x^2 + y^2 + z^2 - 2\,xy + 2xz - 2\,yz.$
2. $a^2 - 8\,ab + 16\,b^2 - 2\,ac + c^2 + 8\,bc.$
3. $9\,x^2 + 4\,y^2 + z^2 - 12\,xy + 6\,xz - 4\,yz.$
4. $y^4 - 4\,y^2 - 8\,xy^2 + 16\,x + 16\,x^2 + 4.$
5. $a^2 + a^2b^2 - 2\,a^2b + 2\,abc - ab^2c + b^2c^2.$
6. $x^6 - 4\,x^5 + 4\,x^4 + 6\,x^3 - 12\,x^2 + 9.$
7. $x^2 + 16\,x^2y^2 + 289 + 8\,x^2y + 34\,x + 136\,xy.$

Case X: THE REMAINDER THEOREM*

112. It is possible to tell whether a binomial like $x-2$ will exactly divide a polynomial like x^2-5x+4 without actually performing the division.

Examples. (1)
$$\begin{array}{r|l} x^2-5x+4 & x-2 \\ \underline{x^2-2x} & \overline{x-3} \\ -3x+4 & \\ \underline{-3x+6} & \\ -2 & \end{array}$$

(2)
$$\begin{array}{r|l} x^2-5x+4 & x-1 \\ \underline{x^2-x} & \overline{x-4} \\ -4x+4 & \\ \underline{-4x+4} & \\ 0 & \end{array}$$

If we substitute 2 for x in x^2-5x+4, we get $4-10+4=-2$, which is the remainder in Example (1).

If we put 1 for x in x^2-5x+4, we get $1-5+4=0$, which is the remainder in Example (2).

These examples illustrate the **remainder theorem.**

Rule: *If we substitute a number k for x in a polynomial involving x, the resulting number is the remainder arising from dividing the polynomial by $x-k$.*

To make this more evident, let us divide x^2-5x+4 by $x-k$.

$$\begin{array}{l|l} x^2-5x+4 & x-k \\ \underline{x^2-kx} & \overline{x+(k-5)} \\ (k-5)x+4 & \\ \underline{(k-5)x-k^2+5k} & \\ \qquad\qquad k^2-5k+4. & \text{Remainder.} \end{array}$$

We thus see that this remainder is exactly like the dividend with x replaced by k according to the rule.

The object of this rule is to find remainders which are zero, for then the division is exact, and the **divisor is a factor** of the given polynomial.

Hence this rule is also called the **factor theorem.**

For instance, if we put $x=2$ in x^2-5x+4, we get $4-10+4=-2$, which is the remainder, according to the rule, when we divide x^2-5x+4 by $x-2$. Hence $x-2$ is not a factor of x^2-5x+4.

But if we put $x=1$ in x^2-5x+4, the remainder is $1-5+4=0$. Hence $x-1$ is a factor of x^2-5x+4.

---

* Articles 112–115 may be omitted without destroying the continuity.

113. Finding Factors by the Remainder Theorem. Since one expression is divisible by another only when the remainder is zero, we use the remainder theorem to find factors as in the following examples:

Example 1. Is $x-1$ a factor of x^2-3x+2?

Solution. Substitute 1 for x in x^2-3x+2 and we have $1-3+2=0$. Hence there is no remainder when we divide x^2-3x+2 by $x-1$. That is, $x-1$ is a factor of x^2-3x+2.

Example 2. Show by the remainder theorem that $x-1$ and $x+2$ are factors of x^2+x-2.

Solution. If we substitute 1 for x in x^2+x-2, we get $1+1-2=0$.

Hence there is no remainder when we divide x^2+x-2 by $x-1$. That is, $x-1$ is a factor of x^2+x-2.

To apply this test to $x+2$, we write it in the form $x-(-2)$ and then substitute -2 for x in x^2+x-2 and get $4-2-2=0$.

Hence, $x-(-2)=x+2$ is an exact divisor of x^2+x-2. That is, $x+2$ is a factor of x^2+x-2.

Example 3. Factor x^2+4x-5 by the remainder theorem.

We know that if a binomial exactly divides x^2+4x-5, its last term must be a factor of 5.

Hence the only possible binomial divisors are $x-5$, $x+5$, $x-1$, and $x+1$.

If we substitute 5 for x in x^2+4x-5, we get $25+20-5=40$.

Hence the remainder is not zero when we divide x^2+4x-5 by $x-5$, and therefore $x-5$ is not a factor of x^2+4x-5.

If we substitute -5 for x we get $25-20-5=0$. Hence $x-(-5)=x+5$ is a factor of x^2+4x-5.

Likewise, we find that $x-1$ is a factor and $x+1$ is not.

Hence, $x^2+4x-5=(x+5)(x-1)$.

Example 4. Is $x-2$ a factor of x^3-5x^2+7x-2?

By the remainder theorem the remainder is $8-20+14-2=0$. Hence $x-2$ is a factor. To find the other factor, we divide by $x-2$ and get as the quotient x^2-3x+1.

Example 5. Is $x-1$ a factor of x^7-1?

By the remainder theorem, $x=1$ gives $1^7-1=1-1=0$. Hence, $x-1$ is a factor of x^7-1.

ORAL EXERCISES

1. Is $x-1$ a factor of x^3-1?
2. Is $x-1$ a factor of x^3+1?
3. Is $x+1$ a factor of x^3-1?

Suggestion. Put $x=-1$ in x^3-1.

4. Is $x+1$ a factor of x^3+1?
5. Is $x-1$ a factor of x^5-1? of x^5+1?
6. Is $x+1$ a factor of x^5-1? of x^5+1?
7. Is $x-1$ a factor of x^6-1? of x^6+1?
8. Is $x+1$ a factor of x^6-1? of x^6+1?
9. Is $x-1$ a factor of x^7-1? of x^7+1?
10. Is $x+1$ a factor of x^7-1? of x^7+1?
11. Is $x+1$ a factor of x^8-1? of x^8+1?

WRITTEN EXERCISES

1. Is $x+1$ a factor of x^2+3x+2?
2. Is $x-1$ or $x+1$ a factor of x^2-4x+1?
3. Is $x-1$ a factor of x^3-2x^2+2x-1?
4. Find the factors of $x^3-7x^2+11x-5$.
5. Is $x-2$ a factor of x^3-8? of x^3+8?
6. Is $x-y$ a factor of x^4-y^4? of x^4+y^4? of x^5-y^5? of x^5+y^5? of x^6-y^6? of x^6+y^6?

Suggestion. In each case put y in place of x and see whether the expression is reduced to zero.

7. Is $x+y$ a factor of x^4-y^4? of x^4+y^4? of x^5-y^5? of x^5+y^5? of x^6-y^6? of x^6+y^6?
8. Is $x-1$ a factor of $x^{11}-1$? of $x^{12}-1$?
9. Is $x+1$ a factor of $x^{11}-1$? of $x^{12}-1$?
10. Factor a^3-7a+6.
11. Factor x^3+2x^2-x-2.
12. Factor $a^4-a^3-7a^2+a+6$.

FACTORS OF x^n+y^n AND x^n-y^n

114. Applying the Remainder Theorem. By use of the remainder theorem we may find under what conditions $x+y$ and $x-y$ are factors of x^n+y^n and x^n-y^n.

1. Is $x+y$ a factor of x^n+y^n?

If we substitute $-y$ for x we have $(-y)^n+y^n$. This is zero only when $(-y)^n=-y^n$; that is, when n is an *odd integer.*

E.g. $(-y)^5=-y^5$ but $(-y)^6=+y^6$.

Hence $x+y$ is a factor of x^5+y^5, but *not* of x^6+y^6.

2. Is $x+y$ a factor of x^n-y^n?

Here we have $(-y)^n-y^n$. This reduces to zero only when $(-y)^n=+y^n$; that is, when n is an *even integer.*

E.g. $(-y)^6=+y^6$ but $(-y)^7=-y^7$.

Hence $x+y$ is a factor of x^6-y^6, but *not* of x^7-y^7.

3. Is $x-y$ a factor of x^n+y^n?

Since y^n+y^n is never zero, $x-y$ is not a factor of x^n+y^n.

E.g. $x-y$ is not a factor of x^3+y^3, nor of x^4+y^4.

4. Is $x-y$ a factor of x^n-y^n?

Since $y^n-y^n=0$, $x-y$ is a factor of x^n-y^n, for all integral values of n.

E.g. $x-y$ is a factor of x^3-y^3 and also of x^4-y^4.

Summary. From the foregoing examples, we conclude that:

(1) *$x+y$ is a factor of x^n-y^n if n is **even** but not if n is **odd**.*

(2) *$x-y$ is a factor of x^n-y^n whether n is **even** or **odd**.*

(3) *$x+y$ is a factor of x^n+y^n if n is **odd** but not if n is **even**.*

(4) *$x-y$ is not a factor of x^n+y^n in **any** case.*

115. Special Case. When n is an *even* integer, it is best to factor x^n-y^n as the *difference of two squares.*

E.g.
$$\begin{aligned} x^6-y^6 &= (x^3+y^3)(x^3-y^3) \\ &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2). \end{aligned}$$

Also
$$\begin{aligned} x^8-y^8 &= (x^4-y^4)(x^4+y^4) \\ &= (x-y)(x+y)(x^2+y^2)(x^4+y^4). \end{aligned}$$

EXERCISES

1. Find one factor of $x^9 - y^9$; also of $x^{29} + y^{29}$.

2. Find two factors of $x^{20} - y^{20}$; also of $x^{30} - y^{30}$.

3. Find all the factors of $x^4 - y^4$; also of $a^5 - b^5$.

4. Find all the factors of $x^{16} - y^{16}$; also of $x^{10} - y^{10}$.

5. Find all the factors of $x^{12} - y^{12}$.

6. Make a rule for reading at sight the following quotients, $(x^5 + y^5) \div (x + y)$ and $(x^5 - y^5) \div (x - y)$.

7. Does a similar rule apply to $(x^7 + y^7) \div (x + y)$ and $(x^7 - y^7) \div (x - y)$?

8. Factor $x^6 + y^6$. Show that $x^2 + y^2$ is one factor by substituting $-y^2$ for x^2 in $(x^2)^3 + (y^2)^3$.

9. Is $x^3 - y^3$ a factor of $x^{15} - y^{15}$? Why?

REVIEW AND SUMMARY

1. What is meant by *factoring?* Is $x(a + b) + y(a + b)$ factored? Why?

2. By what principle is the monomial factor removed from

$$ax + ay + az?$$

3. What are the characteristics of the trinomial squares:

$$a^{2n} + 2\,a^nb^n + b^{2n};\quad a^{2n} - 2\,a^nb^n + b^{2n}?$$

Are the following trinomials squares? If not, state why they are not. $x^2 + xy + y^2$; $x^4 + x^2y^2 + y^4$; $a^2 - 2\,ab - b^2$; $4\,a^2 + 4\,ab + 4\,b^2$.

4. What are the factors of the difference of two squares:

$$x^{2n} - y^{2n}?$$

Factor $x^6 - y^6$ as the difference of two squares

How can $$x^{4n} + x^{2n}y^{2n} + y^{4n}$$

be changed into the difference of two squares?

5. What are the factors of the sum of two cubes:

$$x^3 + y^3?$$

Can $x^6 + y^6$ be factored in this way? $x^9 + y^9$?

6. What are the factors of the difference of two cubes:

$$x^3 - y^3?$$

Can $x^6 - y^6$ be factored in this way? $x^9 - y^9$?

7. State the conditions under which

$$x^2 + px + q$$

can be factored by inspection. Can $x^2 - 10x + 16$ be so factored? $x^2 - 10x - 16$?

8. Tell how to decide whether

$$ax^2 + bx + c$$

can be factored by inspection. Can $6x^2 + 13xy + 6y^2$ be so factored?

9. What are the factors of

$$ax + ay + bx + by?$$

Can $x^2 + ax + bx + ab$ be factored in this way? Show that $x^3 - x - 3x^2 + 3$ can be factored in this way.

10. What are the factors of

$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz?$$

State the characteristics of the square of a trinomial.

11.* State the **remainder theorem.** For what values of n is $x^n + y^n$ divisible by $x + y$? by $x - y$? For what values of n is $x^n - y^n$ divisible by $x - y$? by $x + y$?

Why is this theorem also called the **factor theorem**?

12. If k is an integer, what kind of a number is $2k$? What kind of a number is $2k + 1$?

13.* Is $x - y$ a factor of $x^{2k} - y^{2k}$? of $x^{2k+1} - y^{2k+1}$?

14.* Is $x + y$ a factor of $x^{2k} + y^{2k}$? of $x^{2k+1} + y^{2k+1}$?

MISCELLANEOUS ORAL EXERCISES

Factor the following:

1. $a^2b^2 + a^3b + a^4$.
2. $a^2b^2 - c^2$.
3. $x^2 + 4x + 4$.
4. $a^2 - 6a + 9$.
5. $x^2 + 9x + 20$.
6. $a^2 - 5a + 6$.
7. $x^2 + 5x + 6$.
8. $x^2 + 11x + 30$.
9. $4x^2 - y^2$.
10. $x^2 + 7x + 6$.
11. $a^4 - 25$.
12. $2ab + b^2 + a^2$
13. $a^2 + 15a - 16$.
14. $x^2 - 11x + 30$.
15. $-2xy + y^2 + x^2$.
16. $4a^2 + b^2 + 4ab$.
17. $(a + b)^2 - c^2$.
18. $5x^2 + 3x^3 + x^4$.
19. $5x^5 - 3x^6 + 2x^8$.
20. $a^2 - 13a + 30$.
21. $a^2 + 13a + 30$.
22. $x^2 - (y + 3)^2$.
23. $(x - 1)^2 - y^2$.
24. $1 + 6a + 9a^2$.
25. $a^2 + 4ab + 4b^2$.
26. $x^3 + y^3$.
27. $x^3 - y^3$.
28. $1 - y^3$.
29. $y^3 - 1$.
30. $x^3 + 1$.

MISCELLANEOUS WRITTEN EXERCISES

Classify the following expressions according to the types for factoring, and find the factors:

1. $x^2 - 13x + 42$.
2. $1 - 8x^3$.
3. $x^2 + 17x + 72$.
4. $4x^2 + 9y^2 + 12xy$.
5. $4x^2 + 9y^2 - 12xy$.
6. $5x^2 + 4ax + 7xy$.
7. $2n^2 - 6nc - 3ny + 9cy$.
8. $(x + y)^2 - (y - 2x)^2$.
9. $a^3 + b^3$.
10. $9x^2 + y^4 + 6xy^2$.
11. $2y^2a^3 + 4ya^2 - 8ya$.
12. $x^2 - 15x + 36$.
13. $9x^2 + 36y^4 + 36xy^2$.
14. $9y - 9z - 2xy + 2xz$.
15. $a^3 - 1$.
16. $a^4 + b^4 + 2a^2b^2$.
17. $a^2 + 2ab + b^2 - c^4$.
18. $27a^3 - 125$.

19. $4a^2+4ab+b^2$.
20. $4a^2+9x^4-12ax^2$.
21. $1+x^3$.
22. $2x^2+5x+3$.
23. $36+4x^6+24x^3$.
24. $(x-1)^2-(x+1)^2$.
25. $8+64a^6$.
26. $ac-ax-4bc+4bx$.
27. $27-216a^3$.
28. $3^3+6^3a^3$.
29. $25(x+1)^2-4$.
30. $5cx-10c+4dx-8d$.
31. $4(x+2)^2+y^2+4(x+2)y$.
32. $ra+2rh-5sa-10sh$.
33. $-2a^2b+a^4+b^2$.
34. $2ha-hb+6a-3b$.
35. $3(a+1)^3+4(a+1)^2+a+1$.
36. $(x+a)^2-(x-a)^2$.
37. $15m^2+34m+15$.
38. $3x^2+27x+42$.
39. $x^4+49a^2+14ax^2$.
40. $27a^6-a^3x^3$.
41. $3^3a^6+a^3x^3$.
42. $8bd-40be+3cd-15ce$.
43. $x^2-x-240$.
44. $(x+2)^2-4(x-2)^2$.
45. $x^4+9y^2-6x^2y$.
46. $4a^2-7ca^2-4d^2+7cd^2$.
47. $x^2+31x+240$.
48. $18-27c+16b-24cb$.
49. $4-(a^2+b^2-2ab)$.
50. $10r+3bs-6br-5s$.
51. $25+64x^6+80x^3$.
52. $1000-x^3$.
53. 10^3+x^3.
54. $8a^3+a^3b^3+b^2a^2$.
55. $100-49x^4$.
56. $100+625+500$.
57. $a^2-17a+72$.
58. $a^2+17a+72$.
59. a^2+16b^2-8ab.
60. x^6-y^9.
61. $4a^2+23a-72$.
62. x^4+15x^2-100.
63. 9^3+8^3.
64. $9x^4+16y^2+24x^2y$.
65. $1-1000=1-10^3$.
66. $16a^2b^2+24ab+36b^3$.
67. $64+8=4^3+2^3$.
68. $16a^2b^2+9a^2c^2+24a^2bc$.
69. $a^2+4b^2+4ab-4x^2$.
70. $a^3b^6+c^3$.
71. $5x^3+10x^3y^2+30x^3y^4$.
72. $16a^2c^2+4c^2x^2+16ac^2x$.
73. $a^6y^3-z^3$.
74. x^4-7x^2-120.
75. $9a^4b^2-12a^3b+4a^2$.
76. $8ab+27ab^7$.
77. $x^4+4x^2+4-x^6$.
78. $1-125a^3b^6$.

79. $16+16ab+4a^2b^2$.
80. $64a^3+8a^2b^3$.
81. $65r^2+8r-1$.
82. $a^2-13a-140$.
83. x^6+17x^3+30.
84. $25-(a^4-2a^2b^3+b^6)$.
85. $36a^2-29ab+5b^2$.
86. $a^2-a-380$.
87. $24a^4c^4+a^6+144c^8a^2$.
88. $9x^2+4y^4-12xy^2-16$.
89. $81+100x^8-180x^4$.
90. a^4+27a^2+180.
91. a^4+3a^2-180.
92. a^4-3a^2-180.
93. $144-(a^4+b^2-2a^2b)$.
94. $81a^2b^4+49c^2-126ab^2c$.
95. $12s^2-23st+10t^2$.
96. $36x^4+12x^2y^4+y^8$.
97. $16x^6+9y^4+24x^3y^2-49$.
98. $y^2+35y+300$.
99. $5y^2-80y+300$.
100. $39x^4-16x^2+1$.
101. $ac-bc+ad-bd$.
102. $625-(31-4a^2)^2$.
103. $z^3+ya-y^3z^3-ay^4$.
104. $x^4+2x^2+1-x^2$.
105. $60x^2+7xy-y^2$.
106. $x^2-20xy+75y^2$.
107. $x^2-17x-60$.
108. $36a^2b^4+c^2b^4+12ab^4c$.
109. $4a^2+9b^4+12ab^2-16a^4$.
110. $100-(16x^2+y^6-8xy^3)$.
111. $6rd-15re+22cd-55ce$.
112. $-112a^2c^3+49a^4+64c^6$.
113. x^3-2x^2+x-2.
114. $4x^2+9y^2+z^2-12xy-4xz+6yz$.
115. $a^2b^2+a^2c^2+b^2c^2-2a^2bc+2acb^2-2abc^2$.
116. x^3+2x^2-x-2.
117. x^3+2x^2-5x+2.
118. $4a^2-12ab+4ac+9b^2+c^2-6bc$.
119. $a^6+10a^3+9a^2+25-6a^4-30a$.
120. x^3-3x^2-x+3.
121. x^3+3x^2-x-3.

CHAPTER VIII

EQUATIONS SOLVED BY FACTORING

116. Quadratic Equations. The equations solved in the preceding chapters have all been of the *first degree*, that is, they have involved the first power of the unknown and no higher power, after all possible reductions have been made.

An equation which involves the second power, but no higher power of the unknown, is called a **quadratic equation.**

E.g. $x^2 + x = 30$ is a quadratic equation.

But $x^2 + 5x + 2 = x(x+1)$ is of the first degree, since it may be reduced to $5x + 2 = x$.

117. One method of solving quadratic equations is now to be considered, namely, the **method by factoring.**

Example. Solve the equation $x^2 + (x+1)^2 = 61$.

Solution. By F, $$x^2 + x^2 + 2x + 1 = 61. \quad (1)$$

By F, S, $$2x^2 + 2x = 60. \quad (2)$$

By $D \mid 2$, $S \mid 30$ $$x^2 + x - 30 = 0. \quad (3)$$

Factoring the left member,

$$(x+6)(x-5) = 0. \quad (5)$$

This equation is satisfied by $x = 5$ since $(5+6)(5-5) = 11 \cdot 0 = 0$, and also by $x = -6$ since $(-6+6)(-6-5) = 0 \cdot (-11) = 0$.

It thus appears that equation (4) has two solutions, namely, 5 and -6.

Each of these values also satisfies equation (1). Thus, $5^2 + (5+1)^2 = 61$, and $(-6)^2 + (-6+1)^2 = 61$.

In the above solution, equation (5) has one member zero and the other member is in the *factored form.* The solution, then, consists in finding the values of x which make *either factor equal to zero,* since we know that if one of two factors is zero, then the product is zero.

ORAL EXERCISES

1. What is the product of 3 and zero? of 6 and zero? of 10 and zero? of 275 and zero?

2. If one of two factors is zero, what is the product? Does it matter what the other factor is?

3. What is the value of $k(x-1)$ if $x=1$?

4. What is the value of $(x-1)(x-5)$ if $x=1$? If $x=5$?

5. If $x=2$, what is the value of $(x-2)(x^2+4x-8)$?

6. If $x=-3$, what is the value of $(x+3)(x^3-2x+7)$?

7. Find a value of x which makes $(x-3)(x+2)$ equal to zero. Does this value of x make both factors equal to zero?

8. Find a value of x which satisfies the equation

$$(x-7)(x^2+2x-3)=0;$$

also one which satisfies $(x+8)(x^3+x+4)=0$.

Suggestion. Find a value of x which makes the first factor zero in each case.

9. Find two values of x which satisfy $(x-3)(x+4)=0$, also two which satisfy $(x+8)(x-3)=0$.

10. Find two values of x which satisfy $5x(x+7)=0$. Does $x=0$ satisfy this equation?

11. Find two values of x which satisfy $(3x-2)(2x+5)=0$.

118. Rule for Solving Equations by Factoring. The method of solution suggested by the foregoing examples consists of three steps:

(1) *Transform the equation so that all terms are collected in the left member, with similar terms united, leaving the right member zero.*

(2) *Factor the expression on the left.*

(3) *Find the values of the unknown, by setting each factor in turn equal to zero and solving.*

ORAL EXERCISES

Find two solutions of each of the following equations.

1. $(x-1)(x-2)=0$.
2. $(x-4)(x-5)=0$.
3. $(x-3)(x-7)=0$.
4. $(x+3)(x+2)=0$.
5. $(x+1)(x+2)=0$.
6. $(x+1)(x-3)=0$.
7. $(x+2)(x-5)=0$.
8. $(x-3)(x+3)=0$.
9. $(x+4)(x+5)=0$.
10. $(x+5)(x-5)=0$.
11. $(x-\frac{1}{2})(x+1)=0$.
12. $(2x-1)(x+7)=0$.
13. $(3x-1)(2x+1)=0$.
14. $(2x-1)(x+12)=0$.
15. $(x-\frac{1}{3})(x+\frac{1}{2})=0$.
16. $(4x-1)(2x+1)=0$.
17. $(5x-1)(3x-1)=0$.
18. $(2x-3)(3x-2)=0$.

WRITTEN EXERCISES

Find two solutions for each of the following equations:

1. $x^2-3x+2=0$.
2. $x^2+7x=30$.
3. $a^2-11a=-30$.
4. $a^2+13a=30$.
5. $3x+x^2=20x-72$.
6. $17x+30=-x^2-40$.
7. $7x^2+2x=30x-21$.
8. $11x+3x^2=20$.
9. $a^2+10a+8=-3a-34$.
10. $a^2+3a=10a+18$.
11. $a^2+10a=-24-4a$.
12. $2x^2-6x=-40+12x$.
13. $x^2-16=0$.
14. $x^2-1=0$.
15. $x^2-x=0$.
16. $x^2+x=0$.
17. $16-5x+x^2=-2x^2-20x-2$.
18. $4x^2=25$.
19. $x^2+5x+4=0$.
20. $x^2-10x+16=0$.
21. $x^2+12x+6=5x-4$.
22. $2x^2-7x=60+7x$.
23. $60x+4x^2+144=8x$.
24. $18x=63-x^2$.
25. $24x^2=12x+12$.
26. $2x=63-x^2$.
27. $22x+x^2=363$.
28. $3x^2+7x=6$.
29. $2x^2=2-3x$.
30. $x-2=-3x^2$.
31. $x^2-10=3x$.

119. Equations of Higher Degree than the Second. It is sometimes possible by the preceding process to solve equations in which the exponent of the highest power of the unknown is greater than 2.

Example 1. Solve the equation $x^3 + 30x = 11x^2$. (1)

By *S*, $x^3 - 11x^2 + 30x = 0$. (2)

By § 95, $x(x^2 - 11x + 30) = 0$. (3)

By § 105, $x(x-5)(x-6) = 0$. (4)

(4) is satisfied if $x = 0$, and also if $x - 5 = 0$, and if $x - 6 = 0$. Hence the roots are $x = 0$, $x = 5$, $x = 6$.

Example 2. Solve the equation $x(x+1)(x-2)(x+3) = 0$.

Any value of x which makes one of these factors zero reduces the product to zero and hence satisfies the equation. Hence the roots of the equation are

$$x = 0,\ x = -1,\ x = 2,\ x = -3.$$

Example 3. Solve the equation $x^3 - 5x^2 + 11x - 6 = 0$.

Solution. Using the remainder theorem we get,

$$x^3 - 5x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$$
$$= (x-1)(x-2)(x-3) = 0.$$

Hence the roots are $x = 1$, $x = 2$, and $x = 3$.

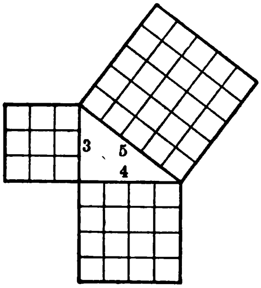
Notice that this process is applicable only when one member of the equation is *zero* and the other member can be factored.

WRITTEN EXERCISES

Solve the following equations by factoring:

1. $x^3 - x^2 = 6x$.
2. $5x = 4x^2 + x^3$.
3. $x^3 - 25x = 0$.
4. $x^3 - 3x^2 = -2x$.
5. $3x^3 = 15x^2 + 42x$.
6. $5x^3 + 315x = 80x^2$.
7. $x^2 + ax + bx + ab = 0$.
8. $x^2 - ax - bx + ab = 0$.
9. $4(x-2)^2 - (x+3)^2 = 0$.
10. $x^2 - ax + bx - ab = 0$.
11. $x^2 + ax - bx - ab = 0$.
12. $9(x+2)^2 - 4(x-3)^2 = 0$.
13. $x^3 - x - 3x^2 + 3 = 0$.
14. $x^3 - 4x - 8x^2 + 32 = 0$.
15. $x^3 - 2x^2 - x + 2 = 0$.
16. $x^3 + 2x^2 - x - 2 = 0$.
17. $x^3 - 3x^2 - x + 3 = 0$.
18. $x^3 + 3x^2 - x - 3 = 0$.

120. Pythagorean Proposition. It is proved in geometry that if squares are constructed on the two sides, and also on the hypotenuse of a right-angled triangle, then the *sum of the squares on the sides is equal to the square on the hypotenuse.*



HISTORICAL NOTE

The Pythagorean Theorem. The theorem that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides was discovered by Pythagoras and is usually called the Theorem of Pythagoras. See next page. The Egyptians had discovered that, if the sides of a triangle are 3, 4, and 5, one angle is a right angle ($3^2 + 4^2 = 5^2$), and had used this in constructing right angles. Pythagoras extended the theorem to all right-angled triangles. The Theorem of Pythagoras is one of the most important in all mathematics.

The following example illustrates the use of this theorem:

Example. The difference between the two sides of a right triangle is 2 feet, and the length of the hypotenuse is 10 feet. Find the two sides.

Solution. Let $x =$ the length of one side in feet.
Then $x + 2 =$ the length of the other side.
Hence $x^2 + (x + 2)^2 = 10^2 = 100.$ (Pythagorean theorem.)
Simplifying, $x^2 + 2x - 48 = 0.$
Factoring, $(x - 6)(x + 8) = 0.$
Hence $x = 6$ and $x = -8.$

It is clear that the length of a side of the triangle cannot be -8 feet. Hence 6 is the only one of these two results which has a meaning in this problem.

Therefore 6 and $6 + 2 = 8$ are the two sides of the triangle.

It happens frequently when a quadratic equation is used to solve a problem that one of the two numbers which satisfy the equation will not satisfy the conditions of the problem, as in this example.

PROBLEMS SOLVED BY FACTORING

In each of the following problems find all the solutions possible for the equations and then determine whether or not each solution has a reasonable interpretation in the problem.

1. The sum of the sides about the right angle of a right triangle is 35 inches, and the hypotenuse is 25 inches. Find the sides of the triangle.

2. The sum of the length and width of a rectangle is 17 rods, and the diagonal is 13 rods. Find the dimensions of the rectangle.

3. A room is 3 feet longer than it is wide, and the length of the diagonal is 15 feet. Find the dimensions of the room.

4. The length of the molding around a rectangular room is 46 feet, and the diagonal of the room is 17 feet. Find its dimensions.

5. The longest rod that can be placed flat on the bottom of a certain trunk is 45 inches. The trunk is 9 inches longer than it is wide. What are its dimensions?

6. The floor space of a rectanglar room is 180 square feet, and the length of the molding around the room is 56 feet. What are the dimensions of the room?

7. A rectangular field is 20 rods longer than it is wide, and its area is 2400 square rods. What are its dimensions?

8. A ceiling requires 24 square yards of paper, and the border is 20 yards long. What are the dimensions of the ceiling?

9. The area of a triangle is 18 square inches, and the sum of the base and altitude is 12 inches. Find the base and altitude.

10. The altitude of a triangle is 7 inches less than the base, and the area is 130 square inches. Find the base and altitude.

Pythagoras (569–500 B.C.), born on the Island of Samos, was the first of the great Greek mathematicians. He studied in Egypt, where no doubt he learned the practical geometry of the Egyptians. Later he returned to Samos to teach, but soon migrated westward to Sicily, and finally settled in the Greek colony of Croton in Southern Italy.

Here Pythagoras became the center of a widespread and influential organization, a sort of brotherhood for the moral education and purification of the community.

Pythagoras is famous for a system of Philosophy and for his studies in mathematics. The name *mathematics* and the name *philosophy* have been ascribed to him.

11. The sum of two numbers is 17, and the sum of their squares is 145. Find the numbers.

12. The difference of two numbers is 8, and the sum of their squares is 274. Find the numbers.

13. The difference of two numbers is 13, and the difference of their squares is 481. Find the numbers.

14. The sum of two numbers is 40, and the difference of their squares is 320. Find the numbers.

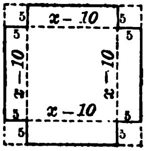
15. The sum of two numbers is 45, and their product is 450. Find the numbers.

16. The difference of two numbers is 32, and their product is 833. What are the numbers?

17. An open box is made from a piece of paper 20 inches square by cutting out a 5-inch square from each corner and turning up the sides. What is the volume of the box? What is the volume if the original square is x inches on a side?

18. An open box is made from a square piece of tin by cutting out a 5-inch square from each corner and turning up the sides. How large is the original square if the box contains 180 cubic inches?

If $x =$ length of a side of the tin, then the volume of the box is: $5(x-10)(x-10) = 180$. (See the figure.)



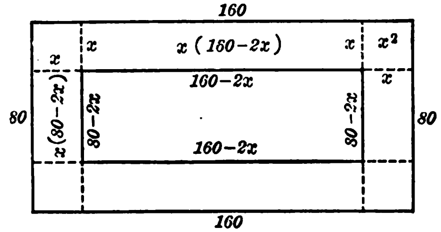
19. A rectangular piece of paper is 20 inches long and 16 inches wide. A box is made by cutting a 3-inch square out of each corner and turning up the sides. What is the volume of the box? What is the volume if the original paper is x inches wide and $x + 4$ inches long?

20. A rectangular piece of tin is 4 inches longer than it is wide. An open box containing 840 cubic inches is made by cutting a 6-inch square from each corner and turning up the ends and sides. What are the dimensions of the box?

21. The dimensions of a picture inside the frame are 8 by 10 inches. Find the area of the frame if its width is 2 inches. If its width is x inches.

22. The dimensions of a picture inside the frame are 12 by 16 inches. What is the width of the frame if its area is 288 square inches?

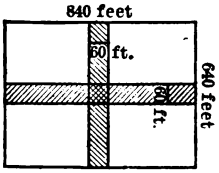
23. A farmer has a rectangular wheat field 160 rods long by 80 rods wide. In cutting the grain, he cuts a strip of equal width around the field. How many acres has he cut when the width of the strip is 8 rods?



24. How wide is the strip around the field of problem 23, if it contains $27\frac{1}{2}$ acres?

25. In the Northwest a farmer using a steam plow starts plowing around a rectangular field 640 by 320 rods. If the strip plowed the first day lacks 16 square rods of being 24 acres, how wide is it?

26. A rectangular piece of ground 840 by 640 feet is divided into 4 city blocks by two streets 60 feet wide running through it at right angles. How many square feet are contained in the streets?



27. A farmer lays out two roads through the middle of his farm, one running lengthwise of the farm and the other crosswise. How wide are the roads if the farm is 320 by 240 rods, and the area of the roads is 1671 square rods?

CHAPTER IX

COMMON FACTORS AND MULTIPLES

121. Common Factors. If an expression is a factor of each of two or more expressions, it is said to be a *common factor* of these expressions.

Thus, 8 is a common factor of 16 and 48, and 12 is a common factor of 12, 36, and 48.

If each of a given set of expressions is separated into prime factors, any common factor which they may have is at once apparent.

Illustrative Example. Find the common factors of

(1) $10(x+y)^2(x-y)$; (2) $5(x+y)(x^2-y^2)$; and (3) $15(x+y)(x^3-y^3)$.

Factoring,
$$10(x+y)^2(x-y) = 2\cdot 5(x+y)(x+y)(x-y). \quad (1)$$
$$5(x+y)(x^2-y^2) = 5(x+y)(x+y)(x-y). \quad (2)$$
$$15(x+y)(x^3-y^3) = 3\cdot 5(x+y)(x-y)(x^2+xy+y^2). \quad (3)$$

The common *prime* factors are 5, $x+y$, and $x-y$. The highest common factor is the product of these common factors, namely, $5(x+y)(x-y)$.

122. Highest Common Factor. The product of all the common prime factors of two or more expressions is called their *highest common factor.* This is usually abbreviated to H. C. F.

The name *highest*, instead of *greatest*, common factor is used in algebra referring to the number of prime factors which enter. Thus, x^2 is of *higher degree* than x, although if $x = \frac{1}{2}$, x^2 is not greater than x.

ORAL EXERCISES

Find the H. C. F. of the following:

1. $3, 6, 12.$
2. $6, 48, 24.$
3. $a, a^2 + a.$
4. $ab, ac, a^2.$
5. $3a^2, 2a^3, 6a.$
6. $x - y, x^2 - y^2.$
7. $x + y, x^3 + y^3.$
8. $x - y, x^3 - y^3.$
9. $x^2 - y^2, x^3 - y^3.$
10. $x^2 - y^2, x^3 + y^3.$
11. $x^2 - y^2, x^4 - y^4.$
12. $a - b, a^2 - 2ab + b^2.$
13. $a^2 - b^2, a^2 + 2ab + b^2.$
14. $a^3 + b^3, a^2 + 2ab + b^2.$

WRITTEN EXERCISES

Find the H. C. F. of the following:

1. $x - y,\ x^2 - y^2,\ x^2 - 2xy + y^2.$
2. $x^2 + 2x + 1,\ 3x + 6x^2 + 3x^3.$
3. $x^2 + 4x + 4,\ x^2 - 6x - 16.$
4. $x^2 - 8x + 16,\ x^2 + 10x - 56.$
5. $a^3 - b^3,\ a^2 - 2ab + b^2.$
6. $x^3 + y^3,\ x^2 - y^2,\ x^2 + 2xy + y^2.$
7. $x^2 - 7x + 12,\ ax - 3a - bx + 3b.$
8. $a^2 - 13a + 42,\ a^3 - 216,\ a^2 - a - 30.$
9. $27 + y^3,\ y^2 + 9y + 18,\ y^2 - 9.$
10. $b^2 + 7b - 30,\ b^2 + 11b - 42,\ b^2 - b - 6.$
11. $a^3 + 2a^2 + a,\ a^2 + a,\ a^3 + 5a^2 + 4a.$
12. $x^3 + y^3,\ x^3 + x^2y + xy^2 + y^3.$
13. $x^4 + 3x^3 + 2x^2,\ x^3 + x^2,\ x^4 + 7x^3 + 6x^2.$
14. $x^2 - 11x + 30,\ xz - 5z + x^2 - 5x.$
15. $m^3 - n^3,\ 2x^2m^2 + 2x^2mn + 2x^2n^2.$
16. $x^2 - 1,\ x^3 - 1,\ x^2 - 13x + 12.$
17. $1 - 64x^3,\ 1 - 16x^2,\ 5 - 2z - 20x + 8xz.$
18. $1 + 125a^3,\ 1 + 10a + 25a^2,\ 1 - 25a^2.$

Find the H. C. F. of the following:

19. $ac - ax + 3bc - 3bx, \quad a^3 + 27b^3.$
20. $5c - 2, \quad 5ac + 20c - 2a - 8.$
21. $4x^4 - x^2, \quad 2x^4 + x^3 - x^2, \quad 2x^4 - 3x^3 + x^2.$
22. $3a^3 - 3a, \quad 3a^3 - 6a^2 + 3a, \quad 6a^3 + 9a^2 - 15a.$
23. $6x - 10xy + 4xy^2, \quad 18x - 8xy^2, \quad 54x - 16xy^3.$
24. $16x^2 - 9y^2, \quad 12x^2 - 9xy, \quad 16xy - 12y^2.$
25. $36a^2 - 25b^2, \quad 18a^2 + 15ab, \quad 24ab + 20b^2.$
26. $3x^5 + 9x^4 - 3x^3, \quad 5x^2y^2 + 15xy^2 - 5y^2, \quad 7ax^2 + 21ax - 7a.$
27. $18x^3 - 57x^2 + 30x, \quad 9x^3 - 15x^2 + 6x, \quad 18x^3 - 39x^2 + 18x.$

COMMON MULTIPLES

123. Multiples. An algebraic expression is said to be a **multiple of any of its factors.** In particular, any expression is a multiple of itself and of one.

Thus, 18 is a multiple of 1, 2, 3, 6, 9, and 18, but not of 8 or of 12. $3a^2x^2$ is a multiple of 3, $3x$, $3x^2$, etc.

Since a multiple of an expression is divisible by that expression, it must contain as a factor every factor of that expression.

E.g. 108 is a multiple of 54 and contains as factors all the prime factors of 54; namely, 3, 3, 3, and 2.

124. Common Multiples. An expression is a **common multiple** of two or more expressions if it is a multiple of each of them. The **lowest common multiple** of a set of expressions is that one of their common multiples which contains the *smallest number* of prime factors. The lowest common multiple is usually abbreviated to L. C. M.

Thus, $x^2 - y^2$ is a common multiple of $x + y$ and $x - y$; also $x^4 - y^4$ is a common multiple of $x + y$ and $x - y$; but evidently $x^2 - y^2$ contains a smaller number of prime factors than $x^4 - y^4$, and hence is the *lowest common multiple.*

The process of finding the lowest common multiple of a set of expressions is shown in the following example:

Illustrative Example. Find the lowest common multiple of

(1) x^2-y^2; (2) $x^2+2xy+y^2$; and (3) $x^2-2xy+y^2$.

Factoring,
$$x^2-y^2=(x-y)(x+y). \quad (1)$$
$$x^2+2xy+y^2=(x+y)(x+y). \quad (2)$$
$$x^2-2xy+y^2=(x-y)(x-y). \quad (3)$$

In order that an expression may be a multiple of (1) it must contain the factors $x-y$ and $x+y$. To be a common multiple of (1) and (2) it must contain an additional factor $x+y$; that is, it must contain $(x-y)$, $(x+y)$, $(x+y)$.

To be a common multiple of (1), (2), and (3) it must contain an additional factor $x-y$; that is, it must contain the factors $(x-y)$, $(x+y)$, $(x+y)$, $(x-y)$.

The product $(x-y)(x+y)(x+y)(x-y)=(x-y)^2(x+y)^2$ is the lowest common multiple of (1), (2), and (3).

Rule for finding Lowest Common Multiple. *To obtain the lowest common multiple of a set of expressions:*

(1) *Find the prime factors of each expression.*

(2) *Use all factors of the first expression, together with those factors of the next expression which are not in the first, those of the third which are in neither the first nor in the second, etc.*

It is evident that in this manner we obtain a product which is a common multiple of the given expressions, but such that if any one of these factors is omitted, it will cease to be a multiple of some one of the expressions; that is, it will no longer be a common multiple of them *all*.

Thus, if in the example above either of the factors $x-y$ is omitted, the product will no longer be a multiple of $x^2-2xy+y^2$.

ORAL EXERCISES

Find the L. C. M. of the following sets of expressions:

1. 3, 5.
2. 3, 4, 6.
3. 6, 48, 24.
4. a, b.
5. a, bk.
6. ab, bc, ac.
7. $3a$, $2b$, $4c$.
8. $a+1$, $a-1$.
9. $b+2$, $b+3$.
10. $1-a$, $1-2a$.
11. $2x+3$, $x-4$.
12. $m+3$, $m-5$.
13. ax^2, b^2x, a^2b^2.
14. $x+4$, $x-1$.
15. xyz, yzv, $2vx$.

WRITTEN EXERCISES

Find the L. C. M. of the following expressions:

1. $2 \cdot 3 \cdot 4$; $3 \cdot 7 \cdot 8$; $2^3 \cdot 3 \cdot 4$.
2. $5x^2y^4$, $10x^3y$, $25x^2y$.
3. $2ab$, $6a^2$, $4b^2c$.
4. $x^2 - y^2$, $x^2 - 2xy + y^2$.
5. $x - y$, $x + y$, $x^2 - y^2$.
6. $4 - x^2$, $2 - x$, $2 + x$.
7. $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$.
8. $x^2 + 3x + 2$, $x^2 - 4$, $x^2 - 1$.
9. $25x^2 - 1$, $125x^3 - 1$.
10. $2x^2 - 7x + 6$, $4x^2 - 11x + 6$.
11. $x^3 - y^3$, $x - y$, $x^2 + xy + y^2$.
12. $x^3 - y^3$, $x^3 + y^3$, $x^2 - y^2$.
13. $5x^2 + 7x - 6$, $x^2 - 15x - 34$.
14. $x^3 + y^3$, $x^2 - y^2$, $(x - y)^2$.
15. $3abc$, $a^2 - 4ac + 4c^2$, $a - 2c$.
16. $x^2 - 1$, $x + 1$, $x^2 + 8x + 7$.
17. $4x^3y - 44x^2y + 120xy$, $3a^3x^2 - 22a^3x + 35a^3$.
18. $x^2 + 2xy + y^2$, $2ax^2 - 10ax + 12a$.
19. $3bx^2 - 21bx + 36b$, $x^2 - 5x + 4$.
20. $5a^2b^2 - 5a^2c^2$, $b^2 + 2bc + b + c + c^2$.
21. $15c^2ax^2 + 16c^2ax + c^2a$, $2cax^2 + 10cax + 8ca$.
22. $x^3 - 6x + 4$, $x - 4$, $x^2 - 16$.

Solution. By means of the remainder theorem, we find,

$$x^3 - 6x + 4 = (x - 2)(x^2 + 2x - 2).$$

Hence the L. C. M. is

$$(x - 2)(x + 4)(x - 4)(x^2 + 2x - 2).$$

23. $x^3 - 6x^2 + 5$, $x + 3$, $x^2 - 1$.
24. $x^2 + 5x - 1$, $x^2 - 6x + 3$, $x - 1$.
25. $x^2 - 5x + 6$, $x^2 - 7x + 12$, $x^2 - 9x + 20$.

CHAPTER X

ALGEBRAIC FRACTIONS

125. Fractions. In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; *i.e.* $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

In algebra a fraction is usually regarded as an **indicated division** in which the numerator is the dividend and the denominator is the divisor.

Thus, $\frac{a}{b}$ is understood to mean $a \div b$.

Terms of a Fraction. The numerator and denominator are together called the *terms of the fraction.*

Operations on algebraic fractions are performed in accordance with the same rules as in arithmetic.

For example, just as $\frac{2}{3} = \frac{4}{6}$, so $\frac{a}{b} = \frac{2a}{2b}$, $\frac{m}{n} = \frac{mx}{nx}$,

and $$\frac{a+b}{c+d} = \frac{(x-y)(a+b)}{(x-y)(c+d)}.$$

ORAL EXERCISES

Supply the missing numerator or denominator in each of the following:

1. $\frac{1}{2} = \frac{\ }{4}$.

2. $\frac{3}{4} = \frac{\ }{12}$.

3. $\frac{1}{a} = \frac{\ }{ka}$.

4. $\frac{a}{b} = \frac{\ }{cb}$.

5. $\frac{m}{n} = \frac{\ }{3n}$.

6. $\frac{1}{a} = \frac{\ }{-2a}$.

7. $\frac{6}{9}=\frac{2}{\quad}.$

8. $\frac{x^2}{xy}=\frac{\quad}{y}.$

9. $\frac{ax+ay}{a^3}=\frac{\quad}{a^2}.$

10. $\frac{15}{20}=\frac{\quad}{4}.$

11. $\frac{kx}{xy}=\frac{\quad}{y}.$

12. $\frac{x^3+y^3}{a(x+y)}=\frac{\quad}{a^2}.$

13. $\frac{x^2-y^2}{a(x-y)}=\frac{\quad}{a}.$

14. $\frac{1}{a-1}=\frac{\quad}{1-a}.$

15. $\frac{1}{a+1}=\frac{\quad}{(a+1)(a-1)}.$

16. $\frac{x^2+3x+2}{x^2-1}=\frac{x+2}{\quad}.$

17. $\frac{x^3-y^3}{x^2-y^2}=\frac{\quad}{x+y}.$

18. $\frac{1}{b+2}=\frac{\quad}{(b+3)(b+2)}.$

19. $\frac{x^2-y^2}{(x+y)^2}=\frac{\quad}{x+y}.$

20. $\frac{x^2-y^2}{(x-y)^2}=\frac{\quad}{x-y}.$

The preceding examples illustrate

Principle XVII

126. Rule. *Both terms of a fraction may be multiplied or divided by the same number without changing its value.*

HISTORICAL NOTE

Fractional Notation. Our present method of representing fractions is the result of a long historical development. The **Babylonians** used 60ths only. Thus $\frac{1}{2}$ was called 30 sixtieths, $\frac{1}{3}$ was called 20 sixtieths, etc.

The **Egyptians** used only fractions with a numerator 1. The denominator only was written, a dot over it indicating that it represented a fraction. Thus $\dot{3}$ would mean $\frac{1}{3}$. The use of "unit" fractions only necessitated the reduction of other fractions to this form. Thus $\frac{2}{5}=\frac{1}{3}+\frac{1}{15}$, and $\frac{2}{7}=\frac{1}{4}+\frac{1}{28}$.

The **Greeks** used one accent to indicate the numerator and two to indicate the denominator. Thus $2'\ 3''$ meant $\frac{2}{3}$.

The **Romans** used a duodecimal system of fractions. That is, all their fractions were twelfths. The Hindus wrote $\frac{2}{3}$ in the form $\begin{smallmatrix}2\\3\end{smallmatrix}$, not using the bar. Alkarismi (see page 101) used the bar, writing the fraction in its present form. Decimal fractions came in much later than the common fractions. Simon Stevinus of Bruges in Belgium (1548–1620) was the first to treat decimal fractions systematically.

REDUCTION OF FRACTIONS TO LOWEST TERMS

127. By Principle XVII any factor common to the numerator and denominator of a fraction may be cancelled. That is

$$\frac{ak}{bk}=\frac{a}{b}.$$

Thus,

$$\frac{2\cdot\cancel{3}\cdot 4\cdot 5}{\cancel{3}\cdot 7\cdot 11}=\frac{2\cdot 4\cdot 5}{7\cdot 11};\qquad \frac{\cancel{2^3}\cdot\overset{3}{\cancel{3^2}}\cdot\cancel{4}\,x}{\underset{2}{\cancel{2^4}}\cdot\cancel{3}\cdot\underset{4}{\cancel{4^2}}}=\frac{3\,x}{2\cdot 4}=\frac{3\,x}{8};$$

$$\frac{x^2-7\,x+12}{x^2-5\,x+6}=\frac{\cancel{(x-3)}(x-4)}{(x-2)\cancel{(x-3)}}=\frac{x-4}{x-2}.$$

Lowest Terms. If the terms of a fraction have no common factor, the fraction is said to be reduced to its *lowest terms.*

ORAL EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{4}{6}$.
2. $\frac{14}{21}$.
3. $\dfrac{4}{2\,x+4}$.
4. $\dfrac{x}{x^3-3\,x^2+x}$.
5. $\dfrac{x-1}{(x-3)(x-1)}$.
6. $\dfrac{x+2}{x^2+3\,x+2}$.
7. $\dfrac{x-2}{x^2-3\,x+2}$.
8. $\dfrac{x+1}{x^2+3\,x+2}$.
9. $\dfrac{x-y}{x^2-y^2}$.
10. $\dfrac{x-y}{x^2-2\,xy+y^2}$.
11. $\dfrac{x+y}{x^2+2\,xy+y^2}$.
12. $\dfrac{(x+y)^2}{(x+y)^3}$.
13. $\dfrac{x+3}{x^2+5\,x+6}$.
14. $\dfrac{x-5}{x^2-4\,x-5}$.
15. $\dfrac{x-4}{x^2-7\,x+12}$.
16. $\dfrac{x-1}{x^2-5\,x+4}$.
17. $\dfrac{x+1}{x^2+5\,x+4}$.
18. $\dfrac{x+1}{x^2+4\,x+3}$.
19. $\dfrac{x+3}{x^2+4\,x+3}$.
20. $\dfrac{x-2}{x^2-5\,x+6}$.
21. $\dfrac{x+2}{x^2+5\,x+6}$.
22. $\dfrac{x+2}{x^2+6\,x+8}$.
23. $\dfrac{x-2}{x^2-6\,x+8}$.
24. $\dfrac{x+3}{x^2+5\,x+6}$.
25. $\dfrac{x-3}{x^2-5\,x+6}$.
26. $\dfrac{x+3}{x^2+8\,x+15}$.
27. $\dfrac{x-3}{x^2-8\,x+15}$.

WRITTEN EXERCISES

Reduce the following fractions to lowest terms:

1. $\frac{3 \cdot 9^2 \cdot 2^6}{2^4 \cdot 5^3 \cdot 9^4}.$

2. $\frac{4\,a^4b^{12}c^3}{8\,a^3b^4c^4}.$

3. $\frac{x^2y^3z^4}{xy^2z^3}.$

4. $\frac{a^4b^3}{a^2b^2}.$

5. $\frac{x^2+2\,xy+y^2}{x^2-y^2}.$

6. $\frac{x^2+7\,x-30}{x^2-7\,x+12}.$

7. $\frac{x^3-y^3}{2\,x^2-3\,xy+y^2}.$

8. $\frac{64-b^3}{16-8\,b+b^2}.$

9. $\frac{x^3+27\,z^3}{xy-5\,x+3\,yz-15\,z}.$

10. $\frac{1-216\,c^3}{x-4\,y-6\,cx+24\,cy}.$

11. $\frac{14\,bz-2\,bx+ax-7\,az}{x^2-49\,z^2}.$

12. $\frac{3\,a^2-29\,a+56}{63-9\,a-7\,m+ma}.$

13. $\frac{a(x-y)^3}{(x^2-y^2)(x-y)}.$

14. $\frac{x^3+27}{4\,x^2+24\,x+36}.$

15. $\frac{a^2-3\,a-3\,b+ab}{(a^2-b^2)(a-3)}.$

16. $\frac{5\,x^3y^4-12\,x^2y^5+7\,x^3y^2}{6\,x^5y^2+3\,x^2y^2}.$

17. $\frac{5\,c+10\,b-bc-2\,b^2}{8\,c^3+64\,b^3}.$

18. $\frac{4\,x^4-28\,x^3+48\,x^2}{2\,x^4-8\,x^3+6\,x^2}.$

19. $\frac{9\,a^2b^4+18\,a^2b^3c+9\,a^2b^2c^2}{3\,ab^3-3\,abc^2}.$

20. $\frac{7\,xy^2-133\,xy+126\,x}{15\,xy^2-36\,xy+21\,x}.$

21. $\frac{20\,x^3+20\,x^2y+5\,xy^2}{60\,x^5-15\,x^3y^2}.$

22. $\frac{3\,ab^4-3\,ab^2c^2}{27\,a^3b^2+27\,a^3bc}.$

23. $\frac{4\,a^3-42\,a^2+20\,a}{2\,a^4b^6-20\,a^3b^6}.$

24. $\frac{(x-1)(x-2)(x-3)(x-4)}{(x-1)(x-3)(x-3)(x-4)}.$

25. $\frac{(x^2-y^2)(x^2+2\,xy+y^2)}{(x^2-2\,xy+y^2)(x+y)}.$

26. $\frac{(x^2-1)(x^2+1)(3\,x^2+3)}{3(x^4-1)}.$

27. $\frac{(x^2-4)(x^2+4)(x+5)}{(x-2)(x^2+3\,x-10)(x^4-16)}.$

28. $\frac{(2\,x^2+3\,x-2)(2\,x^2+7\,x+6)}{(4\,x^2+4\,x-3)(x^2+4\,x+4)}.$

29. $\frac{(2\,x^2-3\,x+1)(2\,x^2+3\,x+1)}{(x^2-1)(4\,x^2-1)}.$

30. $\frac{(2\,x^2+5\,x+2)(3\,x^2+10\,x+3)}{(3\,x^2+7\,x+2)(2\,x^2+7\,x+3)}.$

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR

128. Illustrative Examples. 1. Reduce $\frac{1}{2}$ and $\frac{1}{3}$ to a common denominator.

Solution. $\frac{1}{2}=\frac{3}{6}$; $\frac{1}{3}=\frac{2}{6}$. Is 6 a common multiple of 2 and 3? Is it the least common multiple of 2 and 3?

2. Reduce $\frac{1}{a}$ and $\frac{1}{b}$ to a common denominator.

Solution. $\frac{1}{a}=\frac{b}{ab}$; $\frac{1}{b}=\frac{a}{ab}$.

How is the common denominator related to the denominators a and b?

3. Reduce $\frac{2}{a+1}$, $\frac{3}{(a+1)(a+2)}$, $\frac{4}{a+2}$ to a common denominator.

Solution. The required denominator is the L.C.M. of the given denominators; that is, $(a+1)(a+2)$.

Hence, $\frac{2}{a+1}=\frac{2(a+2)}{(a+1)(a+2)}$ and $\frac{4}{a+2}=\frac{4(a+1)}{(a+1)(a+2)}$.

$\frac{3}{(a+1)(a+2)}$ already has the required denominator.

ORAL EXERCISES

Reduce each of the following to a common denominator:

1. $\frac{1}{a}, \frac{1}{b}$.
2. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.
3. $\frac{1}{a}, \frac{1}{-a}$.
4. $\frac{1}{ab}, \frac{1}{ac}$.
5. $\frac{a}{bc}, \frac{b}{ac}$.
6. $\frac{a}{xy}, \frac{b}{xz}$.
7. $\frac{1}{ab}, \frac{1}{bc}, \frac{1}{ac}$.
8. $\frac{a}{bc}, \frac{b}{ac}, \frac{c}{ab}$.
9. $\frac{m}{xy}, \frac{n}{xz}, \frac{s}{yz}$.
10. $\frac{1}{x+1}, \frac{1}{x+2}$.
11. $\frac{1}{a-3}, \frac{1}{a+4}$.
12. $\frac{a}{a-b}, \frac{c}{c-d}$.
13. $\frac{1}{1-k}, \frac{1}{k-1}$.
14. $\frac{1}{x+2}, \frac{1}{x-2}$.
15. $\frac{a}{a-b}, \frac{b}{a+b}$.

129. By the formula $\frac{a}{b}=\frac{ak}{bk}$, Principle XVII, both terms of a fraction may be multiplied by the same number.

In this manner any fraction may be changed into an equal fraction whose denominator is any given multiple of the denominator of the given fraction.

E.g. $$\frac{3}{4}=\frac{3\cdot 5}{4\cdot 5},\quad \frac{a-b}{a+b}=\frac{(a-b)(a+b)}{(a+b)(a+b)}=\frac{a^2-b^2}{(a+b)^2}.$$

Any two or more fractions may therefore be changed into equal fractions which shall have a *common denominator*, namely, a common multiple of the denominators of the given fractions.

Illustrative Example. Reduce

$$(1)\ \frac{x-1}{x+1};\quad (2)\ \frac{x+1}{x-1};\ \text{and}\ (3)\ \frac{2x+3}{x^2-1}$$

to fractions having a common denominator.

The L. C. M. of the denominators is $(x-1)(x+1)$. Multiply the numerator and denominator of each fraction by an expression which will make the denominator of each new fraction $(x-1)(x+1)$.

Thus, $$\frac{x-1}{x+1}=\frac{(x-1)(x-1)}{(x+1)(x-1)}=\frac{x^2-2x+1}{(x+1)(x-1)};\qquad (1)$$

$$\frac{x+1}{x-1}=\frac{(x+1)(x+1)}{(x-1)(x+1)}=\frac{(x+1)^2}{(x+1)(x-1)};\qquad (2)$$

$$\frac{2x+3}{x^2-1}=\frac{2x+3}{(x+1)(x-1)}.\qquad (3)$$

It is best to *indicate* the multiplication in the common denominator, since this makes it more easily apparent by what expression the terms of a fraction must be multiplied in order to reduce it to a fraction with the required denominator.

ORAL EXERCISES

Supply the missing numerators:

1. $\frac{1}{x}=\frac{\quad}{x^2(x-1)}$.

2. $\frac{x}{x+1}=\frac{\quad}{x^2+8x+7}$.

3. $\frac{3}{x+5}=\frac{\quad}{x^2+7x+10}$.

4. $\frac{2}{x-5}=\frac{\quad}{x^2-2x-15}$.

130. The Three Signs of a Fraction. There are three signs in connection with a fraction: (1) the sign of the fraction itself, (2) the sign of the numerator, and (3) the sign of the denominator. Any two of these signs may be changed simultaneously without changing the value of the fraction.

Thus, (1) $\frac{a}{b}=\frac{-a}{-b}$. (2) $\frac{a}{b}=-\frac{a}{-b}$. (3) $\frac{a}{b}=-\frac{-a}{b}$.

If either the numerator or the denominator is a polynomial, its sign can be changed only by changing the sign of each term in it.

E.g. $\frac{a+b}{c-d}=\frac{-a-b}{-c+d}=-\frac{a+b}{-c+d}$.

If either the numerator or the denominator is in the factored form, its sign can be changed only by changing the sign of an *odd* number of its factors.

E.g. $\frac{(x+y)(x-y)}{(a+b)(a-b)}=\frac{(x+y)(y-x)}{(a+b)(b-a)}$.

This is useful in examples like the following:

Example. Reduce $\frac{x+1}{1-x}$, $\frac{x}{x^2-1}$, and $\frac{1}{x+1}$ to fractions having a common denominator.

$$\frac{x+1}{1-x}=\frac{-x-1}{x-1}=\frac{(x+1)(-x-1)}{(x+1)(x-1)}=\frac{-x^2-2x-1}{x^2-1},$$
$$\frac{x}{x^2-1}=\frac{x}{x^2-1}, \text{ and } \frac{1}{x+1}=\frac{x-1}{(x+1)(x-1)}=\frac{x-1}{x^2-1}.$$

ORAL EXERCISES

Apply the sign changes shown in (1), (2), and (3) above to each of the following fractions:

(1) $\frac{a}{b-c}$.

(2) $\frac{a-b}{c-d}$.

(3) $\frac{-a}{c+d}$.

(4) $\frac{(a-b)(c-d)}{-x}$.

(5) $\frac{1}{(1-x)(x+2)}$.

(6) $-\frac{b-a}{d-c}$.

(7) $-\frac{-c}{(d-a)(b-a)}$.

(8) $\frac{-(a-b)(b-c)}{(c-d)(d-a)}$.

(9) $-\frac{(-a)(-b)(-c)}{d(-e)(-f)}$.

WRITTEN EXERCISES

Reduce each of the following sets of fractions to equivalent fractions having a common denominator:

1. $\frac{x+3}{x-y}, \frac{4}{x^2-2xy+y^2}.$
2. $\frac{a-1}{a^2-b^2}, \frac{a+1}{a^2+2ab+b^2}.$
3. $\frac{3-x}{x^2-9x+20}, \frac{x+4}{7x^2-26x-8}.$
4. $\frac{2}{3-x}, \frac{x-1}{x+1}, \frac{x+1}{x-3}.$
5. $\frac{a+b}{b-a}, \frac{a-b}{(a+b)^2}, \frac{a}{a^2-b^2}.$
6. $\frac{1}{x^2-3x-4}, \frac{1}{x^2+3x+2}.$
7. $\frac{a+1}{a^2-2ab+b^2}, \frac{a-1}{a^2+2ab+b^2}.$
8. $\frac{1}{a^3-b^3}, \frac{1}{b-a}, \frac{1}{a^2+ab+b^2}.$
9. $\frac{a}{5a^2-4a-12}, \frac{b}{a^2+4a-12}, \frac{c}{a-2}.$
10. $\frac{mr}{m-1}, \frac{d}{1+n}.$
11. $\frac{Rr}{(R+r)(m-1)}, \frac{1}{R+r}.$
12. $\frac{a}{n-a}, \frac{b}{n-b}.$
13. $\frac{W(T-Q)}{w(Q-t)}, \frac{V}{w}.$
14. $\frac{V}{V-v}, \frac{V}{V+v}, \frac{1}{V^2-v^2}.$
15. $\frac{RA}{a-A}, \frac{1}{R+A}, \frac{1}{A-a}.$
16. $\frac{R}{x}, \frac{r}{y}, \frac{1}{x-y}, \frac{Rx}{x+y}.$

131. Reducing Integral Expressions to the Fractional Form. Since any number may be written as a fraction with the denominator 1, the above process may be used to reduce an integral expression to the form of a fraction having any desired denominator.

Thus, $3 = \frac{3 \cdot 5}{5}$; $x - y = \frac{(x-y)(x^2-1)}{x^2-1}$, etc.

Illustrative Example. Reduce $5x$, $\frac{5x-1}{x^2-1}$, and $\frac{2x-y}{x-1}$ to fractions having a common denominator. The lowest common denominator is x^2-1.

Thus, $5x = \frac{5x(x^2-1)}{x^2-1} = \frac{5x^3-5x}{x^2-1}$, $\frac{5x-1}{x^2-1} = \frac{5x-1}{x^2-1}$, and

$$\frac{2x-y}{x-1} = \frac{(2x-y)(x+1)}{(x-1)(x+1)} = \frac{2x^2+2x-xy-y}{x^2-1}.$$

WRITTEN EXERCISES

Reduce the following expressions to fractions having a common denominator:

1. $5,\ x - y,\ \dfrac{5x-3}{x^2+2xy+y^2}.$ 6. $x^4+x^2y^2+y^4,\ \dfrac{x}{x-y},\ \dfrac{y}{x+y}.$

2. $\dfrac{3a-c}{x-y},\ \dfrac{2b-c}{x+y},\ 2c+2.$ 7. $3a-2b-c,\ \dfrac{5}{a-b},\ \dfrac{2}{b-c}.$

3. $1+a+a^2,\ \dfrac{a+1}{a-1}.$ 8. $x^4-1,\ x^2-1,\ \dfrac{x+1}{x-1}.$

4. $x^2+xy+y^2,\ \dfrac{x+y}{x-y}.$ 9. $x^2+2xy+y^2,\ \dfrac{1}{x+y},\ \dfrac{1}{x-y}.$

5. $x^2-xy+y^2,\ x^2-y^2,\ \dfrac{1}{x+y}.$ 10. $x+y,\ x-y,\ \dfrac{x-y}{x^2+y^2},\ \dfrac{x+1}{x-y}.$

11. $\dfrac{RA}{a-A},\ r.$ 12. $\dfrac{RA}{a-A},\ R-r.$ 13. $T,\ \dfrac{Q+t}{2}.$

ADDITION AND SUBTRACTION OF FRACTIONS

132. Rule for Adding or Subtracting Fractions. By Principle III, § 16, we know that

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ and } \frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}.$$

Reading this in the reverse order we have the formula for adding and subtracting fractions:

$$\boldsymbol{\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}} \text{ and } \boldsymbol{\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}}.$$

From this formula we get the following rule.

(1) *Reduce the fractions to a common denominator.*

(2) *To add the fractions, add the numerators and place the sum over the common denominator.*

(3) *To subtract the fractions, subtract the numerators and place the remainder over the common denominator.*

(4) *Reduce the resulting fraction in each case to its lowest terms.*

Example. Add $\frac{1}{a+1}$ and $\frac{2}{a+3}$.

Solution. $\frac{1}{a+1}=\frac{a+3}{(a+1)(a+3)}$; $\frac{2}{a+3}=\frac{2(a+1)}{(a+1)(a+3)}$.

Hence, $\frac{1}{a+1}+\frac{2}{a+3}=\frac{a+3}{(a+1)(a+3)}+\frac{2(a+1)}{(a+1)(a+3)}$

$$=\frac{a+3+2(a+1)}{(a+1)(a+3)}=\frac{3a+5}{(a+1)(a+3)}.$$

ORAL EXERCISES

Perform the following additions:

1. $\frac{1}{a}+\frac{1}{b}$.
2. $\frac{1}{ab}+\frac{1}{ac}$.
3. $\frac{1}{a+1}+\frac{1}{a-1}$.
4. $\frac{1}{x+1}+\frac{1}{x+2}$.
5. $\frac{2}{a}+\frac{1}{-a}$.
6. $\frac{1}{1-k}+\frac{1}{k-1}$.
7. $\frac{1}{n-4}+\frac{1}{n-1}$.
8. $\frac{1}{x+2}+\frac{1}{x-2}$.

Example. From $\frac{3}{x-4}$ subtract $\frac{2}{x+3}$.

Solution.

$$\frac{3}{x-4}-\frac{2}{x+3}=\frac{3(x+3)}{(x-4)(x+3)}-\frac{2(x-4)}{(x-4)(x+3)}$$

$$=\frac{3(x+3)-2(x-4)}{(x-4)(x+3)}=\frac{x+17}{(x-4)(x+3)}.$$

ORAL EXERCISES

Perform the following subtractions:

1. $\frac{1}{a}-\frac{1}{b}$.
2. $\frac{1}{ab}-\frac{1}{ac}$.
3. $\frac{1}{a+1}-\frac{1}{a-1}$.
4. $\frac{1}{x+1}-\frac{1}{x+2}$.
5. $\frac{2}{-a}-\frac{1}{a}$.
6. $\frac{1}{1-k}-\frac{1}{k-1}$.

Example. Add $\dfrac{a-b}{a+b}$ and $\dfrac{a^2+2ab+b^2}{a^2-2ab+b^2}$.

Solution. The L. C. M. of the denominators is

$$(a-b)(a-b)(a+b).$$

$$\frac{a-b}{a+b}=\frac{(a-b)(a-b)(a-b)}{(a+b)(a-b)(a-b)}=\frac{a^3-3a^2b+3ab^2-b^3}{(a+b)(a-b)(a-b)},$$

and $$\frac{a^2+2ab+b^2}{a^2-2ab+b^2}=\frac{(a+b)(a^2+2ab+b^2)}{(a+b)(a-b)(a-b)}=\frac{a^3+3a^2b+3ab^2+b^3}{(a+b)(a-b)(a-b)}.$$

Adding the numerators, we have $2a^3+6ab^2$; whence the sum of the fractions is

$$\frac{2a^3+6ab^2}{(a+b)(a-b)(a-b)}.$$

WRITTEN EXERCISES

Perform the following additions and subtractions:

1. $\dfrac{1}{x-2}-\dfrac{1}{x+1}$.
2. $\dfrac{1}{a+4}-\dfrac{2}{a-3}$.
3. $\dfrac{1}{x-1}-\dfrac{1}{1-x}$.
4. $\dfrac{1}{x}-\dfrac{1}{y}$.
5. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$.
6. $\dfrac{3}{k-1}-\dfrac{2}{k+1}$.
7. $\dfrac{4}{c-2}-\dfrac{3}{(c-1)(c-2)}$.
8. $\dfrac{1}{a}-\dfrac{1}{b}+\dfrac{1}{c}$.
9. $\dfrac{1}{ab}+\dfrac{1}{bc}-\dfrac{1}{abc}$.
10. $\dfrac{3}{a+b}-\dfrac{2}{a-b}+\dfrac{1}{a}$.
11. $\dfrac{a}{1-a}+\dfrac{1}{a-1}+\dfrac{1}{a}$.
12. $\dfrac{3}{x-y}-\dfrac{1}{y}+\dfrac{1}{x}$.
13. $\dfrac{x}{x-y}+\dfrac{x}{y}-\dfrac{1}{y}$.
14. $\dfrac{1}{x}+\dfrac{a}{y}-\dfrac{b}{z}$.
15. $\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}$.
16. $\dfrac{a}{(x-1)(x+3)}-\dfrac{b}{x+3}$.
17. $\dfrac{1}{x+y}+\dfrac{1}{x-y}+\dfrac{1}{(x+y)(x-y)}$.
18. $\dfrac{1}{a+1}+\dfrac{1}{a-1}-\dfrac{1}{(a+1)(a-1)}$.
19. $\dfrac{3}{4}+\dfrac{2}{7}+\dfrac{a+b}{3a-b}$.
20. $\dfrac{x-y}{(x+y)^2}-\dfrac{x+y}{x^2-y^2}$.

21. $\dfrac{x^2-9x+18}{x^2-13x+36}+\dfrac{x}{4-x}.$

22. $\dfrac{2}{3}+\dfrac{a}{a+b}+\dfrac{b}{a-b}.$

23. $\dfrac{3}{2^3\cdot 3^2}+\dfrac{5}{2^2\cdot 3^4}-\dfrac{2}{2^4\cdot 3^3}.$

24. $\dfrac{a^2-9b^2}{a^2+6ab+9b^2}-\dfrac{a^2-6ab}{a^2-9b^2}.$

25. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}-2.$

26. $\dfrac{2x^2}{x^2-y^2}-\dfrac{y}{x-y}-\dfrac{y}{x+y}.$

27. $\dfrac{a+1}{a^2+a+1}-\dfrac{a-1}{a^2-a+1}.$

28. $\dfrac{b}{1-b}+\dfrac{b}{1+b}-\dfrac{b^2}{1-b^2}.$

29. $\dfrac{x+1}{x-2}+\dfrac{x+1}{x+2}+\dfrac{3x+2}{x^2-4}.$

30. $\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}+\dfrac{x^2-5}{x^2-1}.$

31. $\dfrac{y^2}{y^2-1}+\dfrac{y}{y+1}-\dfrac{y}{1-y}.$

32. $\dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{x-y}+\dfrac{-1}{x+y}.$

33. $\dfrac{1}{2}-\dfrac{b}{b-a}+\dfrac{2a^2}{b^2-a^2}+\dfrac{-a}{b+a}.$

34. $\dfrac{x^2+4xy}{x^3+y^3}+\dfrac{1}{x+y}-\dfrac{x}{x^2-xy+y^2}.$

35. $\dfrac{a-3}{a^2-3a+2}-\dfrac{a-1}{a^2-5a+6}+\dfrac{a}{a^2-4a+3}.$

36. $\dfrac{x}{x^2-5x-14}+\dfrac{2}{x-7}-\dfrac{x}{x^2-9x+14}.$

37. $\dfrac{a}{ac+ad-bc-bd}-\dfrac{b}{a^2-2ab+b^2}.$

38. $\dfrac{1}{y^2+8y+16}-\dfrac{1}{y(y+4)}+\dfrac{4}{y^2(y+4)}.$

39. $\dfrac{a+2}{a^2-a-6}+\dfrac{a-4}{a^2-7a+12}-\dfrac{a+2}{a^2-2a-8}.$

40. $\dfrac{2}{x^2-11x+30}-\dfrac{1}{x^2-36}+\dfrac{1}{x^2-25}.$

41. $\dfrac{1}{(x-1)(x+2)}+\dfrac{1}{(x+2)(x-3)}+\dfrac{1}{(x-1)(3-x)}.$

42. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-a)(c-d)}+\dfrac{1}{(b-c)(c-d)}.$

43. $\dfrac{4}{a-3}-\dfrac{a-1}{a^2+3a+9}+\dfrac{a^2-11a-3}{a^3-27}.$

MULTIPLICATION OF FRACTIONS

133. Fractions are multiplied as in arithmetic.

Thus $$\frac{2}{3} \times \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}.$$

Similarly, $$\frac{a}{b} \times \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}.$$

That is, we have the following

Rule. *The product of two algebraic fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

134. Steps in Multiplying Fractions. The steps in multiplying one fractional expression by another are:

(1) *Reduce mixed expressions to fractions.*

(2) *Factor each numerator and each denominator.*

(3) *Cancel all factors common to numerators and denominators.*

(4) *Multiply the remaining factors in the numerators for the numerator of the product, and those of the denominators for the denominator of the product.*

These steps are illustrated in the following examples.

(1) $$\frac{3}{4} \times \frac{8}{9} = \frac{\cancel{3} \cdot \overset{2}{\cancel{8}}}{\cancel{4} \cdot \underset{3}{\cancel{9}}} = \frac{2}{3};$$ (2) $$1\tfrac{2}{3} \times 2\tfrac{2}{5} = \frac{5}{3} \times \frac{12}{5} = \frac{\cancel{5} \cdot \overset{4}{\cancel{12}}}{\cancel{3} \cdot \cancel{5}} = 4.$$

(3) $$\left(1 + \frac{a}{b}\right) \times \frac{b^2}{b+a} = \frac{b+a}{b} \times \frac{b^2}{b+a} = \frac{\overset{b}{\cancel{b^2}}\cancel{(b+a)}}{\cancel{b}\cancel{(b+a)}} = b.$$

(4) $$\frac{4x+12}{x^2+x-2} \times \frac{3x-3}{4x^2+4x-24} = \frac{\cancel{4}\cancel{(x+3)} \cdot 3\cancel{(x-1)}}{\cancel{(x-1)}(x+2) \cdot \cancel{4}\cancel{(x+3)}(x-2)}$$
$$= \frac{3}{(x+2)(x-2)} = \frac{3}{x^2-4}.$$

ORAL EXERCISES

Perform the following multiplications:

1. $\frac{ab}{c} \times \frac{c}{a}$.

2. $\frac{5\,ab}{2\,cx} \times \frac{4\,c^2x^2}{10\,a}$.

3. $\frac{abc}{xy} \times \frac{bx}{ac}$.

4. $\frac{cx^2}{by^2} \times \frac{cy}{dx}$.

5. $\frac{mxy}{nyc} \times \frac{ny}{mx}$.

6. $\frac{rst}{xy} \times \frac{xy^2}{r^2s}$.

7. $(a+b) \times \frac{1}{a^2-b^2} = \frac{\cancel{a+b}}{(\cancel{a+b})(a-b)} = \frac{1}{a-b}$.

8. $(a-b) \times \frac{1}{a^2-b^2}$.

9. $\frac{a-b}{a+b} \times \frac{a^2-b^2}{a-b}$.

10. $(a-1) \times \frac{1}{a^3-1}$.

11. $(x+y) \times \frac{1}{x^3+y^3}$.

12. $\frac{x+y}{x-y} \times \frac{x-y}{x^3+y^3}$.

13. $\left(1+\frac{1}{x}\right) \times \frac{a}{x+1}$.

14. $\frac{2}{3(n+1)} \times 6(n+1)$.

15. $\frac{2}{(a-2)(a+3)} \times 3(a-2)$.

WRITTEN EXERCISES

Find the following indicated products and reduce the fractions to simplest forms:

1. $(1-a) \times \frac{1+a+a^2}{a-1}$.

2. $(x^3-y^3) \times \frac{x+y}{x-y}$.

3. $(x^2-2xa+a^2) \times \frac{x+a}{x-a}$.

4. $\frac{3\,x-1}{x^2-5\,x+6} \times (x^2-11\,x+18)$.

5. $\frac{a^2-4\,a+3}{a^2-8\,a+16} \times (a^2-5\,a+4)$.

6. $(x^2+9\,x+18) \times \frac{x-5}{x^2-2\,x-15}$.

7. $(1-x^3) \times \frac{1-x}{1+x+x^2}$.

8. $(27\,a^3-1) \times \frac{a+1}{9\,a^2+3\,a+1}$.

Perform the following multiplications:

9. $(a^2+ab+b^2)\times\frac{a-b}{a^3-b^3}.$ **11.** $\frac{x+1}{x^4-1}\times\frac{x^2-1}{(x+1)^2}.$

10. $(1-a+a^2)\times\frac{a+1}{a^3+1}.$ **12.** $\frac{a+b}{a^6-b^6}\times\frac{a^3+b^3}{a^2-b^2}.$

13. $\frac{3c}{(3-c)(3+c)}\times\frac{(c-3)(c+3)}{6cd}.$

14. $\frac{3a+b}{2+3x}\times\frac{(3x+2)(2a-b)}{(a+b)(3a+b)}.$ **15.** $\frac{a-b}{a^6-b^6}\times\frac{a^3-b^3}{a^2-b^2}.$

16. $\frac{(a+2b)(a+2b)}{a-2b}\times\frac{(2a-b)(a-2b)}{(3a+b)(a+2b)}.$

17. $\frac{3x^2y^2}{2yz^2}\times\frac{6az}{9x^3}.$ **18.** $\frac{5a(a-b)}{3c(a+b)}\times\frac{9(a+b)^2}{15(a^2-b^2)}.$

19. $\frac{12c^3b}{5(c^3-b^3)}\times\frac{35(c^2+cb+b^2)}{14c^2b^2}.$

20. $\frac{y^2+3y+2}{y^2-5y+6}\times\frac{y^2-7y+12}{y^2+8y+7}.$

21. $\frac{x^2-x}{x^2-1}\times\frac{2x^2+4x+2}{3x^2+6x}.$ **22.** $\frac{a^2-10a+16}{a^2+6a+9}\times\frac{a+3}{a^2-4}.$

23. $\frac{(x+y)^2-z^2}{x^2+xy-xz}\times\frac{x}{(x-z)^2-y^2}\times\frac{(x-y)^2-z^2}{xy-y^2+yz}.$

24. $\frac{3(x+4)^2}{4(x+4)(x-7)}\times\frac{(x-7)^2}{3(x+4)(x-7)}.$

25. $\frac{a^3+5a^2-36a}{a^2-7a-144}\times\frac{(a-16)(a-3)}{a(a-4)(a+2)}.$

26. $\frac{3a(a+7)(a-5)}{7b(a+3)(a+7)}\times\frac{b(a+3)(a+10)}{a(a-5)(a-10)}.$

27. $\frac{3t^2-2t-1}{2t^2+t-1}\times\frac{2t^2+5t-3}{3t^2+7t+2}\times\frac{4t^2+10t+4}{4t^2-2t-2}.$

28. $\frac{6x^2-7x+2}{10x^2-7x+1}\times\frac{6x^2-5x-1}{6x^2+x-1}\times\frac{10x^2+3x-1}{5x^2-4x-1}.$

DIVISION OF FRACTIONS

135. Fractions are divided by the same rule as in arithmetic.

Thus, $$\frac{2}{3} \div \frac{8}{9} = \frac{2}{3} \times \frac{9}{8} = \frac{\cancel{2} \cdot \overset{3}{\cancel{9}}}{\cancel{3} \cdot \underset{4}{\cancel{8}}} = \frac{3}{4}.$$

Similarly in algebra, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$

That is, we have the following

Rule. *To divide by a fraction, invert the divisor and multiply by the fraction thus obtained.*

Steps in Dividing Fractions. The steps in dividing by a fraction are:

(1) *Reduce all mixed expressions to the fractional form.*
(2) *Factor each numerator and each denominator.*
(3) *Invert the fraction by which you are dividing.*
(4) *Now proceed as in multiplying fractions.*

These steps are illustrated in the following examples.

(1) $$\frac{ab}{x^2 - 3x + 2} \div \frac{b}{x - 1} = \frac{a\cancel{b}}{\cancel{(x-1)}(x-2)} \times \frac{\cancel{x-1}}{\cancel{b}} = \frac{a}{x-2}.$$

(2) $$\frac{ax - ay}{b(x + y)} \div (x - y) = \frac{a\cancel{(x-y)}}{b(x+y)} \times \frac{1}{\cancel{x-y}} = \frac{a}{b(x+y)}.$$

(3) $$\left(1 + \frac{b}{a}\right) \div (a^2 - b^2) = \frac{\cancel{a+b}}{a} \times \frac{1}{\cancel{(a+b)}(a-b)} = \frac{1}{a(a-b)}.$$

In Example (3), the divisor is an integral expression, $a^2 - b^2$. This may be written $\frac{a^2 - b^2}{1}$, and this inverted is $\frac{1}{a^2 - b^2}$.

ORAL EXERCISES

Perform the following divisions:

1. $\frac{a}{b} \div \frac{a^2}{b^2}.$
2. $\frac{x^2}{y^2} \div \frac{x}{y}.$
3. $\frac{x^3}{y^3} \div \frac{x^2}{y^2}.$
4. $\frac{x^3}{y^5} \div \frac{x^4}{y^3}.$
5. $\frac{a^4}{b^4} \div \frac{a^3}{b^5}.$
6. $\frac{a^5}{b^3} \div \frac{a^4}{b^4}.$

WRITTEN EXERCISES

Perform the following divisions:

1. $\frac{ac}{b} \div c.$

2. $\frac{ab}{c} \div \frac{bx}{c}.$

3. $\frac{abc}{xy} \div \frac{bc}{ax}.$

4. $\frac{ax^2}{by^2} \div \frac{ax}{cy}.$

5. $\frac{nxy}{my} \div \frac{ny}{mx}.$

6. $\frac{rst}{xy} \div \frac{r^2s}{x^2y}.$

7. $\frac{abc}{x-1} \div ab.$

8. $\frac{xy}{a-b} \div xy.$

9. $\frac{a+1}{a-2} \div \frac{3(a+1)}{a+4}.$

10. $\frac{24\,ab}{25\,c} \div \frac{16\,ab}{15}.$

11. $\frac{4\,k^2}{-9\,abc^2} \div -\frac{2\,k}{ab}.$

12. $\frac{ab^2}{(x+y)^2} \div \frac{ab}{x+y}.$

Find the following indicated quotients and reduce the fractions to their lowest terms:

13. $\frac{x^3-y^3}{x+y} \div (x^2+xy+y^2).$

14. $\frac{x^3+y^3}{x-y} \div (x^2-y^2).$

15. $\frac{x^2+4\,x+4}{x^2-2\,x+1} \div (x^2-4).$

16. $\frac{1-27\,x^3}{1-9\,x^2} \div (1+3\,x+9\,x^2).$

17. $\frac{x^2+2\,x-35}{x^2+10\,x+21} \div (x^2-4\,x-5).$

18. $\frac{x^2-16\,x+39}{x^2-8\,x+15} \div (x^2-x-156).$

19. $\frac{ac+bc-ad-bd}{xy-4\,x-3\,y+12} \div (cx-3\,c-dx+3\,d).$

20. $\frac{x^2+ax+bx+ab}{x^2+bx-3\,x-3\,b} \div (x^2+ax-5\,x-5\,a).$

21. $\frac{mr+ms-nr-ns}{mx-m-nx+n} \div (3\,r-xr+3\,s-xs).$

22. $\frac{x^2-3\,x-88}{x^2-9\,x-22} \div (x^2+9\,x+8).$

EXERCISES IN MULTIPLICATION AND DIVISION

Perform the following indicated operations:

1. $\dfrac{a^3+b^3}{a^2-9b^2}\times\dfrac{a+3b}{a+b}.$

2. $\dfrac{x^2+x-2}{x^2-3x}\div\dfrac{x^3+2x^2}{x^2+9x-36}.$

3. $\dfrac{x^2-6x-16}{x^2+4x-21}\div\dfrac{x^2+9x+14}{x^2-8x+15}.$

4. $\dfrac{x^2-1}{x^2-4x-5}\times\dfrac{x^2-25}{x^2+2x-3}.$

5. $\dfrac{x^2+9xy+18y^2}{x^2-9xy+20y^2}\div\dfrac{x^2+6xy+9y^2}{xy^2-4y^3}.$

6. $\dfrac{3a^4-9a^3-54a^2}{9a^3-117a^2+378a}\div\dfrac{a^3+8a^2+15a}{3a^2-33a+84}.$

7. $\dfrac{a^2-11a+30}{a^3-6a^2+9a}\times\dfrac{a^2-3a}{a^2-25}\div\dfrac{a^2-9}{a^2+2a-15}.$

8. $\dfrac{x^2-10x+21}{x^2+x-56}\div\dfrac{x^2-8x+15}{x^2+4x-32}.$

9. $\dfrac{a^2-b^2}{ab^2x}\times\dfrac{b(a-b)}{a^2+2ab+b^2}\div\dfrac{a^2-2ab+b^2}{b(a+b)}.$

10. $\dfrac{a+b}{ab}\times\dfrac{a^2-b^2}{3(a^2+b^2)}\div\dfrac{(a-b)^2(a+b)^2}{3a^3y+3ay^3}.$

11. $\dfrac{5x^4-5x^3}{7x^2-56x-63}\div\dfrac{x^4-9x^3+8x^2}{14x^2+14x-1260}.$

12. $\dfrac{8y^2(y+4)(y+5)}{2^4(y+5)(y-7)}\div\dfrac{y(y+4)(y+8)}{2^3(y-7)(y+11)}.$

13. $\dfrac{x^3(x-2)^2}{(x+2)^2}\times\dfrac{(x+2)(x-3)}{(x-2)(x-7)}\div\dfrac{x^2(x-3)(x-5)}{(x-5)(x-7)}.$

14. $\dfrac{a^2b^2(c+5)(c-4)}{(c-4)(c-8)}\times\dfrac{(c-8)(c+9)}{(c+4)(c+7)}\div\dfrac{ab(c+9)(c-1)}{(c+7)(c+1)}.$

15. $\dfrac{21x^2+23x-20}{10x^2-27x+5}\times\dfrac{6x^2-11x-10}{3x^2+2x-5}\div\dfrac{7x^2+17x-12}{5x^2+9x-2}.$

16. $\dfrac{4b^2-17b+4}{6b^2-7b+2}\times\dfrac{10b^2-21b+9}{5b^2-23b+12}\times\dfrac{3b^2-5b+2}{4b^2-5b+1}.$

COMPLEX FRACTIONS*

136. A complex fraction is one which contains one or more fractions in its numerator or denominator or in both.

E.g.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} \quad \text{and} \quad \frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}}.$$

Simplifying a Complex Fraction.

A complex fraction is said to be *simplified* when it is reduced to an equal fraction, in its lowest terms, whose numerator and denominator are in the integral form. The following examples show how such reductions are made:

1.
$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} = \frac{\frac{a+1}{a}}{\frac{a-1}{a}} = \frac{a+1}{a} \div \frac{a-1}{a} = \frac{a+1}{\cancel{a}} \times \frac{\cancel{a}}{a-1} = \frac{a+1}{a-1}.$$

This result may also be obtained directly by multiplying both terms of the given fraction by a which is the L. C. D. of all the small fractions in the complex fraction. This gives

$$\frac{1+\frac{1}{a}}{1-\frac{1}{a}} = \frac{a\left(1+\frac{1}{a}\right)}{a\left(1-\frac{1}{a}\right)} = \frac{a+\frac{a}{a}}{a-\frac{a}{a}} = \frac{a+1}{a-1}.$$

2.
$$\frac{\frac{1}{x+1}+\frac{1}{x-1}}{\frac{1}{x-1}-\frac{1}{x+1}} = \frac{\frac{x-1+x+1}{x^2-1}}{\frac{x+1-(x-1)}{x^2-1}} = \frac{2x}{x^2-1} \div \frac{2}{x^2-1} = x.$$

By multiplying the terms of the given fraction by $(x+1)(x-1)$, we may also get directly

$$\frac{\frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}+\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}}{\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}-\frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}} = \frac{x-1+x+1}{x+1-x+1} = \frac{2x}{2} = x.$$

After a little practice, this cancellation can be done *mentally*, thus writing $\frac{x-1+x+1}{x+1-x+1}$ at once.

---

* §§ 136, 137 may be omitted without destroying the continuity.

137. Rules for Reducing Complex Fractions.

Rule 1. *Reduce both numerator and denominator to a single fraction by performing the operations indicated.*

Then divide the numerator by the denominator.

Rule 2. *Multiply both numerator and denominator by the lowest common denominator of all the small fractions.*

Then reduce the resulting fraction to its simplest form.

Rule 2 is applied in each of the following:

1. $$\frac{\frac{1}{2}+\frac{1}{3}-\frac{1}{6}}{\frac{2}{3}+\frac{5}{6}-\frac{1}{2}}=\frac{3+2-1}{4+5-3}=\frac{4}{6}=\frac{2}{3}.$$

Here we multiply both numerator and denominator of the given fraction by 6, the L. C. M. of the small denominators, 2, 3, and 6.

2. Similarly, $$\frac{\frac{1}{4}+\frac{1}{8}+\frac{1}{2}}{\frac{3}{2}-\frac{1}{4}-\frac{1}{8}}=\frac{2+1+4}{12-2-1}=\frac{7}{9}.$$

Here we multiply numerator and denominator by 8, the L. C. M. of 2, 4, and 8.

WRITTEN EXERCISES

Reduce each of the following complex fractions to its simplest form:

1. $\dfrac{1+x}{1+\frac{1}{x}}$.

2. $\dfrac{1-\frac{a}{b}}{1+a}$.

3. $\dfrac{x+\frac{x}{2}}{x-\frac{x}{2}}$.

4. $\dfrac{\frac{m+n+1}{3}}{\frac{m-n-1}{2}}$.

5. $\dfrac{4+\frac{a+b}{2}}{4-\frac{a-b}{2}}$.

6. $\dfrac{\frac{x^2-y^2}{4}}{\frac{x+y}{2}}$.

7. $\dfrac{\frac{a^3-8b^3}{27}}{a-2b}$.

8. $\dfrac{\frac{M}{D^3}}{\frac{1+d^2}{D^2}}$.

9. $\dfrac{\frac{H}{c}-\frac{hd}{cD}}{\frac{1+t}{c}}$.

10. $\dfrac{\frac{x}{x-3}-\frac{x}{x+3}}{\frac{x+2}{x-3}-\frac{x-2}{x+3}}$.

11. $\dfrac{\frac{x-3}{x+4}}{\frac{x+2}{x-4}}\div\dfrac{x-4}{x+2}$.

Simplify the following:

12. $\dfrac{\dfrac{1}{x-4}+\dfrac{1}{x-3}}{\dfrac{x}{x^2-7x+12}}$.

13. $\dfrac{1}{1+\dfrac{1}{1+x}}$.

14. $\dfrac{a}{a+\dfrac{a}{a+x}}$.

15. $\dfrac{\dfrac{2}{x-1}-\dfrac{1}{x-2}}{\dfrac{3}{x-3}-\dfrac{4}{x-1}}$.

16. $\dfrac{\dfrac{2}{x-3}+\dfrac{2x^2+2}{x-2x^3}}{\dfrac{1}{x-3}-\dfrac{1}{1-2x^2}}$.

17. $\dfrac{\dfrac{b}{a-b}+\dfrac{b}{a+b}}{\dfrac{a}{a+b}-\dfrac{a}{a-b}}$.

REVIEW QUESTIONS

1. How is a fraction defined in algebra?

2. In what ways may a fraction be changed in form without changing its value? State Principle XVII.

3. How is a fraction reduced to lowest terms?

4. How are fractions reduced to a common denominator?

5. What are the three signs of a fraction? How may these be changed without affecting the value of the fraction? Explain

$$\frac{8}{4}=-\frac{8}{-4}=-\frac{-8}{4}=\frac{-8}{-4} \text{ and } \frac{a}{b}=-\frac{a}{-b}=-\frac{-a}{b}=\frac{-a}{-b}.$$

6. State the following principles in words:

(1) $\frac{a}{b}=\frac{ma}{mb}$ (2) $\frac{a}{c}\pm\frac{b}{c}=\frac{a\pm b}{c}$

(3) $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ (4) $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$

7.* Explain how a complex fraction may be reduced to a simple fraction in two different ways. Which is the shorter method?

CHAPTER XI

EQUATIONS INVOLVING FRACTIONS

138. Example 1. Solve $n + \frac{n}{2} + \frac{n}{3} = 88.$ (1)

Solution. Multiplying both members of equation (1) by 6, which is a common multiple of 2 and 3, these denominators are cancelled, and we get at once

$$6n + 3n + 2n = 528. \qquad (2)$$

Uniting terms, $11n = 528.$ (3)

Dividing both sides by 11, $n = 48.$ (4)

The object is to multiply both members of the equation by a number that will cancel each denominator. Hence the multiplier must contain each denominator as a factor.

Evidently 12 or 18 might have been chosen for this purpose, but not 8 or 10. 6 is the smallest number which will cancel both 2 and 3, because 6 is the L. C. M. of 2 and 3.

Example 2. Solve $\frac{1}{x} + \frac{2}{x-1} = \frac{1}{x(x-1)}.$ (1)

Solution. Multiplying both members of equation (1) by the L. C. D., $x(x-1)$, we have,

$$\frac{\cancel{x}(x-1)}{\cancel{x}} + \frac{2x\cancel{(x-1)}}{\cancel{x-1}} = \frac{\cancel{x}\cancel{(x-1)}}{\cancel{x}\cancel{(x-1)}}. \qquad (2)$$

Cancelling, we have

$$x - 1 + 2x = 1. \qquad (3)$$

Hence $3x = 2$ and $x = \frac{2}{3}$.

Here, as in Example 1, the members of the given equation are multiplied by an expression which cancels each denominator.

139. Clearing of Fractions. The process explained in the foregoing solution is called *clearing of fractions.*

As another illustration solve the equation

$$\frac{4}{x+1}-\frac{1}{x-1}=\frac{4}{(x+1)(x-1)}. \qquad (1)$$

Here the lowest common multiple is $(x+1)(x-1)$.

When we multiply $\frac{4}{x+1}$ by $(x+1)(x-1)$, the denominator $x+1$ is cancelled, thus,

$$\frac{4\cancel{(x+1)}(x-1)}{\cancel{x+1}}=4(x-1).$$

Similarly,

$$-\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}=-(x+1),$$

and

$$\frac{4\cancel{(x+1)}\cancel{(x-1)}}{\cancel{(x+1)}\cancel{(x-1)}}=4.$$

Hence, equation (1) becomes $4(x-1)-(x+1)=4.$ (2)

By F, $4x-4-x-1=4.$ (3)

By F, and $A \mid 5$, $3x=9$ (4)

By $D \mid 3$, $x=3.$ (5)

Check. Substituting $x=3$ in (1) we get

$$1-\tfrac{1}{2}=\tfrac{4}{8} \text{ or } \tfrac{1}{2}=\tfrac{1}{2}$$

After a little practice the cancelling of the denominators can be done *mentally,* as we multiply by the least common denominator in clearing of fractions. For example, in this case, equation (2) may be written at once.

The foregoing examples illustrate the following

Rule. ***To clear an equation of fractions, multiply both members by the L. C. D. of all the fractions involved, and then cancel the denominators.***

WRITTEN EXERCISES

Solve the following equations, indicating the principles used at each step in Examples 1 to 10:

1. $\frac{n}{2}+\frac{n}{3}=5.$

2. $n+\frac{n}{2}+\frac{n}{3}+\frac{n}{4}=25.$

3. $\frac{n}{4}+4n-\frac{5n}{3}=\frac{3n}{2}+26.$

4. $\frac{n}{2}+\frac{n}{3}-\frac{n}{4}+\frac{n}{10}=82.$

5. $7x+\frac{4x}{7}+\frac{3x}{5}+23=\frac{x}{5}+\frac{2x}{5}+5x+113.$

6. $4n+\frac{6n}{7}=\frac{3n+2}{2}+46.$

7. $12+\frac{4(9x+6)}{3}-\frac{2(3+11x)}{5}=\frac{5(4x+4)}{3}+14.$

8. $13b+7-\frac{9b+8}{7}=\frac{2b+9}{5}+38.$

9. $\frac{5a+7}{2}+\frac{2a+4}{3}=\frac{3a+9}{4}+5.$

10. $\frac{17a-5}{3}-\frac{10a+2}{4}=\frac{5a+7}{2}-5.$

11. $\frac{5x+3}{2}+\frac{5(2x+10)}{5}=2x+24.$

12. $\frac{3}{x}-\frac{1}{2}=\frac{1}{x}.$

13. $\frac{1}{3x}+\frac{1}{2x}=\frac{1}{6}.$

14. $\frac{1}{4x}+\frac{1}{3x}=\frac{5}{6x}+\frac{1}{2}.$

15. $\frac{1}{x}+\frac{2}{x-1}=\frac{8}{x(x-1)}.$

16. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{-2}{x^2-1}.$

17. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}.$

18. The sum of two numbers is 12, and the first number is $\frac{1}{3}$ as great as the second. What are the numbers?

19. The smaller of two numbers is $\frac{3}{8}$ of the larger. If their sum is 66, what are the numbers?

20. Find two consecutive integers such that $\frac{5}{7}$ of the first minus $\frac{3}{11}$ of the second equals 9.

21. Find three consecutive integers such that $\frac{1}{3}$ of the sum of the first and second minus $\frac{1}{2}$ the third equals 5.

22. Find three consecutive integers such that $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second minus $\frac{1}{10}$ of the third equals 28.

140. Special Cases. I. Sometimes it is best to *add fractions before multiplying* by the L. C. D. as in the following example:

Solve
$$\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{x-4}-\frac{1}{x-3}. \tag{1}$$

Adding fractions on the right and left,
$$\frac{x-1-(x-2)}{(x-2)(x-1)}=\frac{x-3-(x-4)}{(x-4)(x-3)}.$$

Simplifying the numerators,
$$\frac{1}{(x-2)(x-1)}=\frac{1}{(x-4)(x-3)}. \tag{2}$$

Multiplying by the L. C. D. of all the fractions,
$$(x-4)(x-3)=(x-2)(x-1). \tag{3}$$

Hence, solving,
$$x=2\tfrac{1}{2}. \tag{4}$$

Check by substituting $x=2\frac{1}{2}$ in equation (1).

Note.—If we attempt to solve this equation by first clearing of fractions, we have
$$\begin{aligned}&(x-1)(x-4)(x-3)-(x-2)(x-4)(x-3)\\&=(x-2)(x-1)(x-3)-(x-2)(x-1)(x-4).\end{aligned}$$

The solution of this equation is much more laborious than the solution of equation (2) above.

II. Sometimes it is best to multiply by the L. C. D. of *part of the denominators first*, and, after simplifying, multiply by the L. C. D. of the remaining denominators.

Solve
$$\frac{4t-3}{16}-\frac{t-2}{4}=\frac{2t-2}{5t+2}. \tag{1}$$

Multiplying by 16,
$$4t-3-4(t-2)=\frac{32t-32}{5t+2}. \tag{2}$$

Hence,
$$5=\frac{32t-32}{5t+2}. \tag{3}$$

Multiplying by $5t+2$,
$$25t+10=32t-32. \tag{4}$$

Hence, solving,
$$t=6. \tag{5}$$

Check by substituting $t=6$ in equation (1).

In this example, try to solve by clearing of fractions completely at the outset to see whether the plan of solving here given is simpler.

WRITTEN EXERCISES

1. $\frac{3x+6}{5}-\frac{9x+3}{15}=\frac{x+7}{6x-8}+4.$ *Ans.* $x=\frac{17}{19}.$

2. $\frac{7x+1}{12}-\frac{14x-22}{24}=\frac{11x+5}{8x-28}.$

3. $\frac{3x+4}{2}-\frac{12x+1}{8}=\frac{5x-1}{3x+2}.$ *Ans.* $x=-7\frac{3}{5}.$

4. $\frac{7t+3}{5}-\frac{21t+9}{15}=\frac{17t-3}{3t+11}+2.$ *Ans.* $t=-\frac{19}{23}.$

5. $\frac{11v-15}{10}-\frac{33v+15}{30}=\frac{5v+5}{v-5}.$ *Ans.* $v=\frac{5}{7}.$

6. $\frac{x-1}{x-2}+\frac{x-2}{3-x}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$ *Ans.* $x=3\frac{1}{2}.$

7. $\frac{1}{x-1}-\frac{2}{2x+1}=\frac{1}{x-2}-\frac{2}{2x-1}.$ *Ans.* $x=\frac{3}{4}.$

8. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-4}{x-5}+\frac{x-5}{6-x}.$

9. $\frac{9}{x-7}-\frac{9}{x-2}=\frac{5}{x-8}-\frac{5}{x+1}.$

10. $\frac{x+17}{x+5}-\frac{2(x+6)}{x+3}=-\frac{x-1}{x+3}.$

11. $\frac{3x-4}{x+5}-\frac{4x-1}{x+4}+\frac{x^2+44}{x^2+9x+20}=0.$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

1. What number must be subtracted from each term of the fraction $\frac{11}{17}$ so that the result shall be equal to $\frac{1}{2}$?

2. What number must be subtracted from each term of the fraction $\frac{21}{35}$ so that the result shall be equal to $\frac{2}{9}$?

3. What number must be added to each term of the fraction $\frac{1}{2}$ to obtain a fraction equal to $\frac{16}{17}$?

4. What number must be added to each of the terms in the fractions $\frac{2}{9}$ and $\frac{16}{37}$ in order to make the resulting fractions equal?

5. There are three numbers such that the second is 4 more than 9 times the first, and the third is 2 more than 6 times the first. If $\frac{1}{8}$ of the third is subtracted from $\frac{1}{7}$ of the second, the remainder is 3. Find the numbers.

6. There are three numbers such that the second is 2 more than 9 times the first and the third is 5 more than 11 times the first. The remainder when $\frac{1}{3}$ of the third is subtracted from $\frac{1}{2}$ of the second is one. Find the numbers.

7. What number must be subtracted from both the numerator and the denominator of the fraction $\frac{5}{7}$ in order to make the result equal to $\frac{1}{2}$?

8. What number must be subtracted from both numerator and denominator of the fraction $\frac{5}{8}$ in order that the fraction may be increased threefold? *Ans.* $2\frac{1}{2}$.

9. What number added to both numerator and denominator of the fraction $\frac{3}{4}$ will double the fraction?

10. Find a number of two digits in which the tens' digit is 3 greater than the units' digit, and such that if the number is divided by the sum of the digits, the quotient is 7.

11. In a number of two digits the units' digit exceeds the tens' digit by 4, and when the number is divided by the sum of its digits the quotient is 4. Find the number.

12. **Illustrative Problem.** B can do a piece of work in 8 days, and A can do it in 10 days. In how many days can they do it working together?

Since B can do the work in 8 days, in 1 day he can do $\frac{1}{8}$ of it, and since A can do it in 10 days, in 1 day he can do $\frac{1}{10}$ of it. If x is the number of days required when both work together, then in 1 day they can do $\frac{1}{x}$ of it. Hence we have the equation,

$$\frac{1}{8}+\frac{1}{10}=\frac{1}{x}.$$

13. B can do a piece of work in 12 days and A can do it in 9 days. How long will it take both working together to do it?

14. A pipe can fill a cistern in 11 hours and another in 13 hours. How long will it require both pipes to fill it?

Ans. $5\frac{23}{24}$ hours.

15. B can do a piece of work in a days and A can do it in b days. How long will it take both together to do it?

16. A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take to fill the cistern when both are running together?

17. A pipe can fill a cistern in 12 hours, another in 10 hours, and a third can empty it in 8 hours. How long will it require to fill the cistern when they are all running?

18. A man can do a piece of work in 18 days, another in 21 days, a third in 24 days, and a fourth in 10 days. How long will it require them when all are working together?

Ans. $4\frac{52}{617}$ days.

REVIEW QUESTIONS

1. What is meant by *clearing an equation of fractions?*

2. What principle is used in clearing an equation of fractions?

3. By what expression must both members of an equation be multiplied in order to clear the equation of fractions?

4. Is it always best to multiply *at once* by the least common multiple of all the denominators?

5. In the first example solved on page 184, state what advantage was gained by adding the fractions on each side before clearing of fractions.

6. In the second example solved on page 184, what advantage was gained by clearing of fractions *partially* at first?

CHAPTER XII

RATIO AND PROPORTION

141. Ratio. A fraction is often called a *ratio*. Thus $\frac{a}{b}$ may be read *the ratio of a to b*, and it may also be written $a : b$.

Terms of a Ratio. The numerator is called the *antecedent* of the ratio, and the denominator the *consequent*. The antecedent and consequent are called the *terms* of the ratio.

Proportion. An equation, each of whose members is a ratio, is called a *proportion*.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion. A proportion is usually written

$$a : b = c : d, \quad \text{or} \quad a : b :: c : d.$$

It is read *the ratio of a to b equals the ratio of c to d*, or briefly, *a is to b as c is to d*.

Means and Extremes. The four numbers a, b, c, and d, when written $\frac{a}{b} = \frac{c}{d}$, or $a : b :: c : d$, are said to be *in proportion*. Then the two end terms, a and d, are called the *extremes* of the proportion, and the two middle terms, b and c, the *means*.

HISTORICAL NOTE

Ratio and Proportion. The subject of ratio and proportion was studied fully by Euclid (300 B.C.) in connection with geometric magnitudes, but of course all his results apply equally well to numbers. Euclid, however, did not regard a ratio as a fraction and his treatment is exceedingly complicated as compared with the modern treatment in which a ratio is recognized as a fraction.

In 1631 Oughtred, an Englishman, wrote the proportion $a : b = c : d$ in the form $a : b :: c : d$. Before his time this proportion was written $a - b - c - d$. John Wallis (see opposite page) brought the symbol $::$ into common use and it has survived to the present time, though its meaning is exactly the same as that of $=$.

John Wallis (1616–1703) was an English clergyman who made contributions to mathematics, logic, and grammar. He was educated at Emmanuel College, Cambridge, and was afterward chosen fellow of Queen's College.

During the period of the conflict between King Charles I and Parliament, Wallis was an adherent of the party of the latter, and displayed surprising talent in deciphering intercepted papers and letters of the Royalists.

Wallis was the author of numerous works relating to logic, English grammar, and especially to mathematics. His work on algebra, *De Algebra Tractatus*, contains a history of the subject.

IMPORTANT PROPERTIES OF A PROPORTION

142. Transforming a Proportion. Starting each time with the proportion, $a:b::c:d$, we deduce the following results:

Case I: $\quad ad = bc.$

Proof. Writing $a:b::c:d$, in the fractional form, $\frac{a}{b}=\frac{c}{d}$, we clear of fractions and obtain $ad = bc$.

That is: *If four numbers are in proportion, the product of the means equals the product of the extremes.*

Case II: $\quad b:a::d:c.$

We are to show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{b}{a}=\frac{d}{c}$.

Proof. From Case I, we have $bc = ad$.

Dividing both sides by ac, we get $\frac{b}{a}=\frac{d}{c}$, or $b:a::d:c$.

This process is called taking the proportion by **inversion.**

Case III: $\quad a:c::b:d.$

We are to show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$.

Proof. From Case I, we have $ad = bc$.

Dividing both sides by cd, we get $\frac{a}{c}=\frac{b}{d}$, or $a:c::b:d$.

This process is called taking the proportion by **alternation.**

Case IV: $\quad a+b:b::c+d:d.$

We are to show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$.

Proof. We have $\frac{a}{b}=\frac{c}{d}$.

Adding 1 to both sides, we get $\frac{a}{b}+1=\frac{c}{d}+1$.

Hence $\frac{a+b}{b}=\frac{c+d}{d}$, or $a+b:b::c+d:d$.

This process is called taking the proportion by **addition,** or by **composition,** as it is sometimes called.

Case V: $\boldsymbol{a-b:b::c-d:d.}$

We are to show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a-b}{b}=\frac{c-d}{d}$.

Proof. We have $\frac{a}{b}=\frac{c}{d}$.

Subtracting 1 from each side, we get $\frac{a}{b}-1=\frac{c}{d}-1$.

Hence $\frac{a-b}{b}=\frac{c-d}{d}$, or $a-b : b :: c-d : d$.

This process is called taking the proportion by **subtraction,** or by **division,** as it is sometimes called.

Case VI: $\boldsymbol{a+b : a-b :: c+d : c-d.}$

We are to show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

Proof. We have from Cases IV and V,

(1) $\frac{a+b}{b}=\frac{c+d}{d}$; and (2) $\frac{a-b}{b}=\frac{c-d}{d}$.

Dividing equation (1) by (2),

$$\frac{a+b}{\cancel{b}}\times\frac{\cancel{b}}{a-b}=\frac{c+d}{\cancel{d}}\times\frac{\cancel{d}}{c-d}.$$

Hence $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, or $a+b : a-b :: c+d : c-d$.

This process is called taking the proportion by **addition** and **subtraction,** or by **composition** and **division.**

Case VII. If $\boldsymbol{\frac{a}{b}=\frac{c}{d}=\frac{e}{f}}$, then $\boldsymbol{\frac{a+c+e}{b+d+f}=\frac{a}{b}}$.

Proof. Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=k$; then $a=bk$, $c=dk$, $e=fk$.

Hence $a+c+e=bk+dk+fk=(b+d+f)k$,

and $$\frac{a+c+e}{b+d+f}=k=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$$

That is, *The sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Mean Proportional. If $a : b :: b : x$, then b is called a *mean proportional* between a and x.

Fourth Proportional. If $a : b :: c : x$, then x is called a *fourth proportional* to a, b, and c.

WRITTEN EXERCISES

1. If $ad = bc$, show that $\frac{a}{b} = \frac{c}{d}$. *Hint.* Divide by bd.

2. If $ad = bc$, show that $\frac{a}{c} = \frac{b}{d}$. **3.** If $ad = bc$, show that $\frac{d}{c} = \frac{b}{a}$.

4. If $ad = bc$, show that $\frac{d}{b} = \frac{c}{a}$.

5. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

6. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a} = \frac{c-d}{c}$.

7. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{a+b} = \frac{c-d}{c+d}$.

8. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a-b}{c-d}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{c+d} = \frac{a}{c}$.

10. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+c}{b+d} = \frac{a}{b}$.

11. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-b}{c-d} = \frac{a}{c}$.

12. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a-c}{b-d} = \frac{a}{b}$.

13. Solve the equation $\frac{a}{b} = \frac{c}{d}$ for each letter in terms of all the others. If $a = 3$, $b = 5$, $c = 8$, find d. If $b = 7$, $c = 9$, $d = 3$, find a. If $c = 13$, $d = 2$, $a = 5$, find b. If $d = 50$, $a = 3$, $b = -7$, find c.

14. Find a fourth proportional to 3, 5, and 7; also to 9, 5, and 1; and to 3, -2, and -5.

15. Is 4 a mean proportional between 2 and 8? Why? Is 16 a mean proportional between 4 and 32? Why?

PROBLEMS INVOLVING RATIOS

1. Which is the greater ratio, $\frac{5+3d}{5+4d}$ or $\frac{5+4d}{5+5d}$?

Hint. Reduce the fractions to a common denominator and compare numerators. (d is a positive number.)

2. Which is the greater ratio, $\frac{a+7b}{a+8b}$ or $\frac{a+9b}{a+10b}$?

3. Which is the greater ratio, $\frac{a}{b}$ or $\frac{a+c}{b+c}$, if b and c are positive, and a less than b? a equal to b? a greater than b?

4. Find two numbers in the ratio of 3 to 5 whose sum is 160.

Hint. Call the numbers x and $160 - x$.

5. Find two numbers in the ratio of 2 to 7 whose sum is -108.

6. Find two numbers in the ratio of 3 to -4 whose sum is -15.

7. What number added to each of the terms of the ratio $\frac{5}{7}$ makes it equal to $\frac{65}{75}$?

8. What number must be added to each term of the ratio $\frac{1}{11}$ to make it equal to the ratio $\frac{3}{5}$?

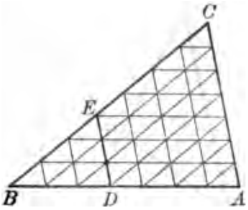
9. What number added to each of the numbers 3, 5, 7, 10, will make the sums in proportion, when taken in the given order?

10. Two numbers are in the ratio of 2 to 3, and their sum is 60. Find the numbers.

11. What number must be subtracted from each of the numbers 7, 8, 9, and 12, so that the resulting differences shall form a proportion when taken in the given order?

SIMILAR TRIANGLES

143. Similar Triangles. Triangles are called *similar* if they have the *same shape*.



Thus the triangles ABC and DBE of the figure are similar. Note that AB and DB have been divided into 7 and 3 equal parts, respectively. Hence the ratio of these sides is $\frac{7}{3}$.

What is the ratio of the sides BC and BE? Of the sides CA and ED? This is stated as follows: *The lengths of the pairs of corresponding sides of two similar triangles form a proportion.*

That is, we may write $\frac{AB}{DB} = \frac{CB}{EB} = \frac{CA}{ED}$.

Note that AB, BC, ⋯ represent the **lengths** of these sides.

WRITTEN EXERCISES

1. If in two similar triangles the sides of the first are 11, 13, and 16, and the side of the second which corresponds to 11 in the first is 33, find the other sides of the second triangle.

Solution. Let x represent the length of the side corresponding to the one whose length is 13. Then $\frac{x}{13} = \frac{33}{11}$, and $11x = 13 \cdot 33$, or $x = 39$.

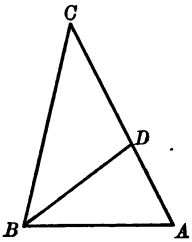
In this manner find the remaining side.

2. If the sides of a triangle are 4, 6, and 10, and one side of a similar triangle is 9, find the remaining sides of the second triangle, if the given side corresponds to the side 4.

3. Solve Example 2 if the given side in the second triangle corresponds to 6.

4. Solve Example 2 if the given side of the second triangle corresponds to 10.

5. A triangular field, one of whose sides is 20 rods, has an area of 80 square rods. Find the area of a similar field whose corresponding side is 45 rods.



It is found in geometry that if a *line divides one angle of a triangle into two equal angles, it divides the opposite side into two parts which are in the same ratio as the other two adjacent sides of the triangle.*

That is, in the figure $\frac{AD}{DC} = \frac{AB}{BC}$.

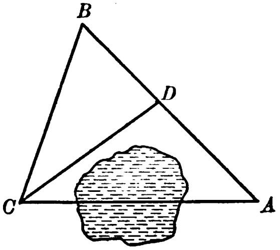
6. Knowing this fact, how many of the lines AD, DC, BC, and AB must you measure in order to find the rest of them?

7. If in the figure $AD = 6$, $DC = 9$, and $BC = 12$, find AB.

8. If in the figure $BC = 18$, $AB = 12$, and $AD = 6$, find DC.

This fact about geometry enables us in some cases to find the distance between two points without measuring it directly.

9. If in the figure CD divides the angle at C into two equal parts and if you know the lengths of AD, DB, and BC, show fully how to find the length of a straight line across the pond from C to A without measuring it directly.



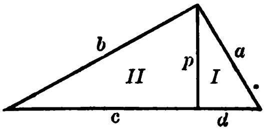
10. If $BC = 75$ rods, $BD = 50$ rods, and $AD = 60$ rods, compute AC.

11. If the sides of a triangle are a, b, c, and the corresponding sides of a similar triangle are a', b', c', show that $\frac{a+b+c}{a'+b'+c'} = \frac{a}{a'}$.

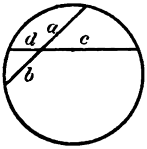
12. Two corresponding sides of two similar triangles are in the ratio 13 : 14. Show that the perimeters (sum of the sides) of the triangles are in the same ratio.

13. The perimeters of two similar triangles are in the ratio $32:35$. Two sides of the first triangle are 8 and 12. Find two sides of the second triangle corresponding to the given sides of the first.

14. In the figure each of the triangles I and II is similar to the original triangle. From these triangles read three proportions, using the principle of § 143.



15. It follows that the triangles I and II are similar to each other. Read three proportions from this fact.



16. If in a circle two intersecting chords are drawn, as in the figure, it is known that $ab = cd$. Form a proportion from this equation. If $a = 9$, $b = 8$, $d = 6$, find c.

REVIEW QUESTIONS

1. Define *ratio, proportion, means, extremes.*

2. State in words the way in which the following proportions are derived from $\frac{a}{b} = \frac{c}{d}$.

(1) $\frac{b}{a} = \frac{d}{c}$. (2) $\frac{a}{c} = \frac{b}{d}$. (3) $\frac{a+b}{b} = \frac{c+d}{d}$.

(4) $\frac{a-b}{b} = \frac{c-d}{d}$. (5) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

3. Define *mean proportional, fourth proportional.*

4. What can you say of the corresponding sides of two similar triangles?

Draw a large figure like that in Example 14, above. Measure the sides and verify the proportions.

CHAPTER XIII

LITERAL EQUATIONS AND THEIR USES

144. Advantage in using Literal Equations. Some of the advantages of algebra over arithmetic in solving problems have been pointed out in the preceding chapters.

A further advantage is set forth in the present chapter; namely, the opportunity offered in algebra to *summarize the solution of a whole class of problems* by solving what is called a **literal equation,** thus obtaining **a formula** which may be used in solving all particular problems of this type.

For example, in arithmetic we solved many problems obtaining the **interest** when the **principal, rate,** and **time** were given. We now see that all of these can be summarized in the one literal equation, which is our *first formula for interest problems:*

$$\boldsymbol{i = prt}. \tag{1}$$

Furthermore, the rules for obtaining the principal, the rate, and the time may now be derived directly from this equation by Principle VI, thus obtaining:

$$p = \frac{i}{rt}, \quad r = \frac{i}{pt}, \quad t = \frac{i}{pr}.$$

Translate each of these formulas into a rule of arithmetic.

145. Solving a Literal Equation. The process of deriving $p = \frac{i}{rt}$ from $i = prt$ is called *solving* the equation $i = prt$ for p in terms of i, r, and t, or simply *solving the equation for p.* Similarly, the process of deriving $r = \frac{i}{pt}$ from $i = prt$ is called *solving the equation for i*; and deriving $t = \frac{i}{pr}$ is called *solving the equation for t.*

In arithmetic a problem is said to be solved when a *numerical answer* is obtained which satisfies the conditions given. The solutions thus far found in algebra have, for the most part, been of this sort.

It is customary, however, to say that a problem has been solved in the *algebraic sense* when a formula is found which gives complete directions for deriving the numerical answer.

Thus, $p = \frac{i}{tr}$ is a solution for the principal since it states precisely how to find the principal in terms of interest, rate, and time.

It is thus seen that from the literal equation $i = prt$ we obtain the complete solution of every problem which calls for any one of these four numbers in terms of the other three.

In modern times machines are used extensively for computation. The algebraic solution of a literal equation *gets the problem ready for the computing machine;* that is, it gets the **formula** which the computer must use.

I. INTEREST PROBLEMS

Oral

1. If $200 is invested at 5 %, what is the amount at the end of one year?

This problem calls for the *amount*, which is the sum of the principal and interest.

2. If $500 is invested at 6 %, what is the amount at the end of one year?

3. If $1000 is invested at 4 %, what is the amount at the end of one year?

4. If $ 500 is invested at 5 %, what is the amount at the end of 2 years?

5. If $1000 is invested at 6 %, what is the amount at the end of 5 years?

6. If $i = prt$, how would you represent the amount in terms of p, r, and t?

Another Interest Formula. If a = amount, then

$$a = p + prt, = p(1 + rt). \qquad (2)$$

This is our *second formula for interest problems.*

This equation may now be solved for any one of the four letters.

Thus, solving for p, we have $p = \frac{a}{1 + rt}$.

Using this equation we may find the principal when the amount, rate, and time are given.

Translated into a rule, the formula $p = \frac{a}{1 + rt}$ reads:

To find the principal, divide the amount by 1 plus the product of the rate and the time.

Written Problems

1. Find the principal if the rate is 5 %, the time 2 years, and the amount $ 880.

Solution. $p = \frac{a}{1 + rt} = \frac{880}{1 + .10} = \frac{880}{1.1} = 800.$

2. Find the principal if the rate is 6 %, the time 4 years, and the amount $ 1488.

3. Solve the equation $a = p + prt$ for t. Translate the resulting formula into words.

4. Find the time if the principal is $2500, the amount $ 3100, and the rate 4 %.

5. Find the time if the principal is $5000, the amount $6050, and the rate 6 %.

6. Solve the equation $a = p + prt$ for r and translate the resulting formula into words.

7. Find the rate if the principal is $1800, the amount $2124, and the time 4 years.

8. Find the rate if the principal is $3500, the amount $4340, and the time 8 years.

We have thus used the formula $a = p + prt$ to solve all possible types of problems where the amount, principal, rate, and time are involved. The points to be noticed are: (1) the great ease with which the equation may be solved for any one of its letters, thus obtaining new rules of arithmetic; (2) the convenience of solving problems by direct substitution in a formula. See the solution of Problem 1.

II. PROBLEMS INVOLVING MOTION

146. Formulas for Motion Problems. The space passed over by a moving body is called the **distance**, and the number of units of distance traversed is represented by d. The **rate** of uniform motion, that is, the number of units of space traversed in each unit of time, is called the **speed** or **rate**, and is represented by r. The number of units of time occupied is represented by t.

For example, if a train runs 40 miles per hour, in 5 hours it will run 5 × 40 miles.

Again, if sound travels 1080 feet per second, in 5 seconds it will travel 5 × 1080 feet.

In each of these examples the distance passed over is found by multiplying the rate by the time. Using the symbols d, r, and t, we have *the first formula for motion:*

$$d = rt. \tag{1}$$

1. Solve the equation $d = rt$ for t in terms of d and r, and for r in terms of d and t. See § 145.

Translate each of these formulas into words.

It is to be understood in all problems here considered that the speed remains the same throughout the period of motion; *e g.* sound travels just as far in any one second as in any other second of its passage.

2. If sound travels 1080 feet per second, how far does it travel in 6 seconds?

3. If a transcontinental train has an average of 35 miles per hour, how far does it travel in $2\frac{1}{2}$ days? Here we have given $r = 35$, $t = 2\frac{1}{2} \times 24$, and we are to find d.

13. At what time between 5 and 6 o'clock is the minute hand 15 minute spaces behind the hour hand? At what time is it 15 minute spaces ahead?

Since, at 5 o'clock, it is 25 minute spaces behind the hour hand, in the first case it must gain $25 - 15 = 10$ minute spaces, and in the second case it must gain $25 + 15 = 40$ minute spaces. Make a diagram as in the preceding problem to show both cases.

14. At what time between 9 and 10 o'clock is the minute hand of a clock 30 minute spaces behind the hour hand? At what time are they together?

15. A fast freight leaves Chicago for New York at 8:30 A.M., averaging 32 miles per hour. At 2:30 P.M. a limited express leaves Chicago over the same road, averaging 55 miles per hour. In how many hours will the express overtake the freight?

If the express requires t hours to overtake the freight, the latter had been on the way $t + 6$ hours. Then the distance covered by the express is $55\,t$, and the distance covered by the freight is $32\,(t + 6)$. As these must be equal, we have $55\,t = 32\,(t + 6)$.

16. In a bicycle road race one rider averages $19\frac{1}{2}$ miles per hour, while another, starting 40 minutes later, averages $22\frac{1}{4}$ miles per hour. In how long a time will the latter overtake the former?

III. PROBLEMS INVOLVING THE LEVER

Two boys, A and B, play at seesaw. They find that the teeter board will balance when equal products are obtained by multiplying the weight of each by his distance from the point of support.

Thus, if B weighs 80 pounds and is 5 feet from the point of support, then A, who weighs 100 pounds, must be 4 feet from this point, since $80 \times 5 = 100 \times 4$.

The teeter board is a certain kind of lever; the point of support is called the **fulcrum**.

In each of the following problems make a diagram similar to the figure on the opposite page:

1. A and B weigh 90 and 105 pounds respectively. If A is seated 7 feet from the fulcrum, how far is B from that point?

2. Using the same weights as in the preceding problem, if B is $6\frac{1}{2}$ feet from the fulcrum, how far is A from that point?

3. A and B are 5 and 7 feet respectively from the fulcrum. If B weighs 75 pounds, how much does A weigh?

4. A and B weigh 100 and 110 pounds respectively. A places a stone on the board with him so that they balance when B is 6 feet from the fulcrum and A $5\frac{1}{2}$ feet from this point. How heavy is the stone?

Formula for Lever Problems. If the distances from the boys to the fulcrum are respectively d_1 and d_2, and their weights w_1 and w_2, then

$$d_1w_1 = d_2w_2.$$

This is our *formula for solving lever problems.*

This equation is a statement in the language of algebra of a very important law of nature. The law is the result of a very large number of careful experiments. It is a universal custom among scientific men, to express laws of nature, so far as possible, by means of literal equations of this sort.

If any three of the four numbers d_1, w_1, d_2, w_2, are given, the fourth may be found by means of the equation $d_1w_1 = d_2w_2$.

5. Solve $d_1w_1 = d_2w_2$ for d_1 in terms of the other three letters. Then solve for w_1.

6. Solve $d_1w_1 = d_2w_2$ for d_2 and also for w_2.

7. A and B are seated at the opposite ends of a 13-foot teeter board. Using the weights of problem 1, where must the fulcrum be located so that they shall balance?

If the fulcrum is the distance d from A, then it is $(13 - d)$ from B. Hence, $90\,d = 105(13 - d)$.

8. A, who weighs 75 pounds, sits 7 feet from the fulcrum. If B weighs 105 pounds, at what distance from the fulcrum should he sit in order to make a balance?

9. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

10. A lever 9 feet long carries weights of 17 and 32 pounds at its ends. Where should the fulcrum be placed so as to make the lever balance?

11. A lever of unknown length is balanced when weights of 30 and 45 pounds are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

SUGGESTION. Let x be the distance from the greater weight to the fulcrum.

IV. MISCELLANEOUS LITERAL EQUATIONS

Solve the following for each letter in terms of the others:

1. $F = 32 + \frac{9}{5}C.$

2. $l = a + (n-1)d.$

3. $s = \frac{n}{2}(a + l).$

4. $s = \frac{a - rl}{1 - r}.$

Solve each of the following for x:

5. $ax + 3b = cx + d.$

6. $(a - x)(b + x) = x(b - x).$

7. $(x + a)(x + b) = (x - c)^2.$

8. $\frac{ax + b}{c} - \frac{bx + c}{d} = 1.$

9. $\frac{x}{b} + \frac{x}{a - b} = \frac{a}{a + b}.$

10. $\frac{x + 1}{x - 1} = \frac{a + b}{a - b}.$

11. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1.$

12. $\frac{a + bx}{a + b} = \frac{c + dx}{c + d}.$

13. $\frac{a + b}{1 + x} = \frac{a - b}{1 - x}.$

14. $\frac{3ax}{a - b} - 2a = 5x.$

15. $\frac{ax - 1}{bx} = \frac{1 + bx}{ax} + 1.$

16. $\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2x}{a+b}$.

17. $\frac{a+x}{a}-\frac{b+x}{b}-\frac{c+x}{c}=1$.

18. $\frac{x+1}{x-1}+\frac{x-1}{x+1}=\frac{2x(x+2)}{(x+1)(x-1)}$.

19. $\frac{x+m}{x-m}+\frac{x-m}{x+m}=\frac{2x(x+1)}{(x+m)(x-m)}$.

20. $\frac{x+b}{x-a}+\frac{x-c}{x+a}=\frac{2x(x+a)}{(x+a)(x-a)}$.

21. What number must be added to each term of the fraction $\frac{3}{7}$ to obtain a fraction equal to $\frac{1}{2}$?

22. What number must be added to each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

Solve Example 21 by substituting in the formula obtained under Example 22.

23. What number must be subtracted from each term of the fraction $\frac{3}{10}$ to obtain a fraction equal to $\frac{2}{7}$?

24. What number must be subtracted from each term of the fraction $\frac{a}{b}$ to obtain a fraction equal to $\frac{c}{d}$?

Solve Example 23 by substituting in the formula obtained under Example 24.

25. What number must be added to each of the numbers 2, 5, 4, 12 so that the sums shall be in proportion when taken in the given order?

26. What number must be added to each of the numbers a, b, c, d so that the resulting sums shall be in proportion when taken in the given order?

27. What number must be subtracted from each of the numbers a, b, c, d so that the resulting remainders shall be in proportion when taken in the given order?

Compare the results in Examples 26 and 27 and explain the relation between them.

HISTORICAL NOTE

Literal Equations whose solutions lead to formulas belong to a very late stage in the development of algebra. In the earliest attempts to solve equations, the unknown was represented by some word such as *res*, the thing (see page 41). Later a single letter was used for the unknown, but known quantities were still represented by numerals, and the equations were so-called *numerical* equations. Finally, both known and unknown quantities were represented by letters, and such equations were called *literal* equations. Numerical equations lead only to *special* solutions, while literal equations lead to *general* solutions or **formulas.**

The Greeks solved special cases of general problems (see page 229), but never obtained general formulas. Vieta (see page 101) used letters, as we do, to represent both known and unknown quantities, but he always used capital letters. Harriot (see page 101), following Vieta, likewise used letters, but usually small letters instead of capitals. Newton (see page 62) was the first to let a letter stand for negative as well as positive numbers. For further uses of literal equations see page 217.

REVIEW QUESTIONS

1. State the rules which may be derived from the formula

$$v = lwh,$$

where these letters represent the volume, length, width, and height, respectively, of a rectangular solid.

2. State all the rules of interest which may be derived from the formula $i = prt.$

3. State all the rules of interest which may be derived from the formula $a = p + prt.$

4. Which problems on motion in this chapter can be solved by use of the formula

$$d = rt?$$

5. Which problems on motion in this chapter belong to the class whose solutions are summarized by the solutions of the equation $r_1t = r_2t + n$?

6. State fully the meaning of the equation $w_1d_1 = w_2d_2$ in connection with the lever.

CHAPTER XIV

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE

147. An Equation of the First Degree. An equation of the *first degree* in x and y is one in which neither x nor y is multiplied by itself or by the other. For instance, such an equation must not contain x^2, y^2, or xy.

E.g. $13x - 5y = 14$ is of the first degree in x and y, while $2xy - x = 5$ and $3x - 5y^2 = 13$ are not of the first degree in x and y.

Indeterminate Equations. A single equation in two unknowns is satisfied by indefinitely many pairs of values of the unknowns. Such an equation is called an *indeterminate equation.*

E.g. $x + y = 5$ is satisfied by

$$\left.\begin{matrix} x = 1 \\ y = 4 \end{matrix}\right\};\quad \left.\begin{matrix} x = 2 \\ y = 3 \end{matrix}\right\};\quad \left.\begin{matrix} x = 0 \\ y = 5 \end{matrix}\right\};\quad \left.\begin{matrix} x = -1 \\ y = 6 \end{matrix}\right\};\ \text{etc.}$$

Simultaneous Equations. Two equations in two unknowns which are satisfied by the same pair of values of the unknowns are called *simultaneous equations.*

E.g. $x + y = 5$ and $x - y = 3$ are simultaneous, because both are satisfied by $x = 4$, $y = 1$.

Contradictory Equations. Two equations in two unknowns, which cannot be satisfied by the same pair of values of the unknowns, are called *contradictory equations.*

E.g. $x + y = 5$ and $x + y = 2$ are contradictory, since no pair of values of x and y can make their sum both 5 and 2 at the same time.

Dependent Equations. Two equations in two unknowns are said to be *dependent* if one can be derived from the other. In this case every pair of values which satisfies one will also satisfy the other.

E.g. $x + y = 5$ and $2x + 2y = 10$ are dependent, since the second may be derived from the first by multiplying both members by 2.

Independent Equations. Two equations in two unknowns are *independent* if neither can be derived from the other.

E.g. $x + y = 5$ and $x - y = 3$ are independent equations.

Summary.

The equations $\begin{Bmatrix} x + y = 5 \\ 2x + 2y = 10 \end{Bmatrix}$ are *simultaneous* but not *independent.*

The equations $\begin{Bmatrix} x + y = 5 \\ x + y = 2 \end{Bmatrix}$ are *independent* but not *simultaneous.*

The equations $\begin{Bmatrix} x + y = 5 \\ x - y = 3 \end{Bmatrix}$ are both *simultaneous* and *independent.*

If two equations of the first degree in x and y are both **simultaneous and independent,** they have one and only one pair of values of x and y which satisfies both.

E.g. $x + y = 5$ and $x - y = 3$ are simultaneous and independent, and they are satisfied by $x = 4$, $y = 1$ and by no other pair of values of x and y.

148. Elimination. To solve two simultaneous independent equations of the first degree in x and y is to find the one pair of values of x and y which satisfies both. In this chapter such equations will be solved by the algebraic method called **elimination**. The essential step in elimination consists in combining the equations in such a way as *to get rid of one* of the unknown numbers.

ELIMINATION BY ADDITION OR SUBTRACTION

149. Illustrative Examples.

(1) Solve

$$\begin{cases} x + 2y = 7, & (1) \\ 3x - 4y = 1. & (2) \end{cases}$$

Multiplying both members of equation (1) by 2, we have,

$$2x + 4y = 14. \qquad (3)$$

Adding the members of equations, (2) and (3), $+4y$ and $-4y$ cancel.

Hence, $5x = 15$, and $x = 3$.

Substituting in (1), $3 + 2y = 7$, and $y = 2$.

Verify this by substituting $x = 3$, $y = 2$, in (1) and (2).

(2) Solve
$$\begin{cases} 7x + 3y = 4x - y + 10, & (1) \\ 3x - y = 4y - 3x + 7. & (2) \end{cases}$$

Collecting the unknowns in each equation, we have,

From (1), $3x + 4y = 10,$ (3)

From (2), $6x - 5y = 7.$ (4)

Multiplying both members of equation (3) by 2, we have,

$$6x + 8y = 20. \quad (5)$$

Subtracting equation (4) from equation (5), we have,

$$13y = 13. \quad (6)$$

By $D \mid 13$, $y = 1.$

Substituting in (3), $x = 2.$

Hence, the solution is $x = 2, y = 1.$

The process used in the solution of example (1) is called **elimination by addition**; that used in example (2) is called **elimination by subtraction.**

Rule for Elimination by Addition or Subtraction.

(*A*) *When the coefficients of one of the unknowns are numerically equal in the two equations:*

(1) *If these coefficients have opposite signs, add the equations; if they have like signs, subtract them.*

(2) *Solve the resulting equation, thus finding the value of one of the unknowns. Substitute this value in one of the given equations, thus finding the other unknown.*

(3) *Check the results by substituting in both of the given equations.*

(*B*) *When the coefficients of neither one of the unknowns are numerically equal, make one pair so by multiplying the equations by suitable numbers, and then proceed as in* (*A*).

NOTE. — We commonly say "add two equations" instead of using the longer expression "add the corresponding members of the equations." Similarly for subtraction. Likewise, we say "multiply the equation" and "divide the equation" instead of "multiply both members of the equation" and "divide both members of the equation."

ORAL EXERCISES

Solve the following pairs of equations by addition or subtraction. Check each solution.

1. $\begin{cases} x+y=2, \\ x-y=0. \end{cases}$

2. $\begin{cases} x+y=3, \\ x-y=1. \end{cases}$

3. $\begin{cases} x+y=4, \\ x-y=2. \end{cases}$

4. $\begin{cases} x+y=5, \\ x-y=1. \end{cases}$

5. $\begin{cases} x+y=6, \\ x-y=2. \end{cases}$

6. $\begin{cases} 2x+y=3, \\ x-y=0. \end{cases}$

7. $\begin{cases} x+2y=5, \\ x+y=4. \end{cases}$

8. $\begin{cases} 2x+y=10, \\ x+y=6. \end{cases}$

9. $\begin{cases} 3x+2y=7, \\ 3x+y=5. \end{cases}$

WRITTEN EXERCISES

Solve the following pairs of equations. Check the first six.

1. $\begin{cases} 2x+3y=22, \\ x-y=1. \end{cases}$

2. $\begin{cases} 5x-2y=21, \\ x-y=6. \end{cases}$

3. $\begin{cases} 6x+30=8y, \\ 3y+17=2-3x. \end{cases}$

4. $\begin{cases} 8x-4y=12x, \\ 4x+2y=3+4y. \end{cases}$

5. $\begin{cases} x+6y=2x-16, \\ 3x-2y=24. \end{cases}$

6. $\begin{cases} 5x+10y=-7, \\ 2x+5y=-2. \end{cases}$

7. $\begin{cases} 5x+3y=-2, \\ 3x+2y=-1. \end{cases}$

8. $\begin{cases} 3a+7b=7, \\ 5a+3b=29. \end{cases}$

9. $\begin{cases} r=3s-19, \\ s=3r-23. \end{cases}$

10. $\begin{cases} 2p=5q-16, \\ 7q=-3p+5. \end{cases}$

11. $\begin{cases} 7m=2n-3, \\ 19n=6m+89. \end{cases}$

12. $\begin{cases} 15k=10-20l, \\ 25k-30l=80. \end{cases}$

13. $\begin{cases} 6c+15d=-6, \\ 21d-8c=-74. \end{cases}$

14. $\begin{cases} 2x-3y=4, \\ 2y-3x=-21. \end{cases}$

15. $\begin{cases} u+v=27, \\ \frac{2}{3}v=19-\frac{3}{4}u. \end{cases}$

16. $\begin{cases} 7a=1+10y, \\ 16y=10a-1. \end{cases}$

17. $\begin{cases} 28x+14y=23, \\ 14x-14y=1. \end{cases}$

18. $\begin{cases} 5x+2y=x+18, \\ 2x+3y=3x+27. \end{cases}$

19. $\begin{cases} 7y-x=x-17, \\ 2y+3x=38. \end{cases}$

20. $\begin{cases} 6x+2y=-2, \\ x-4y=-35. \end{cases}$

ELIMINATION BY SUBSTITUTION

150. Illustrative Problem. Solve the equations:

$$\begin{cases} 2x + 3y = 13. & (1) \\ 5x - 6y = -8. & (2) \end{cases}$$

From (1) $3y = 13 - 2x$, or $$y = \frac{13 - 2x}{3}. \quad (3)$$

Substituting in (2), $$5x - \frac{\overset{2}{\cancel{6}}(13 - 2x)}{\cancel{3}} = -8. \quad (4)$$

By *F*, $$5x - 26 + 4x = -8. \quad (5)$$

By I, *A*, $$9x = 18. \quad (6)$$

By *D*, $$x = 2. \quad (7)$$

From (3), $$y = \frac{13 - 2 \cdot 2}{3} = 3. \quad (8)$$

Verify this by substituting these values of x and y in (1) and (2).

The process here used is called **elimination by substitution.**

This process is convenient when no fraction remains after the substitution. This was the case in the above solution because the coefficient of y in (2) is a multiple of that in (1).

Rule for Elimination by Substitution.

(1) *Solve one equation for one unknown in terms of the other unknown.*

(2) *Substitute the expression for this unknown in the second equation.*

(3) *Solve the resulting equation, thus finding the value of one unknown.*

(4) *Substitute this value in one of the given equations, thus finding the second unknown.*

(5) *Check by substituting the results in both of the given equations.*

ORAL EXERCISES

1. If $x = 6$, what is the value of y in $2x + y = 16$?
2. If $x = 4$, what is the value of y in $x + 3y = 16$?
3. If $x = y$, what is the value of y in $x + 2y = 6$?
4. If $x = y$, what is the value of x in $3x + 2y = 15$?
5. If $x = 2y$, what is the value of y in $x + 2y = 8$?

WRITTEN EXERCISES

Solve the following by substitution:

1. $\begin{cases} x+2y=4, \\ 2x+y=5. \end{cases}$
2. $\begin{cases} 3x-y=5, \\ 5x+2y=23. \end{cases}$
3. $\begin{cases} 2x+y=3, \\ 3x-7y=30. \end{cases}$
4. $\begin{cases} 5y+x=7, \\ 5x-3y=4-2x+7. \end{cases}$
5. $\begin{cases} 5x+8y=-1, \\ 6y-x=4y-7. \end{cases}$
6. $\begin{cases} x-y=37, \\ 2x+3y=31x+13y. \end{cases}$
7. $\begin{cases} 2x-y=y+6, \\ x+2y=4y+3. \end{cases}$
8. $\begin{cases} 5x+3y=0, \\ 2x+y=1. \end{cases}$
9. $\begin{cases} 3x-2y=3, \\ 2x+3y=6x-1. \end{cases}$
10. $\begin{cases} 5x-3y=0, \\ 2x-6y=-x. \end{cases}$

WRITTEN EXERCISES

Solve by either method of elimination:

1. $\begin{cases} x+y=4, \\ x-y=10. \end{cases}$
2. $\begin{cases} x-y=-3, \\ x+4y=12. \end{cases}$
3. $\begin{cases} 2x+3y=5, \\ 7x-5y=33. \end{cases}$
4. $\begin{cases} 3x-4y=8, \\ 4x+3y=-6. \end{cases}$
5. $\begin{cases} 2x-4y=8, \\ 3x+2y=4. \end{cases}$
6. $\begin{cases} 3x-4y=8, \\ 2x+3y=11. \end{cases}$
7. $\begin{cases} 4y-2x=2, \\ 2y+5x=7. \end{cases}$
8. $\begin{cases} 3x-y=2x-1, \\ 12x+y=14. \end{cases}$
9. $\begin{cases} 2x+3y=5, \\ 6x+14y=0. \end{cases}$
10. $\begin{cases} 4x+3y=5, \\ 7x-2y=74. \end{cases}$
11. $\begin{cases} 6y+2x=14, \\ 3y+12x=18. \end{cases}$
12. $\begin{cases} 4y+9x=-5, \\ x+y=-5. \end{cases}$
13. $\begin{cases} 3y+5x=12+2x, \\ 17x-y=4y-20. \end{cases}$
14. $\begin{cases} 6+x+y=2x-1, \\ 3y+x=6y+9. \end{cases}$
15. $\begin{cases} y+5x=2x+5, \\ 2y-3x=19. \end{cases}$
16. $\begin{cases} 6x+2y=22, \\ 10x-5y=20. \end{cases}$

17. $\begin{cases} 3x - 7y = 16, \\ 5x + 4y = 11. \end{cases}$

18. $\begin{cases} 7x - 4y = 3, \\ 5x + 8y = 5. \end{cases}$

19. $\begin{cases} 12y - 10x = -6, \\ 7y + x = 99. \end{cases}$

20. $\begin{cases} 7x - 3y = -7, \\ 5y - 9x = 1. \end{cases}$

21. $\begin{cases} 7x + 4y = 3, \\ 2x + 3y = 25. \end{cases}$

22. $\begin{cases} 34x + 70y = 4, \\ 5x - 8y = -36. \end{cases}$

23. $\begin{cases} 7x + 9y = 8, \\ 2x - 3y = 19. \end{cases}$

24. $\begin{cases} 8x + 4y = 49, \\ 5x - 8y = 28. \end{cases}$

25. $\begin{cases} 4x + 7y = 7, \\ 5x - 2y = 41. \end{cases}$

26. $\begin{cases} 8x + 4y = -28. \\ 3x + 9y = 12. \end{cases}$

SIMULTANEOUS FRACTIONAL EQUATIONS

151. Reduction to Standard Form. The equations thus far given have for the most part been written in a standard form, $ax + by = c$, in which all the terms containing x are collected, likewise those containing y, and those which contain neither x nor y. When the equations are not given in this form, they should be reduced to this form at the outset, as in the following solution:

Example. Solve

$$\begin{cases} \dfrac{7y-4}{5} + \dfrac{2x-3}{2} = \dfrac{3}{2}. & (1) \\[2ex] \dfrac{5x-2}{3} + \dfrac{2y+1}{5} = 2. & (2) \end{cases}$$

Solution. (1) × 10, $14y - 8 + 10x - 15 = 15.$ (3)

Transposing in (3), $14y + 10x = 38.$ (4)

(4) ÷ 2, $7y + 5x = 19.$ (5)

(2) × 15, $25x - 10 + 6y + 3 = 30.$ (6)

Transposing in (6), $25x + 6y = 37.$ (7)

(5) × 5, $25x + 35y = 95.$ (8)

(8) − (7), $29y = 58.$ (9)

(9) ÷ 29, $y = 2.$ (10)

Substituting in (7), $x = 1.$

Check. Substitute $x = 1$, $y = 2$ in (1) and (2) and see that each is satisfied.

EXERCISES

After reducing each of the following pairs of equations to the standard form, solve them by means of either process of elimination:

1. $\begin{cases} \dfrac{7x-15}{3} = y, \\ 2x - y = 3. \end{cases}$

2. $\begin{cases} \dfrac{x-3}{5y} = -2, \\ x + 7y = 6. \end{cases}$

3. $\begin{cases} \dfrac{x+y}{2} + \dfrac{x-y}{2} = 10, \\ 2x - y = 16. \end{cases}$

4. $\begin{cases} \dfrac{x-y}{5} + \dfrac{x+y}{3} = 8, \\ \dfrac{2x-y}{2} + \dfrac{3y-x}{4} = 12\frac{1}{2}. \end{cases}$

5. $\begin{cases} \dfrac{7m+8}{5} - \dfrac{7n-1}{4} = -2, \\ \dfrac{2m-4}{2} + \dfrac{n-1}{3} = -\frac{1}{3}. \end{cases}$

6. $\begin{cases} \dfrac{x-3}{4} + \dfrac{y+8}{5} = 2, \\ \dfrac{x+7}{2} + \dfrac{2y-4}{7} = 5. \end{cases}$

7. $\begin{cases} \dfrac{8a-3}{9} + \dfrac{5b-2}{3} = 13, \\ \dfrac{2a+7}{5} - \dfrac{3b+10}{10} = -3\frac{4}{5}. \end{cases}$

8. $\begin{cases} \dfrac{7y-4}{5} + \dfrac{2x-3}{2} = 7, \\ \dfrac{6x-3}{5} + \dfrac{2y+1}{5} = 7. \end{cases}$

   Ans. $x = 5\frac{3}{16}$, $y = 2\frac{15}{16}$.

9. $\begin{cases} \dfrac{3y+7}{2} - \dfrac{5x-7}{3} = 10, \\ \dfrac{2x-4}{3} - \dfrac{2y-1}{4} = -2\frac{1}{4}. \end{cases}$

10. $\begin{cases} \dfrac{5+3p}{7} - \dfrac{5q-2}{4} = -2, \\ 6p + 8q = 108. \end{cases}$

11. $\begin{cases} 3x - 2y = 4, \\ \dfrac{2x-1}{5} - \dfrac{7y-4}{3} = -19. \end{cases}$

12. $\begin{cases} 5x + 7y = 89\frac{1}{2}, \\ \dfrac{2x-4}{3} + \dfrac{6y-1}{5} = 13\frac{1}{3}. \end{cases}$

13. $\begin{cases} 32x - 9y = 299, \\ \dfrac{2x-5}{7} - \dfrac{3y-1}{2} = -16. \end{cases}$

14. $\begin{cases} 5x - 12y = 4, \\ \dfrac{2x-7}{3} - \dfrac{3y-4}{4} = -2. \end{cases}$

15. $\begin{cases} \dfrac{2x-y}{5} + \dfrac{3x-y}{3} = 8, \\ \dfrac{5x+y}{10} - \dfrac{9x-2y}{5} = 5. \end{cases}$ *Ans.* $\begin{cases} x = 1000, \\ y = 2610. \end{cases}$

152. Equations with Literal Denominators.

Solve the equations:

$$\begin{cases} \dfrac{4}{x-y}+\dfrac{6}{x+y}=\dfrac{36}{x^2-y^2}. & (1) \\[2ex] \dfrac{3}{2x-y}-\dfrac{2}{x-3y}=\dfrac{-18}{(2x-y)(x-3y)}. & (2) \end{cases}$$

Multiplying (1) by the L. C. D., x^2-y^2,

$$4(x+y)+6(x-y)=36. \qquad (3)$$

By F, D, $$5x-y=18. \qquad (4)$$

Multiplying (2) by the L. C. D., $(2x-y)(x-3y)$,

$$3(x-3y)-2(2x-y)=-18. \qquad (5)$$

By F, D, $$x+7y=18. \qquad (6)$$

Multiplying (4) by 7, $$35x-7y=126. \qquad (7)$$

Adding (6) and (7), $$36x=144. \qquad (8)$$

By D, $$x=4. \qquad (9)$$

Substituting, $x=4$ in (6), $$y=2. \qquad (10)$$

Check by substituting $x=4$, $y=2$ in (1) and (2).

1. $\begin{cases} \dfrac{8x+24y}{y-2x}=3, \\[2ex] \dfrac{x-7y}{x+2y+2}=17. \end{cases}$

2. $\begin{cases} \dfrac{x}{x+1}=\dfrac{2y}{2y-3}, \\[2ex] \dfrac{3}{x-1}+\dfrac{5}{y+2}=0. \end{cases}$

3. $\begin{cases} \dfrac{3x+2}{3y-5}=\dfrac{x+1}{y-1}, \\[2ex] \dfrac{3x-2}{y+1}=\dfrac{3x-1}{y-1}-\dfrac{2}{(y-1)(y+1)}. \end{cases}$

4. $\begin{cases} \dfrac{2x-1}{x+1}-\dfrac{3y-1}{y+1}=\dfrac{-xy}{(x+1)(y+1)}, \\[2ex] \dfrac{x+2}{2y-1}+\dfrac{2x-1}{y+1}=\dfrac{5xy}{(2y-1)(y+1)}. \end{cases}$

5. $\begin{cases} \dfrac{x-3}{y-2}-\dfrac{x+2}{y+1}=\dfrac{2}{(y+1)(y-2)}, \\[2ex] \dfrac{2x+1}{2y+5}-\dfrac{x+5}{y+4}=\dfrac{7}{(2y+5)(y+4)}. \end{cases}$

153. Special Case. **In examples like the following it is best not to reduce the equations to the integral form.**

Example. Solve the equations:

$$\begin{cases} \frac{2}{x}+\frac{3}{y}=2, & (1) \\ \frac{20}{x}-\frac{21}{y}=3. & (2) \end{cases}$$

Solve the equations for $\frac{1}{x}$ and $\frac{1}{y}$ instead of for x and y.

Multiplying (1) by 7, $\frac{14}{x}+\frac{21}{y}=14.$ (3)

Adding (2) and (3), $\frac{34}{x}=17.$ (4)

Hence by D, $\frac{1}{x}=\frac{1}{2}.$ (5)

Substituting $\frac{1}{x}=\frac{1}{2}$ in (1), $\frac{1}{y}=\frac{1}{3}.$ (6)

From (5) and (6) by M, $x=2,\ y=3.$ (7)

Try to solve these equations by first clearing of fractions.

WRITTEN EXERCISES

Solve the following equations:

1. $\begin{cases} \frac{1}{x}+\frac{1}{y}=4, \\ \frac{1}{x}-\frac{1}{y}=2. \end{cases}$

2. $\begin{cases} \frac{4}{x}-\frac{7}{y}=-13, \\ \frac{3}{x}+\frac{2}{y}=12. \end{cases}$

3. $\begin{cases} \frac{5}{t}-\frac{6}{v}=2, \\ \frac{17}{v}+\frac{4}{t}=67. \end{cases}$

4. $\begin{cases} \frac{3}{x}+\frac{1}{y}=21, \\ \frac{7}{x}-\frac{9}{y}=-19. \end{cases}$

5. $\begin{cases} \frac{7}{a}-\frac{1}{b}=12\frac{1}{2}, \\ \frac{3}{a}+\frac{12}{b}=24. \end{cases}$

6. $\begin{cases} \frac{3}{x}-\frac{2}{y}=-4, \\ \frac{6}{x}+\frac{11}{y}=52. \end{cases}$

7. $\begin{cases} \frac{12}{x}-\frac{10}{y}=1, \\ \frac{9}{x}+\frac{2}{y}=15. \end{cases}$

8. $\begin{cases} \frac{6}{x}+\frac{2}{y}=1, \\ \frac{1}{x}+\frac{9}{y}=1. \end{cases}$

9. $\begin{cases} \frac{1}{x}+\frac{1}{y}=20, \\ \frac{1}{x}-\frac{1}{y}=30. \end{cases}$

SIMULTANEOUS LITERAL EQUATIONS

154. Examples. 1. The sum of two numbers is 35 and their difference is 5. What are the numbers?

Let x represent one number and y the other. Form two equations and solve them.

2. The sum of two numbers is 48 and their difference is 24. What are the numbers?

3. The sum of two numbers is $41\frac{1}{2}$ and their difference is $23\frac{1}{2}$. What are the numbers?

4. The sum of two numbers is 8590 and their difference is 3480. What are the numbers?

5. If the sum of two numbers is s and their difference is d, find the numbers.

Solution. Let x represent one of the numbers and y the other.

Then, $$x + y = s, \tag{1}$$
$$x - y = d. \tag{2}$$

Solving, we get $$x = \frac{s+d}{2} = \frac{s}{2} + \frac{d}{2},$$

and $$y = \frac{s-d}{2} = \frac{s}{2} - \frac{d}{2}.$$

Translated into words these results are:

One of two numbers is equal to half their sum plus half their difference, and the other is equal to half their sum minus half their difference.

6. Test this general solution by applying it to the following:

$$\begin{cases} s = 48 \\ d = 24 \end{cases}; \quad \begin{cases} s = 8590 \\ d = 348 \end{cases}; \quad \begin{cases} s = 40 \\ d = 52 \end{cases}; \quad \begin{cases} s = 38, \\ d = 50. \end{cases}$$

7. Apply the general solution to the following:

$$\begin{cases} s = \frac{3}{4} \\ d = \frac{1}{4} \end{cases}; \quad \begin{cases} s = 3\frac{1}{3} \\ d = 1\frac{1}{3} \end{cases}; \quad \begin{cases} s = 3\frac{1}{2} \\ d = 1\frac{1}{4} \end{cases}; \quad \begin{cases} s = 5\frac{2}{3}, \\ d = 1\frac{2}{3}. \end{cases}$$

This shows again how the solution of literal equations leads to *formulas*. See pages 196 to 206.

WRITTEN EXERCISES

In the following x and y are the unknowns. Solve for them in terms of the other letters.

1. $\begin{cases} ax + by = 1, \\ cx - dy = 1. \end{cases}$

2. $\begin{cases} 2x - 3y = c - d, \\ 3x - 2y = c + d. \end{cases}$

3. $\begin{cases} mx + 2ny = k, \\ 3 + 2mx = + ny. \end{cases}$

4. $\begin{cases} x + y = a, \\ \dfrac{x}{a} - \dfrac{y}{b} = 1. \end{cases}$

5. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = 1, \\ \dfrac{c}{x} + \dfrac{d}{y} = 1. \end{cases}$

6. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a, \\ \dfrac{1}{x} - \dfrac{1}{y} = b. \end{cases}$

7. $\begin{cases} \dfrac{a}{x} + \dfrac{a}{y} = m, \\ \dfrac{b}{x} - \dfrac{b}{y} = n. \end{cases}$

8. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = r, \\ \dfrac{c}{x} + \dfrac{d}{y} = s. \end{cases}$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

In solving the following problems use two equations if two unknowns are involved.

1. Find two numbers whose sum is 51, such that if the greater is divided by their difference, the quotient is $3\frac{1}{3}$.

2. Find two numbers whose sum is 91, such that if the greater is divided by their difference, the quotient is 7.

3. There are two numbers whose sum is s, such that if the greater is divided by their difference, the quotient is q. Find an expression in terms of s and q representing each number. Solve Examples 1 and 2 by substituting in this formula.

4. There are two numbers whose difference is 153. If their sum is divided by the smaller, the quotient is equal to $\frac{47}{15}$. Find the numbers.

5. There are two numbers whose difference is d. If their sum is divided by the smaller, the quotient is q. Find the numbers. Solve Example 4 by substituting in this formula.

6. Find two numbers whose difference is 320, such that the greater divided by their sum is $\frac{3}{4}$.

7. Find two numbers whose difference is 60, such that the greater divided by their sum is $\frac{7}{8}$.

8. Find two numbers whose difference is $\frac{1}{2}$, such that the greater divided by their sum is $\frac{5}{8}$.

9. Find two numbers whose difference is d, such that the greater divided by their sum is $\frac{c}{b}$.

10. A number has two digits whose sum is s. If the number is divided by the difference between the digits, the quotient is q. Find the number, the tens' digit being the larger.

11. There is a number composed of two digits whose sum is 11. If the number is divided by the difference between the digits, the quotient is $16\frac{3}{5}$. Find the number, the tens' digit being the larger.

155. **Many problems** may be solved in two ways, namely, by using one unknown, or by using two unknowns.

Example. Find two numbers whose sum is 20, such that when one of them is subtracted from twice the other, the remainder is 16.

(*a*) *Using one unknown.* Let x represent one number. Then $20 - x$ is the other number, and the equation is $2x - (20 - x) = 16$.

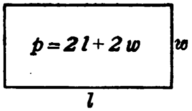
(*b*) *Using two unknowns.* Let x and y represent the two numbers.

Then,
$$\begin{cases} x + y = 20, \\ 2x - y = 16. \end{cases}$$

The translation of problems into equations is usually easier when more than one unknown is permitted. This is due to the fact that in this case each of the given relations between the numbers is put down as a *separate equation.*

PROBLEMS INVOLVING TWO UNKNOWNS

1. If w and l are the width and length of a rectangle, express its perimeter in terms of w and l. Also its area.



2. If the width of the rectangle in the preceding is increased by 10 and its length by 20, express its new perimeter and also its new area in terms of w and l.

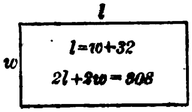
3. If x and y represent the ages of a father and son respectively, represent the sum of their ages 5 years ago in terms of x and y.

4. If a number consisting of two digits is increased by 15 by changing the order of its digits, which is greater, the digit in tens' or in units' place?

5. If a number consisting of two digits is decreased by changing the order of its digits, which is greater, the digit in tens' or in units' place?

6. A rectangular field is 32 rods longer than it is wide. The length of the fence around it is 308 rods. Find the dimensions of the field.

7. Find two numbers such that 7 times the first plus 4 times the second equals 37, while 3 times the first plus 9 times the second equals 45.



8. A certain sum of money was invested at 5 % interest and another sum at 6 %, the two investments yielding \$ 980 per annum. If the first sum had been invested at 6 % and the second at 5 %, the annual income would be \$ 1000. Find each sum invested.

Suggestion. Let x and y represent the sums invested.

Then
$$\begin{cases} \frac{5}{100}x + \frac{6}{100}y = 980. \\ \frac{6}{100}x + \frac{5}{100}y = 1000. \end{cases}$$

9. The combined distance from the sun to Jupiter and from the sun to Saturn is 1369 million miles. Saturn is 403 million miles farther from the sun than Jupiter. Find the distance from the sun to each planet.

10. Find two numbers such that 7 times the first plus 9 times the second equals 116, and 8 times the first minus 4 times the second equals 4.

11. The sum of two numbers is 108. 8 times one of the numbers is 9 greater than the other number. Find the numbers.

12. Two investments of \$24,000 and \$16,000 respectively yield a combined income of \$840. The rate of interest on the larger investment is 1% greater than that on the other. Find the two rates of interest.

Suggestion. $$\begin{cases} x = y + 1, \\ 24{,}000 \times \frac{x}{100} + 16{,}000 \times \frac{y}{100} = 840. \end{cases}$$

13. A father is twice as old as his son. Twenty years ago the father was six times as old as his son. How old is each now?

14. If the length of a rectangle is increased by 3 feet and its width decreased by 1 foot, its area is increased by 3 square feet. If the length is increased by 4 feet and the width decreased by 2 feet, the area is decreased by 3 square feet. What are the dimensions of the rectangle?

Note that if w and l are the original width and length of the rectangle, the term lw will cancel out of both equations.

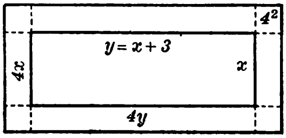
15. A steamer on the Mississippi makes 6 miles per hour going against the current and $19\frac{1}{2}$ miles per hour going with the current. What is the rate of the current and at what rate can the steamer go in still water?

16. One number is three times another If 10 is added to each of the numbers, their quotient is $\frac{5}{11}$. Find the numbers.

17. A starts at 8 A.M. for a walk in the country. At 10 A.M. B starts on horseback to overtake A, which he does at noon. If the rate of B had been one mile per hour less, he would have overtaken A at 1 P.M. At what rate does each travel?

18. A camping party sends a messenger with mail to the nearest post office at 5 A.M. At 8 A.M. another messenger is sent out to overtake the first, which he does in $2\frac{1}{4}$ hours. If the second messenger travels 5 miles per hour faster than the first, what is the rate of each?

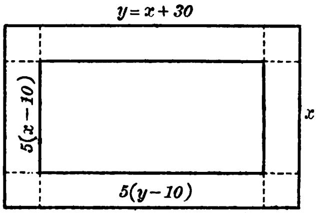
19. There are two numbers such that 3 times the greater is 18 times their difference, and 4 times the smaller is 4 less than twice the sum of the two. What are the numbers?



20. A picture is 3 inches longer than it is wide. The frame, which is 4 inches wide, has an area of 360 square inches. What are the dimensions of the picture?

Suggestion.
$$\begin{cases} y = x + 3. \\ 2\cdot 4x + 2\cdot 4y + 4\cdot 4^2 = 360. \end{cases}$$

21. The difference between two sides of a rectangular wheat field is 30 rods. A farmer cuts a strip 5 rods wide around the field, and finds the area of this strip to be $7\frac{1}{2}$ acres. What are the dimensions of the field?



Suggestion.
$$\begin{cases} y = x + 30. \\ 2\cdot 5(y-10) + 2\cdot 5(x-10) + 4\cdot 5^2 = 7\frac{1}{2}\cdot 160 = 1200. \end{cases}$$

22. In a number consisting of two digits, the sum of the digits is 10. If the order of the digits is reversed, the number is decreased by 54. What is the number?

23. The sum of the length and width of a certain field is 260 rods. If 20 rods are added to the length and 10 rods to the width, the area will be increased by 3800 square rods. What are the dimensions of the field?

24. In a number consisting of two digits the sum of the digits is 12. If the order of the digits is reversed, the number is increased by 36. What is the number?

25. A bird attempting to fly against the wind is blown backward at the rate of $7\frac{1}{2}$ miles per hour. Flying with the wind when it is $\frac{1}{4}$ as strong, the bird makes 48 miles an hour. Find the rate of the wind and the rate at which the bird can fly in calm weather.

26. There is a number whose two digits differ by 2. If the digit in units' place is multiplied by 3 and the digit in tens' place is multiplied by 2, the number is increased by 44. Find the number, the tens' digit being the larger.

27. In a number consisting of two digits the units' digit is equal to twice their difference. If the order of the digits is reversed, the number is increased by 18. Find the number.

28. If the length of a rectangle is doubled and 8 inches added to the width, the area of the resulting rectangle is 180 square inches greater than twice the original area. If the length and width of the rectangle differ by 10, what are its dimensions?

29. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

30. A man rowing against a tidal current drifts back $2\frac{1}{2}$ miles per hour. Rowing with this current, he can make $14\frac{1}{2}$ miles per hour. How fast does he row in still water and how swift is the current?

31. Flying against a wind a bird makes 28 miles per hour, and flying with a wind whose velocity is $2\frac{3}{5}$ times as great, the bird makes 46 miles per hour. What is the velocity of the wind and at what rate does the bird fly in calm weather?

32. A freight train leaves Chicago for St. Paul at 11 A.M. At 3 and 5 P.M. respectively of the same day two passenger trains leave Chicago over the same road. The first overtakes the freight at 7 P.M. the same day, and the other, which runs 10 miles per hour slower, at 3 A.M. the next day. What is the speed of each?

33. Two boys, A and B, trying to determine their respective weights, find that they balance on a teeter board when B is 6 feet and A is 5 feet from the fulcrum. If B places a 30-pound weight on the board beside him, they balance when B is 4 feet and A is 5 feet from the fulcrum. How heavy is each boy?

34. \$ 10,000 and \$ 8000 are invested at different rates of interest, yielding together an annual income of \$ 820. If the first investment were \$ 12,000 and the second \$ 6000, the yearly income would be \$ 840. Find the rates of interest.

SIMULTANEOUS EQUATIONS IN THREE UNKNOWNS

156. An equation of the first degree in x, y, and z is one which contains these letters in such a way that no one of them is multiplied by itself or by any other one. For instance, such an equation must not contain such terms as x^2, y^2, z^2, xy, xz, xyz, etc.

E.g. $2x + 3y - z = 5$ is of the first degree in x, y, and z.

157. Systems of Equations. Two or more equations involving the same unknowns form, when taken together, a *system of equations*. A system of three equations in three unknowns may be *simultaneous* or *contradictory*, *independent* or *dependent*, in the same sense as explained in § 147 for two equations in two unknowns.

Illustrative Problem. Three men were discussing their ages and found that the sum of their ages was 90 years. If the age of the first were doubled and that of the second trebled, the aggregate of the three ages would then be 170. If the ages of the second and third were each doubled, the sum of the three would be 160. Find the age of each.

Solution. Let x, y, and z represent the number of years in their ages in the order named.

Then, $$x + y + z = 90, \quad (1)$$
$$2x + 3y + z = 170, \quad (2)$$
and $$x + 2y + 2z = 160. \quad (3)$$

If we subtract equation (1) from (2), we obtain a new equation from which z is eliminated.

That is, $$x + 2y = 80. \quad (4)$$

Again, multiplying equation (2) by 2 and subtracting (3),
$$3x + 4y = 180. \quad (5)$$

Equations (4) and (5) involve the two unknowns x and y. Solving these by eliminating y, we find $x = 20$. (6)

Substituting $x = 20$ in (4), $y = 30$. (7)

Substituting x and y in (1), $z = 40$. (8)

Check by showing that the values of x, y, and z satisfy the original equations and the conditions of the problem.

The values of x, y, and z as thus found constitute the **solution of the given system** of equations.

Evidently x could have been eliminated first, using equations (1) and (2), and then (1) and (3), giving a new set of two equations in y and z. Let the student find the solution in this manner.

Also find the solution by first eliminating y, using (1) and (2), and then using (2), (3), getting two equations in x and z, from which the values of x and z can be found.

It will be found that the solutions are the same, no matter in what order the equations are combined. This indicates that *a system of three independent and simultaneous equations of the first degree in three unknowns has one and only one solution.*

As in the case of two equations, each should be first reduced to a standard form in which all the terms containing a given unknown are collected and united and all fractions removed.

EXERCISES

Solve the following systems, and check the results:

1. $\begin{cases}2x-y+z=18,\\ x-2y+3z=10,\\ 3x+y-4z=20.\end{cases}$

2. $\begin{cases}5x-3y+z=15,\\ x-3y-z=-3,\\ 2x-y+z=8.\end{cases}$

3. $\begin{cases}4x+2y+z=13,\\ x-y+z=4,\\ x+2y-z=1.\end{cases}$

4. $\begin{cases}6x+4y-4z=-4,\\ 4x-2y+8z=0,\\ x+y+z=4.\end{cases}$

5. $\begin{cases}x+2y+3z=5,\\ 4x-3y-z=5,\\ x+y+z=2.\end{cases}$

6. $\begin{cases}2x-8y+3z=2,\\ x-4y+5z=1,\\ 3x-10y-z=5.\end{cases}$

7. $\begin{cases}x+y+z=1,\\ x+3y+2z=8,\\ 2x+8y-3z=15.\end{cases}$

8. $\begin{cases}2x-3y+z=5,\\ 3x+2y-z=5,\\ x+y+z=3.\end{cases}$

9. $\begin{cases}x+y+z=6,\\ 3x-2y-z=13,\\ 2x-y+3z=26.\end{cases}$

10. $\begin{cases}x+y+z=6,\\ 4x-y-z=-1,\\ 2x+y-3z=-6.\end{cases}$

11. $\begin{cases}2x-3y-4z=17,\\ 4x-4y+2z=-10,\\ 7x+7y+5z=17.\end{cases}$

12. $\begin{cases}x+y+z=0,\\ 5x+3y+4z=-1,\\ 2x-7y+6z=21.\end{cases}$

13. $\begin{cases}x+2y-z=2,\\ 2x-y+z=3,\\ x+2y+z=8.\end{cases}$

14. $\begin{cases}2x-y-z=6,\\ 2x-2y+z=10,\\ x+y-3z=-2.\end{cases}$

15. $\begin{cases}x-y-z=1,\\ 2x+3y+z=20,\\ x-2y+z=0.\end{cases}$

16. $\begin{cases}5x-2y+3z=5,\\ -2x+y-z=-1,\\ -x-y+2z=4.\end{cases}$

17. $\begin{cases}4a-3b+2c=8,\\ a+b-4c=-16,\\ 7a-4b+c=4.\end{cases}$

18. $\begin{cases}5m-4n+r=8,\\ 3m+n-3r=0,\\ 2m-4n+6r=28.\end{cases}$

19. $\begin{cases}9x-4y-z=-4,\\ 2x+5y-6z=-12,\\ -x+2y+4z=30.\end{cases}$

20. $\begin{cases}4x+7y+3z=-3,\\ x-3y+2z=16,\\ 5x+2y-4z=1.\end{cases}$

PROBLEMS INVOLVING THREE UNKNOWNS

158. Illustrative Problem. A broker invested a total of $15,000 in the street railway bonds of three cities, the first investment yielding 3 %, the second $3\frac{1}{2}$ %, and the third 4 %, thus securing an income of $535 per year. If the second investment was one half the sum of the other two, what was the amount of each?

Solution. Suppose x dollars were invested at 3 %, y dollars at $3\frac{1}{2}$ %, and z dollars at 4 %.

$$\begin{array}{lrr}
\text{Then,} & \left\{\begin{array}{r} x + y + z = 15000, \\ .03\,x + .035\,y + .04\,z = 535, \\ x + z = 2\,y. \end{array}\right. & \begin{array}{r}(1)\\(2)\\(3)\end{array} \\
\text{and} & & \\
\text{From equation (3),} & x - 2\,y + z = 0. & (4) \\
\text{Subtracting (4) from (1),} & 3\,y = 15000. & (5) \\
\text{Hence,} & y = 5000. & (6) \\
\text{From (1), by } M, & .035\,x + .035\,y + .035\,z = 525. & (7) \\
\text{Subtracting (7) from (2),} & -.005\,x + .005\,z = 10. & (8) \\
\text{Dividing (8) by .005,} & -x + z = 2000. & (9) \\
\text{Substituting (6) in (4),} & x + z = 10000. & (10) \\
\text{Adding (9) and (10),} & 2\,z = 12000. & (11) \\
 & z = 6000. & (12) \\
\text{Substituting (6) and (12) in (1),} & x = 4000. & (13)
\end{array}$$

Hence, $4000, $5000, and $6000 were the sums invested.

WRITTEN PROBLEMS

Solve the following problems, using three unknowns:

1. The sum of three angles, A, B, and C, of a triangle is 180 degrees. $\frac{1}{3}$ of $A + \frac{1}{4}$ of $B + \frac{1}{5}$ of C is 48 degrees, while $\frac{1}{6}$ of $A + \frac{1}{8}$ of $B + \frac{1}{4}$ of C is 30 degrees. How many degrees in each angle?

2. The combined weight of 1 cubic foot each of compact limestone, granite, and marble is 535 pounds. 1 cubic foot of limestone, 2 of granite, and 3 of marble weigh together 1041 pounds, while 1 cubic foot of limestone and 1 of granite together weigh 195 pounds more than 1 cubic foot of marble. Find the weight per cubic foot of each kind of stone.

3. A number is composed of 3 digits whose sum is 7. If the digits in tens' and hundreds' places are interchanged, the number is increased by 180; and if the order of the digits is reversed, the number is decreased by 99. What is the number?

4. The sum of the angles A, B, and C of a triangle is 180 degrees. If B is subtracted from C, the remainder is $\frac{1}{3}$ of A, and when C is subtracted from twice A, the remainder is 4 times B. How many degrees in each angle?

5. The sum of three numbers, a, b, and c, is 35. Twice a is 5 less than the sum of b and c, and twice c is 4 more than the sum of a and b. What are the numbers?

6. If x is the number of seconds in the Eastern intercollegiate record for a mile run, y the number in the Western record, and z the number in the world's record, then

$$\begin{cases} x + y + z = 768.95, \\ -x + 2y + z = 518.95, \\ 2x - y + z = 502.75. \end{cases}$$

7. If x is the number of seconds in the Eastern intercollegiate record for a half mile run, y the number in the Western intercollegiate record, and z the number in the world's record, then

$$\begin{cases} 2x + 3y + z = 692.9, \\ 3x + 2y + 2z = 804.6, \\ 2x - y + z = 226.5. \end{cases}$$

8. If x is the number of seconds in the world's mile trotting record in 1806, y is the number of seconds in the world's record in 1885, and z is the number of seconds in the world's record in 1911, then

$$\begin{cases} x + y + z = 426.25, \\ 2x + 4y + 6z = 1584, \\ -x + y + 2z = 186.75. \end{cases}$$

9. Diophantus of Alexandria (see page 41) gives the following problem: "Find three numbers such that the sum of each pair is a given number."

It is interesting to note that Diophantus states his problem in words in its general form, but he solves it for a special case; viz., for $x + y = 20$, $y + z = 30$, $z + x = 40$. The Greeks did not use letters to represent numbers in general. Hence they had no formulas such as we now have.

Solve this special case.

10. Solve the preceding problem when the given numbers are a, b, c. That is, solve the system $x + y = a$; $y + z = b$; and $z + x = c$.

REVIEW QUESTIONS

1. Why is a single equation in two unknowns called *indeterminate?*

2. When are two such equations called *simultaneous?* When *independent?* When *contradictory?*

3. May two equations be simultaneous without being independent? May they be independent but not simultaneous?

4. Give the proper description to each of the following systems of equations:

$$(1)\ \begin{cases} x - y = 2 \\ x + y = 10 \end{cases} \quad (2)\ \begin{cases} 2x + y = 7 \\ 6x + 3y = 21 \end{cases} \quad (3)\ \begin{cases} x + 2y = 6 \\ 2x + 4y = 10 \end{cases}$$

5. Describe *elimination* by the process of *addition* or *subtraction;* also by the process of *substitution.* Under what conditions is one or the other of these methods preferable?

6. Describe the solution of a system of three linear equations in three unknowns. Is it immaterial which of the three variables is eliminated first?

7. Can you find a definite solution for two equations such as $4x - 3y - z = 5$ and $x + y + z = 2$?

Eliminate z from these equations. What is the nature of the resulting equation? (See § 147.)

CHAPTER XV

GRAPHIC REPRESENTATION*

159. Graphic Representation of Statistics. A graphic representation of the temperatures recorded on a certain day is shown on the next page. The readings were as follows:

3 P.M.	29°	9 P.M.	21°	3 A.M.	12°	9 A.M.	12°
4 P.M.	29°	10 P.M.	20°	4 A.M.	11°	10 A.M.	13°
5 P.M.	28°	11 P.M.	17°	5 A.M.	10°	11 A.M.	16°
6 P.M.	26°	12 M'T.	16°	6 A.M.	10°	12 Noon	17°
7 P.M.	24°	1 A.M.	14°	7 A.M.	10°	1 P.M.	18°
8 P.M.	22°	2 A.M.	12°	8 A.M.	10°	2 P.M.	20°

In the graph each heavy dot represents the temperature at a certain hour. The distance of the dot from the heavy vertical line indicates the hour of the day counted from noon, and its distance above the heavy horizontal line indicates the thermometer reading at that hour. The lines joining these dots complete the picture representing the gradual changes of temperature from hour to hour.

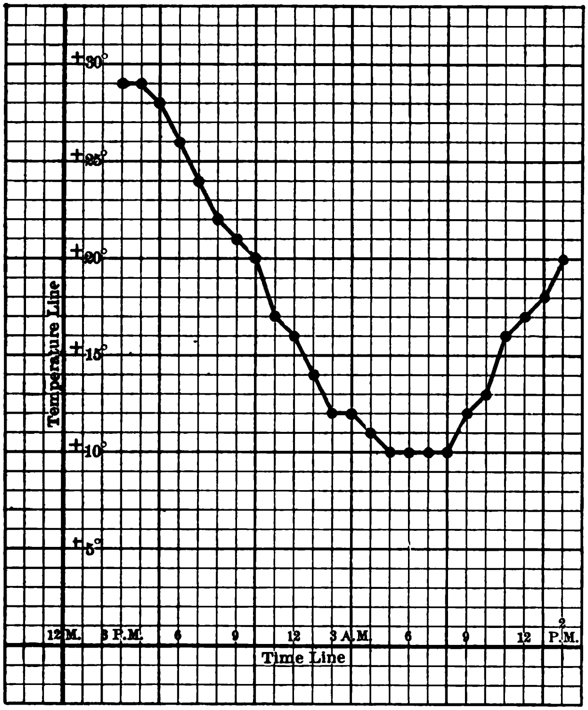
Graphs of this kind are used in commercial houses to represent variations of sales, fluctuations of prices, etc. They are used by the historian to represent changes in population, fluctuations in the amount of mineral productions, etc. In algebra they are used in solving problems and in explaining many difficult processes. In the succeeding exercises cross-ruled paper is essential.

Make a graphic representation of the tables of data on the opposite page:

In each case the number to be represented by one space on the cross-ruled paper should be chosen so as to make the graph go conveniently on a sheet. Thus in Example 1 let one small horizontal space represent two years and one vertical space a million of population; and in Example 3 let one horizontal space represent one year and one large vertical space one hundred thousand of population.

---

* Chapter XV may be omitted without destroying the continuity.



EXERCISES

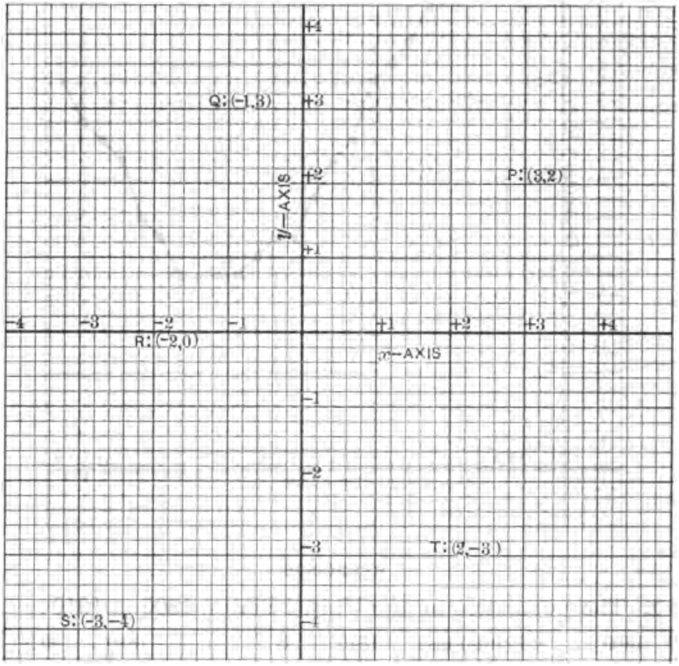
1. The population of the United States, 1800 to 1910:

1800	4.3 (millions)	1840	17.1	1880	50.2
1810	7.2	1850	23.2	1890	62.6
1820	9.6	1860	31.4	1900	76.3
1830	12.9	1870	38.6	1910	92.0

2. The population of the boroughs now constituting Greater New York City:

1800	79 (thousands)	1840	391	1880	1912
1810	119	1850	696	1890	2507
1820	152	1860	1175	1900	3427
1830	242	1870	1478	1910	4767

160. Axes. In the graphs thus far constructed two lines at right angles to each other have been used as reference lines. These lines are called **axes.** The location of a point in the plane of such a pair of axes is completely described by giving its distance and direction from each of the axes.



This scheme of locating points by two reference lines is already familiar to the pupil from geography, where cities are located by latitude and longitude; that is, by degrees north or south of the equator and east or west of the meridian of Greenwich.

The direction to the right of the vertical axis is denoted by a positive sign, and to the left, by a negative sign; while direction upward from the horizontal axis is also called positive, and downward, negative.

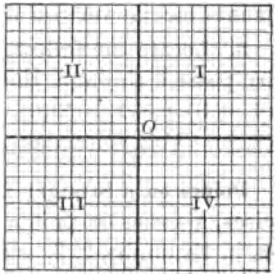
The horizontal line is usually called the **x-axis** and the vertical line the **y-axis**.

Abscissa. The perpendicular distance of any point from the y-axis is called the *abscissa* of the point.

Ordinate. The perpendicular distance of any point from the x-axis is called its *ordinate.*

Coördinates. The abscissa and ordinate of a point are together called its *coördinates.*

E.g. the abscissa of point P in the opposite figure is 3 and its ordinate 2, or we may say the coördinates of P are 3 and 2, and indicate it thus, $P:(3, 2)$, writing the abscissa first. In like manner for the other points we write $Q:(-1, 3)$, $R:(-2, 0)$, $S:(-3, -4)$, and $T:(2, -3)$.



We see that in this manner every point in the plane corresponds to a pair of numbers, and that every pair of numbers corresponds to a point.

Quadrants. The two axes divide the plane into four parts called *quadrants.* These are numbered I, II, III, IV, in counter-clockwise order, as shown in the figure.

Origin. The intersection of the two axes is called the *origin of coördinates.*

Examples. 1. Plot the point $(4, -3)$.

Solution. Since the abscissa is positive, we measure 4 units from the y-axis to the *right.*

Since the ordinate is negative, we measure 3 units *down* from the x-axis.

Hence, the required point is in the fourth quadrant.

The student should use squared paper and plot the point.

2. Plot the point $(-2, -3)$.

Solution. Since the abscissa is negative, we measure 2 units from the y-axis to the *left.*

Since the ordinate is negative, we measure 3 units *down* from the x-axis.

Hence, the point is in the third quadrant.

EXERCISES

1. With any convenient scale, locate the following points: $(2, 6)$, $(-3, 5)$, $(0, 1)$, $(1, 0)$, $(0, 0)$, $(0, -1)$, $(0, -5)$, $(-5, 0)$, $(2\frac{1}{2}, 5\frac{1}{3})$, $(-4, -8)$, $(3, -10)$, $(-10, 3)$.

2. Locate the following series of points and then see if a straight line can be drawn through them: $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(-1, -1)$, $(-2, -2)$, $(-3, -3)$.

3. Locate the following and connect them by a line: $(1, 0)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, -2)$, $(1, -3)$, $(1, -4)$, $(1, -5)$. Name other points in this line.

4. Draw the line every one of whose points has its horizontal distance -2; also draw the line every one of whose points has its vertical distance $+3$.

5. Locate the following points and see if a straight line can be passed through them: $(1, 0)$, $(0, 1)$, $(2, -1)$, $(3, -2)$, $(4, -3)$, $(-1, +2)$, $(-2, 3)$, $(-3, 4)$, $(-4, 5)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{2}{3}, \frac{1}{3})$. Can you name other points on this line?

Points on a Straight Line. In some of the preceding exercises, a series of points has been found to *lie on a straight line*, as in Examples 2, 3, and 5. Evidently this could not happen unless the points were located according to some definite scheme or law.

In Example 5 it is easy to see what the law is; namely, the sum of the abscissa and ordinate is 1 for each point.

Thus, for the first point the sum is $1 + 0 = 1$; for the second point it is $0 + 1 = 1$; for the third, $2 + (-1) = 1$, etc.

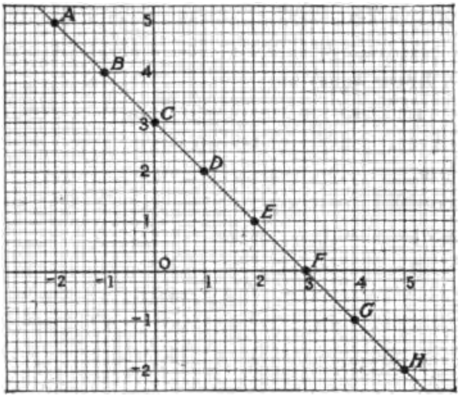
For the last point the sum is $\frac{2}{3} + \frac{1}{3} = 1$, for the next to the last it is $\frac{1}{2} + \frac{1}{2} = 1$, etc.

Hence, in Example 5, if we let x stand for any one of the abscissas, and y for its corresponding ordinate, we have $x + y = 1$ as the law by which all these points are located.

In Example 2, the law is $x = y$, and in Example 3 it is $x = 1$, whatever y may be. In Example 4, the laws are $x = -2$ for the first, and $y = 3$ for the second.

GRAPHIC REPRESENTATION OF EQUATIONS

161. Not only may statistics be represented by graphs, but there is also a way to make a graphic representation of an equation.



Example. Make a graph of the equation $x + y = 3$.

Solution. Writing the equation in the form $y = 3 - x$, we see that for every value which we give to x, we can find a corresponding value of y so as to satisfy the equation $y = 3 - x$. For instance,

$$\text{if } x = 0,\ y = 3 - 0 = 3;$$
$$\text{if } x = 2,\ y = 3 - 2 = 1;$$
$$\text{if } x = 3,\ y = 3 - 3 = 0, \text{ etc.}$$

In this way we may construct the following table of values of x and y, and we may extend this in both directions as far as we like.

If $x =$	-2	-1	0	1	2	3	4	5 etc.,
then $y =$	5	4	3	2	1	0	-1	-2 etc.

If we call each value of x the abscissa of a point and the corresponding value of y its ordinate, this table gives the following points:

A	B	C	D	E	F	G	H
$(-2, 5)$	$(-1, 4)$	$(0, 3)$	$(1, 2)$	$(2, 1)$	$(3, 0)$	$(4, -1)$	$(5, -2)$.

If we plot these points, as in the figure, we see that they all lie on the straight line AH.

Furthermore, it can be shown that all the points whose abscissas and ordinates are values of x and y which satisfy this equation lie on this straight line AH.

Hence the line AH is called the *graph of the equation* $x + y = 3$.

162. Graph of a Linear Equation. If an equation is of the first degree in x and y, all the points whose coördinates are values of x and y which satisfy this equation lie on a straight line. This straight line is called *the graph of the equation.*

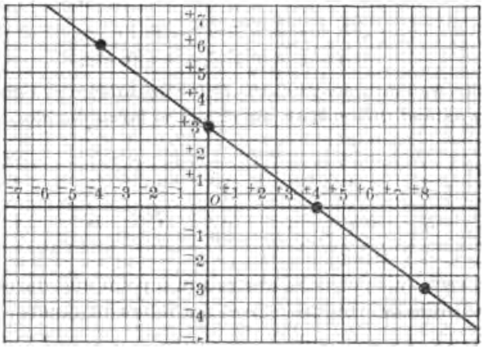
An equation which has a straight line for its graph is called **a linear equation.**

To graph a linear equation it is necessary to locate only two of its points and to draw a straight line through them. If this line is extended indefinitely in both directions, it will pass through all the points whose coördinates satisfy this equation, and through no other points.

E.g. In graphing the equation $x - y = 5$, we choose $x = 0$ and find $y = -5$, and choose $y = 0$ and find $x = 5$ and plot the points $(0, -5)$ and $(5, 0)$. The line through these points is the one required.

Example. Graph the equation $3x + 4y = 12$.

Solution. We see that when $x = 0$, $4y = 12$, and $y = 3$.
Also when $y = 0$, $3x = 12$, and $x = 4$.



Hence two required points are $(0, 3)$ and $(4, 0)$.

Plotting these points, we draw the straight line through them as shown in the figure.

Check. Plot the following points, whose coördinates satisfy the equation, and show that they lie on the graph.

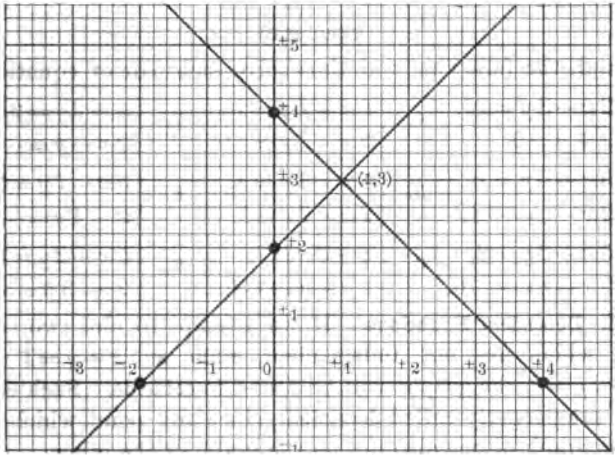
$$\begin{cases} x = 8 \\ y = -3 \end{cases} \quad \begin{cases} x = 12 \\ y = -6 \end{cases} \quad \begin{cases} x = -4 \\ y = 6 \end{cases} \quad \begin{matrix} x = -8 \\ y = 9 \end{matrix}$$

EXERCISES

Construct the graph for each of the following equations:

1. $3x+2y=1.$
2. $5x-3y=-3.$
3. $7x+10y=2.$
4. $x+2y=0.$
5. $5-2y=6x.$
6. $3x+5y=-15.$
7. $2x-y=0.$
8. $3x-4y=7.$
9. $3x-4y=-7.$
10. $3x-4y=-12.$
11. $7y=9x-63.$
12. $x=5y+3.$

163. Graphic Solution. Graph on the same axes $x+y=4$ and $y-x=2$, and thus solve this pair of equations.



Solution. The two graphs are found to intersect in the point (1, 3). Since the point lies on both lines, its coördinates should satisfy both equations, as indeed they do. Since these lines have only one point in common, there is no other pair of numbers which, when substituted for the unknowns x and y, can satisfy both equations.

Check. Solve these equations by elimination, and see if you get these same values for x and y.

Hence, $x=1$, $y=3$ is the **solution of this pair of equations.**

Since $x=1$, $y=3$ represents a point whose abscissa is 1 and whose ordinate is 3, we may write this solution as we would indicate a point; namely, (1, 3).

164. Independent and Simultaneous Equations. Since the graphs of the preceding equations are distinct, the equations are properly called *independent.* Since the graphs intersect, the coördinates of the common point satisfy both equations *simultaneously.*

165. The Graph shows that there is only one Solution. Since two straight lines intersect in but one point, it follows that *two linear equations which are independent and simultaneous have one and only one solution.* Two such equations may be solved by *finding the coördinates of the point where their graphs meet.*

EXERCISES

Graph the following and thus solve each pair of equations:

1. $\begin{cases} 2x - 3y = 25, \\ x + y = 5. \end{cases}$

2. $\begin{cases} 5x + 6y = 7, \\ 2x - y = -4. \end{cases}$

3. $\begin{cases} 5x + 3y = 8, \\ 2x - y = -10. \end{cases}$

4. $\begin{cases} 6x + 8y = 16, \\ 2x - 3y = 11. \end{cases}$

5. $\begin{cases} 3x - 4y = 1, \\ 2x - 7y = 5. \end{cases}$

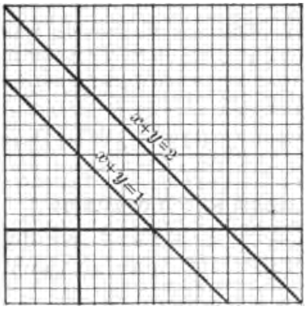
6. $\begin{cases} y + 3x = 7, \\ 2y + x = -6. \end{cases}$

7. $\begin{cases} x - 2y = 2, \\ 2x - y = -2. \end{cases}$

8. $\begin{cases} 5x - 7y = 21, \\ x - 4y = -1. \end{cases}$

9. $\begin{cases} 5x + 2y = 8, \\ 2x - 3y = -12. \end{cases}$

166. Dependent Equations. If we attempt to plot two equations which are not independent, such as $x + y = 1$ and $2x + 2y = 2$, the graphs will be found to **coincide.** Such equations are properly called **dependent,** since any point which lies on the graph of one also lies on the graph of the other.



167. Contradictory Equations. If we attempt to plot two equations which are not simultaneous, such as $x + y = 1$ and $x + y = 2$, the graphs will be found to be parallel and hence they have no point in common. Such equations are properly called **contradictory,** since there is no point which can lie on both graphs.

HISTORICAL NOTE

Graphic Representation of Equations. The representation of equations by means of lines is due to René Descartes (1596–1650). (See next page.) By the time of Descartes the work of Vieta, Harriot, and others had given to algebra the modern form, thus perfecting it as a general instrument of research. The first important use to which it was put was the application to geometry made by Descartes. This must be regarded as one of the greatest contributions of all time to mathematics.

Not only is it possible to graph linear equations as we have done here, but it is found that many important curves of different kinds are graphs of equations of higher degrees. The points where two such curves meet furnish the solutions of the equations of which they are the graphs; and conversely, the algebraic solutions of such equations tell the points where the graphs intersect. This enables us to use the operations of algebra in solving a large range of problems pertaining to lines and curves.

The graphic method is much used by engineers and others in the solution of practical problems.

REVIEW QUESTIONS

1. How may a point in a plane be located by reference to two fixed lines? What are these lines called? What names are given to the distances from the point to the fixed lines?

2. Draw a pair of axes in a plane and locate the following points: $(5, 0)$, $(-2, 0)$, $(0, 3)$, $(0, -1)$, $(0, 0)$.

3. How many pairs of numbers can be found which satisfy the equation $x - 2y = 6$? State five such pairs and plot the corresponding points. How are these points situated with respect to each other? What can you say of all points corresponding to pairs of numbers which satisfy this equation? What is meant by the graph of an equation?

4. How many pairs of numbers will simultaneously satisfy the two equations $3x + 2y = 7$ and $x + y = 3$? Show by means of a graph that your answer is correct.

5. If negative numbers could not be used, how would the graph of $x + y = 3$ be limited? The graph of $x - y = 3$?

CHAPTER XVI

SQUARE ROOTS AND RADICALS

168. Square Root. A **square root** of a number is one of its two equal factors. See § 96.

Thus 3 is a square root of 9, since $3 \cdot 3 = 9$. Similarly $a + b$ is a square root of $a^2 + 2ab + b^2$.

It should be noted that every square **has *two* square roots** which are numerically equal with opposite signs.

E.g. -3 is a square root of 9 as well as $+3$, since $(-3) \cdot (-3) = 9$.

Radical Sign. The positive square root of a number is indicated by the *radical sign* $\sqrt{\ }$ alone or preceded by the sign $+$. The negative square root is indicated by the radical sign preceded by the sign $-$.

E.g. $+\sqrt{9}$ or $\sqrt{9} = +3$ and not -3, and $-\sqrt{9} = -3$, and not $+3$.

The square root of any number is at once evident if we can resolve it into two equal groups of factors.

E.g.

$$\sqrt{576} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3} = \sqrt{(2^3 \cdot 3)(2^3 \cdot 3)} = \sqrt{24 \cdot 24} = 24.$$

ORAL EXERCISES

Find the following indicated square roots:

1. $\sqrt{4}$.
2. $\sqrt{9}$.
3. $-\sqrt{16}$.
4. $\sqrt{25}$.
5. $\sqrt{36}$.
6. $-\sqrt{49}$.
7. $\sqrt{81}$.
8. $\sqrt{121}$.
9. $-\sqrt{169}$.
10. $\sqrt{225}$.
11. $\sqrt{196}$.
12. $-\sqrt{256}$.
13. $-\sqrt{144}$.
14. $\sqrt{400}$.
15. $\sqrt{289}$.
16. $-\sqrt{625}$.
17. $-\sqrt{900}$.
18. $\sqrt{10000}$.
19. $-\sqrt{64}$.
20. $-\sqrt{1600}$.

René Descartes (1596–1650) was born near Tours in France, and died in Stockholm. On leaving school he went to Paris and gave two years to the study of mathematics. After spending some time in the army and in travel he finally settled in Paris and devoted himself to philosophy and mathematics.

In the year 1629 Descartes moved to Holland, in order to pursue his studies without interruption. From that time he lived in seclusion, carrying on his correspondence with learned men through one or two trusted friends who kept his exact whereabouts a profound secret. In 1649 he was invited to Stockholm by Queen Christiana of Sweden, and here he died the following year.

Descartes may be regarded as the first of the modern mathematicians.

Dividing Exponents by 2. *We see that the square root of an even power is obtained by dividing its exponent by 2.*

Thus, we have $\sqrt{a^4} = a^2$, which is $a^{4 \div 2}$.
Similarly, $\sqrt{a^6} = a^3$, which is $a^{6 \div 2}$.
And $\sqrt{a^8} = a^4$, which is $a^{8 \div 2}$.

ORAL EXERCISES

Find the following indicated square roots:

1. $\sqrt{2^4}$.
2. $\sqrt{5^{12}}$.
3. $-\sqrt{7^8}$.
4. $\sqrt{a^{12}}$.
5. $-\sqrt{3^{14}}$.
6. $\sqrt{a^{24}}$.
7. $-\sqrt{3^8}$.
8. $\sqrt{x^{2n}}$.
9. $\sqrt{a^{4n}}$.
10. $\sqrt{4^{2a}}$.
11. $\sqrt{3^{4a}}$.
12. $\sqrt{5^{6ab}}$.
13. $\sqrt{a^{4n}}$.
14. $\sqrt{c^{6m}}$.
15. $\sqrt{x^{12a}}$.
16. $-\sqrt{x^{4a}}$.
17. $\sqrt{y^{6b}}$.
18. $-\sqrt{x^{8c}}$.
19. $\sqrt{m^{16a}}$.
20. $\sqrt{y^{10x}}$.
21. $\sqrt{(a+b)^4}$.
22. $\sqrt{(a+b)^8}$.
23. $-\sqrt{(a+b)^{2n}}$.
24. $\sqrt{(a-b)^6}$.
25. $-\sqrt{(a-b)^{4n}}$.
26. $\sqrt{(a-b)^{10r}}$.
27. $-\sqrt{(a^2-b)^{8c}}$.
28. $-\sqrt{(a-b^2)^{2n}}$.
29. $\sqrt{(a-b+c)^4}$.
30. $\sqrt{(a-b+c)^{8n}}$.

169. The Square Root of a Product. The square root of the product of several factors, each of which is a square, may be found by *taking the square root of each factor separately,* as in the following examples:

(1) $\sqrt{4 \cdot 16 \cdot 25} = \sqrt{4} \cdot \sqrt{16} \cdot \sqrt{25} = 2 \cdot 4 \cdot 5 = 40.$

This is true since $4 \cdot 16 \cdot 25$ can be written as the product of two groups of equal factors $(2 \cdot 4 \cdot 5)(2 \cdot 4 . 5)$. Hence, one of these $(2 \cdot 4 \cdot 5)$ is the square root of $4 \cdot 16 \cdot 25$.

(2) $\sqrt{25\,a^4b^6} = \sqrt{5^2} \cdot \sqrt{a^4} \cdot \sqrt{b^6} = 5\,a^2b^3.$

This is true since $5^2a^4b^6$ can be written as the product of two groups of equal factors $(5\,a^2b^3)(5\,a^2b^3)$. Hence, one of these $(5\,a^2b^3)$ is the square root of $5^2\,a^4b^6$.

From these examples we see that *the square root of a product may be obtained by dividing the exponent of each factor by 2.*

ORAL EXERCISES

Find the following indicated square roots:

1. $-\sqrt{2^2 \cdot 3^2}$.
2. $\sqrt{81 \cdot 121}$.
3. $\sqrt{49 \cdot 25 \cdot 169}$.
4. $-\sqrt{8^2 \cdot 5^2 \cdot 3^2}$.
5. $\sqrt{5^4 \cdot 3^2 \cdot 4^4}$.
6. $\sqrt{25 \cdot 36}$.
7. $-\sqrt{3^{12} \cdot 5^{14}}$.
8. $-\sqrt{2^{22} \cdot 3^{12}}$.
9. $\sqrt{16\,a^2b^2c^2}$.
10. $\sqrt{64\,a^4x^4}$.
11. $-\sqrt{4^4\,a^2b^4}$.
12. $\sqrt{3^2\,x^2y^2}$.
13. $\sqrt{9\,x^4y^{12}}$.
14. $-\sqrt{121\,a^2x^4}$.
15. $-\sqrt{7^4\,a^4b^2}$.

Notice that $\sqrt{9+16}$ is not equal to $\sqrt{9}+\sqrt{16}$.

The preceding exercises illustrate

Principle XVIII

170. Rule. *The square root of a product is obtained by finding the square root of each factor separately and then taking the product of these roots. That is,*

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

HISTORICAL NOTE

Radicals. — The essential elements in the theory of radicals were developed long before the present notation came into use. Alkarismi, the author of the first Arabian algebra (see page 32), states in substance that $a\sqrt{b} = \sqrt{a^2b}$, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ (Principle XVIII).

Nicolas Chuquet, in a French book published in 1484, gives the earliest known use of the radical sign, though the Hindus had used a similar symbol.

Christoff Rudolff, a German writer, used the radical sign in 1525 but he indicated cube root by $\sqrt{\ }\sqrt{\ }\sqrt{\ }$ and fourth root by $\sqrt{\ }\sqrt{\ }$.

Wallis (see page 101) first used fractional exponents to indicate roots.

ORAL EXERCISES

Find the following square roots:

1. $-\sqrt{4\,a^6b^8}$.
2. $-\sqrt{3^2x^{12}y^{14}}$.
3. $\sqrt{5^2 \cdot 3^{22}t^{24}}$.
4. $\sqrt{121\,x^4y^{12}}$.
5. $-\sqrt{64\,a^2b^4}$.
6. $-\sqrt{10^4\,a^4b^4}$.
7. $\sqrt{5^4m^{14}}$.
8. $\sqrt{5^4 \cdot 3^8 \cdot 7^2}$.
9. $\sqrt{3^{14} \cdot 7^{12}\,a^4}$.
10. $-\sqrt{25\,a^2b^4c^{12}}$.
11. $\sqrt{81\,x^4y^8c^{10}}$.
12. $\sqrt{49\,a^6y^{10}z^{14}}$.

THE SQUARE ROOT OF A POLYNOMIAL

171. Relations between a Square and its Square Root. If we square $a+b$ we get $a^2+2ab+b^2$. To find the square root of $a^2+2ab+b^2$ we try to see how $a+b$ can be derived from $a^2+2ab+b^2$.

Step (1) We see that a can be found from a^2 by taking its square root.

Step (2) We see that b can be found from $2ab$ by dividing it by twice the term already found, a.

The steps in the work are arranged as follows:

$$\begin{array}{llcl}
\text{The given square} = & & a^2 + 2ab + b^2 & \underline{\big|\, a + b = \text{the square root.}} \\
\text{The square of } a = & & \underline{a^2 \qquad\qquad\quad} & \\
\text{Trial divisor} & = 2a & 2ab + b^2 & = \text{first remainder.} \\
\text{Complete divisor} & = 2a + b & \underline{2ab + b^2} & = b(2a + b). \\
 & & 0 &
\end{array}$$

Explanation. Having found a, as indicated in step (1) above, we subtract its square from $a^2 + 2ab + b^2$.

The remainder begins with $2ab$, which we use to find b, as in step (2) above.

That is, we multiply a by 2, and divide $2ab$ by this product, $2a$, which we call the *trial divisor*, thus getting b, the second term of the root.

Now, the remainder, $2ab + b^2$, may be written $(2a + b)b$. Hence, if we add b to $2a$ (calling $2a + b$ the *complete divisor*), and multiply this by b, the product, $2ab + b^2$, is the rest of the square.

Thus, in finding the terms, a and b, of the root, we really build up the whole of the square and subtract it piece by piece. The remainder, zero, indicates that the square root is exact.

Example. Find the square root of $36x^2 - 84xy + 49y^2$.

$$\begin{array}{llcl}
\text{Given square} = & & 36x^2 - 84xy + 49y^2 & \underline{\big|\, 6x - 7y = \text{the root.}} \\
\text{Square of } 6x = & & \underline{36x^2 \qquad\qquad\qquad} & \\
\text{Trial divisor} & = 2 \cdot 6x & -84xy + 49y^2. & \\
\text{Complete divisor} & = 12x - 7y & \underline{-84xy + 49y^2} & = -7y(12x - 7y). \\
 & & 0 &
\end{array}$$

Explanation. The first term of the root is $6x$ since $\sqrt{36x^2} = 6x$. The product of twice $6x$ and the second term of the root is $-84xy$. Hence, the second term of the root is $-84xy \div 12x = -7x$. The complete divisor is then $12x - 7y$, and the rest of the square is $-7y(12x - 7y) = -84xy + 49y^2$.

EXERCISES

Find the square root of each of the following:

1. $16x^4 - 64x^2y^2 + 64y^4$.
2. $64x^2 - 32x + 4$.
3. $25 + 49x^2 - 70x$.
4. $a^{10} + 6a^5b + 9b^2$.
5. $36x^2 - 84x + 49$.
6. $121 - 44x^4 + 4x^8$.
7. $9a^2 - 30ab^2 + 25b^4$.
8. $81a^2 - 216a + 144$.
9. $4a^2b^4 - 44ab^2 + 121$.
10. $81 - 270y + 225y^2$.

172. Rule for Finding the Square Root of a Polynomial. From the squares

$$(a+b)^2 = a^2 + 2ab + b^2,$$

$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2,$$

$$(a+b+c+d)^2 = [(a+b+c)+d]$$
$$= (a+b+c)^2 + 2(a+b+c)d + d^2, \text{ etc.,}$$

we see that

(1) If to a^2 we add $2ab + b^2 = (2a+b)b$, we get $(a+b)^2$;

(2) If to $(a+b)^2$ we add $2(a+b)c + c^2$
$= [2(a+b)+c]c$, we get $(a+b+c)^2$;

(3) If to $(a+b+c)^2$ we add $2(a+b+c)d + d^2$
$= [2(a+b+c)+d]d$, we get $(a+b+c+d)^2$, etc.

Hence, in squaring a polynomial:

*For every new term added to the root there is a new **part** added to the power. This new part consists of twice the sum of the **preceding** terms of the root plus the **last** term of the root, all multiplied by the last term of the root.*

This is expressed by the formula:

$$\mathbf{(a+b+c+d)^2}$$
$$\mathbf{= a^2 + (2a+b)b + [2(a+b)+c]c + [2(a+b+c)+d]d.}$$

WRITTEN EXERCISES

Write the following squares in the above form:

1. $(x+y+z)^2$.
2. $(x+y+z+v)^2$.
3. $(x+2y+3z+4v)^2$.
4. $(2a+b+3c+d)^2$.
5. $(b+4c+2d+e)^2$.
6. $(x+2a+3b+c)^2$.

Example 1. Find the square root of

$$9x^4 - 12x^3 + 28x^2 - 16x + 16.$$

Solution.

$$\begin{array}{lr}
 & a \;+\; b \;+ c \qquad\qquad\qquad\qquad\qquad\quad \\
\text{Square root} = & \underline{3x^2 - 2x + 4 \qquad\qquad\qquad\qquad\qquad} \\
\text{Given square} = & 9x^4 - 12x^3 + 28x^2 - 16x + 16 \\
a^2 = (3x^2)^2 = & \underline{9x^4 \qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
2a = 2 \cdot 3x^2 = 6x^2 & -12x^3 + 28x^2 - 16x + 16 \\
(2a + b)b = (6x^2 - 2x)(-2x) = & \underline{-12x^3 + \;\;4x^2 \qquad\qquad\qquad} \\
2(a + b) = 6x^2 - 4x & 24x^2 - 16x + 16 \\
[2(a + b) + c]c = (6x^2 - 4x + 4) \cdot 4 = & \underline{24x^2 - 16x + 16} \\
 & 0
\end{array}$$

Explanation. The first term of the root is $\sqrt{9x^4} = 3x^2$. This is subtracted from the square.

The second term of the root is $-12x^3 \div (2 \cdot 3x^2) = -2x$.

The second part of the square is $(6x^2 - 2x)(-2x) = -12x^3 + 4x^2$, corresponding to $(2a + b)b$. This is now subtracted.

The third term of the root is $24x^2 \div (2 \cdot 3x^2) = 4$.

The third part of the square is $(6x^2 - 4x + 4) \cdot 4 = 24x^3 - 16x + 16$, corresponding to $[2(a + b) + c]c$. This is now subtracted.

Thus at each step a new term of the root is found by dividing the first term of the remainder by twice the first term of the root; and then a new part of the power is built up and subtracted.

Since the final remainder is zero, the square root is exact.

Example 2. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

Solution.

$$\begin{array}{lr}
 & a \qquad +b \qquad +c+d \qquad\qquad\qquad\qquad\qquad\qquad\quad \\
\text{Square root} & \underline{4x^3 - 3x^2 + 2x - 5 \qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
\text{Given square} & 16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
a^2 = (4x^3)^2 = & \underline{16x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
 & -24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
(2a + b)b = & \underline{-24x^5 + \;\;9x^4 \qquad\qquad\qquad\qquad\qquad\qquad} \\
 & 16x^4 - 52x^3 + 34x^2 - 20x + 25 \\
(2(a + b) + c)c = & \underline{16x^4 - 12x^3 + \;\;4x^2 \qquad\qquad\qquad} \\
 & -40x^3 + 30x^2 - 20x + 25 \\
[2(a + b + c) + d]d = & \underline{-40x^3 + 30x^2 - 20x + 25}
\end{array}$$

Explanation. In this solution only the successive parts of the formula are written down. Let the student give the explanation in full.

From the preceding examples we have the following

Rule: (1) *Arrange the polynomial according to ascending or descending powers of some letter.*

(2) *Find the square root of the first term, and write it as the first term of the root.*

(3) *Subtract the square of this first term of the root.*

(4) *Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.*

(5) *Add this second term of the root to twice the first term, and multiply the sum by the second term. This product is the second part of the square and is to be subtracted.*

(6) *Now use the sum of the first two terms of the root to find the third term, just as the first term was used to find the second; and continue in this manner till all the terms of the root are found.*

EXERCISES

Find the square root of each of the following:

1. $x^4+2x^3+3x^2+2x+1$.
2. $1-2a+3a^2-2a^3+a^4$.
3. $1+2b-b^2-2b^3+b^4$.
4. $a^4+4a^3+6a^2+4a+1$.
5. $c^4-4c^3+6c^2-4c+1$.
6. $x^4-2x^3+5x^2-4x+4$.
7. $a^4+4a^3b+6a^2b^2+4ab^3+b^4$.
8. $x^4-4x^3y+6x^2y^2-4xy^3+y^4$.
9. $a^4+53a^2+14a^3+28a+4$.
10. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1$.
11. $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.
12. $4x^6-12x^5+13x^4-14x^3+13x^2-4x+4$.
13. $16a^6+24a^5+25a^4+20a^3+10a^2+4a+1$.
14. $x^6y^6+2x^5y^5+3x^4y^4+4x^3y^3+3x^2y^2+2xy+1$.
15. $1+2x+3x^2+4x^3+5x^4+4x^5+3x^6+2x^7+x^8$.

THE SQUARE ROOT OF AN ARITHMETIC NUMBER

173. Rule for Finding the First Term of the Square Root of an Integral Number. Since $1^2 = 1$ and $9^2 = 81$, the square of a number of one figure contains either *one* or *two* figures.

Since $10^2 = 100$ and $99^2 = 9801$, the square of a number of two figures contains either *three* or *four* figures.

Similarly, the square of a number of three figures contains either *five* or six *figures*, and so on.

Rule. Hence, to find the first figure in the root

(1) *Separate the number into groups of two figures each, counting from units' place toward the left. The last group may contain only one figure.*

(2) *Take the square root of the largest square in the left hand group. This is the first figure of the root, and there are as many figures in the root as there are groups in the number.*

Examples. 1. To find the first figure and the number of figures in the square root of 450,769 we write it thus 45 07 69.

Since there are three groups of two figures each, the square root contains three figures, and hence it starts with the hundreds' figure.

Since 36 is the largest square in 45, the first figure in the root is $\sqrt{36} = 6$.

2. Similarly, the square root of 6,762,436, written 6 76 24 36, contains four figures of which the first one is thousands' figure.

Since 4 is the largest square in 6, the first figure of the root is $\sqrt{4} = 2$.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and state whether it stands in units', tens' or hundreds' place:

1. 8947.
2. 6205.
3. 19,140.
4. 72,048.
5. 90,401.
6. 63,401.
7. 1428.
8. 194,670.
9. 7347.
10. 73,470.
11. 14,051.
12. 140,051.
13. 107.
14. 4091.
15. 10,007.
16. 100,007.

The square root of an arithmetic number may be found by the process just used for polynomials, if we remember that a number like 637 is really a polynomial, namely $600 + 30 + 7$.

Illustrative Example 1. Find the square root of 405769.

Solution.

$$\begin{array}{rrl}
 & \text{SQUARE} & \text{SQUARE ROOT} \\
 & & \quad a + b + c \\
 & 40\ 57\ 69 & \underline{|600 + 30 + 7} = 637 \\
a^2 = \underline{600^2 =} & \underline{36\ 00\ 00} & \\
2a = 1200 & 4\ 57\ 69 & \\
\underline{b = \quad 30} & & \\
2a+b = \underline{1230} & \underline{3\ 69\ 00} & =(2a+b)b \\
2(a+b) = 1260 & 88\ 69 & \\
\underline{c = \quad 7} & & \\
2(a+b)+c = \underline{1267} & \underline{88\ 69} & =[2(a+b)+c]c \\
 & 0 &
\end{array}$$

Explanation. The first figure in the root is the square root of the largest square in the left hand group, and since there are three groups, the root starts with 600, which corresponds to a of the formula (§ 172).

Subtracting the square of 600 we have 4 57 69.

The trial divisor is $2a = 1200$ and when 4 57 69 is divided by 1200, the largest number of tens in the quotient is 3. Hence 30 corresponds to b of the formula.

The complete divisor is $2a + b = 1230$, and this multiplied by b gives $(2a + b)b = 36900$, which is the second part of the square. Subtracting 36900, the remainder is 88 69.

The next trial divisor is $2(a + b) = 1260$ and $8869 \div 1260$ gives 7 as the largest number of units. This is c of the formula. Then

$$2(a + b) + c = 1267,$$

and this multiplied by 7 gives $[2(a + b) + c]c = 88\ 69$.

The remainder is now zero, and hence the square root is

$$600 + 30 + 7 = 637.$$

In case a square consists of a **whole number and a decimal part** the figures in the *integral part* of the square root are found exactly as in Example 1 above. To find the *decimal part* of the root, we proceed as in the next illustrative example.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and tell in which place it stands:

1. 12.645.
2. 1.2645.
3. 126.45.
4. 941.61.
5. 94.16.
6. 9.416.
7. 49.29.
8. 4.929.
9. 492.9.

Illustrative Example 2. Find the square root of 67.7329.

Solution.

$$\begin{array}{rrl}
 & \text{Square} & \text{Square Root} \\
 & & a+b+c \\
 & 67.73\ 29 & 8+.2+.03=8.23 \\
a^2 = \underline{8^2 =} & \underline{64} & \\
2a = 16 & 3.73\ 29 & \\
\underline{b = .2} & & \\
2a+b = \underline{16.2} & \underline{3.24} & =(2a+b)b \\
2(a+b) = 16.4 & .49\ 29 & \\
\underline{c = .03} & & \\
2(a+b)+c = \underline{16.43} & \underline{.49\ 29} & =[2(a+b)+c]c \\
 & 0 &
\end{array}$$

Explanation. There is only one group of two figures to the left of the decimal point. Hence the first figure of the root is units' figure. ***Since the square of a decimal contains twice as many decimal places as the number itself, there will be one decimal figure in the root for every two in the square.***

In getting the second figure of the root, the trial divisor is $2a = 16$. The quotient is $b = .2$ since $.2 \times 16 = 3.2$. The quotient could not be .3 since $.3 \times 16 = 4.8$.

Similarly, in getting the third figure, we divide .4929 by 16.4 and the quotient is .03 since $.03 \times 16.4 = .492$.

Illustrative Example 3. Find the square root of 9.1204

Solution.

$$\begin{array}{rrl}
 & \text{Square} & \text{Square Root} \\
 & & a+b \\
 & 9.1204 & 3+.02=3.02 \\
a^2 = \underline{3^2 =} & \underline{9\ \cdot} & \\
2a = 6 & .12\,04 & \\
\underline{b = .02} & & \\
2a+b = \underline{6.02} & \underline{.12\,04} & =(2a+b)b \\
 & 0 &
\end{array}$$

Explanation. Since there is only one group to the left of the decimal point, the first figure of the root is in units' place.

In this case in dividing .1204 by 6, the quotient is .02 since $.02 \times 6 = 12$; that is, there is a zero in tenths' place, and there are only two terms to the root.

EXERCISES

Find the square root of each of the following:

1. 294,849.
2. 37,636.
3. 872,356.
4. 599,076.
5. 3481.
6. 7569.
7. 1849.
8. 73,441.
9. 100,489.
10. 265.69.
11. 87.4225.
12. 170,569.
13. 357.21.
14. 16,641.
15. 32,761.
16. 2332.89.

174. The First Digit of the Square Root of a Decimal Number. In case a number has no integral part, the first term of its square root is found as in the following examples:

1. In .1742 the first digit in the root is .4 since the square of .4 is .16, the largest square in .17.

2. In .0542 the first digit in the root is .2 since the square of .2 is .04, the largest square in .05.

3. In .0070 the first digit in the root is .08 since the square of .08 is .0064, the largest square in .0070.

4. In .0007 the first digit in the root is .02 since the square of .02 is .0004, the largest square in .0007.

From these examples we have the following

Rule. To find the first digit in the square root of a decimal number:

(1) *Divide the number into groups of two figures each counting from the decimal point toward the right, adding a zero if necessary to complete the last group.*

(2) *Take the square root of the largest square contained in the first group which is not all zeros, and prefix to it as many zeros as there are complete groups of zeros to the right of the decimal point.*

For instance, in the above Examples 1 and 2, there are no groups of zeros to the right of the decimal point. Hence the first digit in the root in each case is tenths' digit.

In Examples 3 and 4, there is one whole group of zeros to the right of the decimal point. Hence the first digit of the root is hundredths' digit.

Similarly, when there are two complete groups of zeros to the right of the decimal point, the first digit in the root is thousandths' digit.

Illustrative Example. Find the square root of .06783.

Solution.

		SQUARE	SQUARE ROOT
			$a + b + c$
		.06 78 30	$.2 + .06 + .0004$
$a^2 = .2^2$		$= .04$	
$2a = 2 \times .2 =$	.4	.02 78	
$b =$	.06		
$2a + b =$	.46	.02 76	$= (2a + b)b$
$2(a + b) =$	.52	.00 02 30 00	
$c =$	.0004		
$2(a + b) + c =$	.5204	.00 02 08 16	$= [2(a + b) + c]c$
		.00 02 21 84	

Explanation. According to the rule, .2 is the first term of the root because 4 is the largest square in 6 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in the process of division in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by 2(.26) = .52 and the quotient lies between .0004 and .0005. Hence $c = .0004$. Zeros are annexed to .00023 to correspond to the number of decimal places in the product .5204 × .0004.

The three terms of the root thus found are .2 + .06 + .0004 = .2604.

To find the next term of the root we would divide .00002184 by 2(.2604) = .5208, finding the quotient .00004. We would then add .00004 to .5208 and multiply the sum by .00004, annexing zeros to the dividend as before.

175. Approximate Square Roots. Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783; in fact, the square of .2604 differs from .06783 by only .00002184. The nearest approximation using three decimal places is .260. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places; and five places must be found in order to be sure of the nearest approximation to four places, and so on.

EXERCISES

Find the square root of each of the following, correct to two decimal places:

1. 387.
2. 5276.
3. 2.92.
4. 27.29.
5. 51.
6. 3.824.
7. 2.
8. 3.
9. 5.
10. 7.
11. 8.
12. 11.
13. .02.
14. .003.
15. .5.
16. .005.

SIMPLIFYING SQUARE ROOTS

176. Approximate Square Root of a Whole Number. When we wish to approximate the square root of a number such as 8 we make use of Principle XVIII as follows:

$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$. We then find the square root of 2 and multiply by 2.

Similarly $$\sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3},$$

$$\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}; \quad \sqrt{32} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2};$$

$$\sqrt{a^3} = \sqrt{a^2} \cdot \sqrt{a} = a\sqrt{a}; \quad \sqrt{a^5} = \sqrt{a^4} \cdot \sqrt{a} = a^2\sqrt{a}; \quad \sqrt{a^2y^4b} = ay^2\sqrt{b}.$$

In general *if the expression under the radical sign contains a factor which is a square this factor may be removed by writing its square root before the radical sign.*

In this manner, the square roots of a few small numbers like 2, 3, 5, etc., are made to do service in finding the roots of many large numbers.

ORAL EXERCISES

Change the following so as to leave no factor which is a square under the radical sign.

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{40}$.
4. $\sqrt{18}$.
5. $\sqrt{20}$.
6. $\sqrt{24}$.
7. $\sqrt{28}$.
8. $\sqrt{32}$.
9. $\sqrt{27}$.
10. $\sqrt{50}$.
11. $\sqrt{72}$.
12. $\sqrt{45}$.
13. $\sqrt{54}$.
14. $\sqrt{a^3}$.
15. $\sqrt{a^5}$.
16. $\sqrt{a^7}$.

17. $\sqrt{a^9}$.
18. $\sqrt{a^{11}}$.
19. $\sqrt{a^{17}}$.
20. $\sqrt{a^{21}}$.
21. $\sqrt{a^2b}$.
22. $\sqrt{a^4b}$.
23. $\sqrt{a^6b}$.
24. $\sqrt{a^3b}$.
25. $\sqrt{a^5b}$.
26. $\sqrt{a^2b^3}$.
27. $\sqrt{a^2b^5}$.
28. $\sqrt{a^3b^3}$.
29. $\sqrt{a^3b^4}$.
30. $\sqrt{a^3b^6}$.
31. $\sqrt{a^2b^2x}$.
32. $\sqrt{a^2b^4x}$.
33. $\sqrt{a^3b^4x}$.
34. $\sqrt{a^4b^3x^2}$.
35. $\sqrt{a^{2n}b}$.
36. $\sqrt{a^{2n}b^{4n}x}$.
37. $\sqrt{a^3b^5x}$.
38. $\sqrt{a^{2n}b^5x^3}$.
39. $\sqrt{8\,a^2b^2}$.
40. $\sqrt{18a^3b}$.

177. Approximate Square Root of a Fraction. A fraction is squared by squaring its numerator and its denominator separately, since $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$. Hence, to extract the square root of a fraction, we find the square root of its numerator and denominator separately.

E.g. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, since $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

When we wish to approximate the square root of a fraction, such as $\frac{1}{2}$, we make use of Principle XVIII as follows, in order to get an integer instead of a fraction under the radical sign:

$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{2}{4}} = \sqrt{\tfrac{1}{4} \cdot 2} = \sqrt{\tfrac{1}{4}} \cdot \sqrt{2} = \tfrac{1}{2}\sqrt{2};$$

$$\sqrt{\tfrac{2}{5}} = \sqrt{\tfrac{10}{25}} = \sqrt{\tfrac{1}{25}} \cdot \sqrt{10} = \tfrac{1}{5}\sqrt{10}.$$

In each case after multiplying both terms of the fraction by the denominator, the fraction is resolved into two factors, one of which is a perfect square.

In this way instead of getting the square root of both numerator and denominator of such a fraction as $\frac{2}{5}$, we get the square root of 10 and divide the result by 5.

In general,

$$\sqrt{\frac{1}{a}} = \sqrt{\frac{a}{a^2}} = \sqrt{\frac{1}{a^2} \cdot a} = \sqrt{\frac{1}{a^2}} \cdot \sqrt{a} = \frac{1}{a}\sqrt{a};$$

$$\sqrt{\frac{b}{a}} = \sqrt{\frac{ab}{a^2}} = \sqrt{\frac{1}{a^2} \cdot ab} = \sqrt{\frac{1}{a^2}} \cdot \sqrt{ab} = \frac{1}{a}\sqrt{ab}.$$

178. Rationalizing the Denominator. Changing a radical expression so as to leave no denominator under a radical sign is called **rationalizing the denominator**; thus, $\sqrt{\frac{1}{a}} = \frac{1}{a}\sqrt{a}$.

ORAL EXERCISES

Rationalize the denominator in each of the following:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{\frac{1}{6}}$.
5. $\sqrt{\frac{2}{3}}$.
6. $\sqrt{\frac{2}{5}}$.
7. $\sqrt{\frac{5}{6}}$.
8. $\sqrt{\frac{1}{c}}$.
9. $\sqrt{\frac{1}{b}}$.
10. $\sqrt{\frac{a}{2}}$.
11. $\sqrt{\frac{x}{3}}$.
12. $\sqrt{\frac{a}{5}}$.
13. $\sqrt{\frac{a}{6}}$.
14. $\sqrt{\frac{2}{a}}$.
15. $\sqrt{\frac{3}{a}}$.
16. $\sqrt{\frac{5}{a}}$.
17. $\sqrt{\frac{3}{2}}$.
18. $\sqrt{\frac{3}{2a}}$.
19. $\sqrt{\frac{3a}{2}}$.
20. $\sqrt{\frac{3}{5}}$.

179. In rationalizing a denominator we should multiply the terms of the fraction by the *smallest number which will make the denominator a perfect square.*

Thus, $\sqrt{\frac{1}{a^3}} = \sqrt{\frac{a}{a^4}} = \sqrt{\frac{1}{a^4}} \cdot \sqrt{a} = \frac{1}{a^2}\sqrt{a}$.

Similarly, $\sqrt{\frac{x}{a^3b^3}} = \sqrt{\frac{abx}{a^4b^4}} = \sqrt{\frac{1}{a^4b^4}} \cdot \sqrt{abx} = \frac{1}{a^2b^2}\sqrt{abx}$.

Again, $\sqrt{\frac{1}{8}} = \sqrt{\frac{2}{16}} = \sqrt{\frac{1}{16} \cdot 2} = \sqrt{\frac{1}{16}} \cdot \sqrt{2} = \frac{1}{4}\sqrt{2}$.

If the fraction is of the form $\frac{a}{\sqrt{b}}$, we multiply both numerator and denominator by $\sqrt{b}$; thus

$$\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b} \cdot \sqrt{b}} = \frac{a\sqrt{b}}{b}, \quad \text{since } \sqrt{b} \cdot \sqrt{b} = \sqrt{b^2} = b.$$

E.g. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3}\,\sqrt{3}} = \frac{\sqrt{3}}{3}$.

ORAL EXERCISES

In the manner just indicated rationalize the denominators in the following:

1. $\sqrt{\frac{1}{a^3}}$.
2. $\sqrt{\frac{1}{a^5}}$.
3. $\sqrt{\frac{1}{a^7}}$.
4. $\sqrt{\frac{b}{a^3}}$.
5. $\frac{1}{\sqrt{2}}$.
6. $\sqrt{\frac{1}{5}}$.
7. $\sqrt{\frac{1}{7}}$.
8. $\frac{1}{\sqrt{10}}$.
9. $\sqrt{\frac{1}{ab}}$.
10. $\frac{1}{\sqrt{abx}}$.
11. $\sqrt{\frac{a}{x^4y}}$.
12. $\sqrt{\frac{5}{ab}}$.

180. Simplified Form of Radicals. An expression in one of the forms $\sqrt{a^2b}$, $\frac{a}{\sqrt{b}}$, $\sqrt{\frac{a}{b}}$, is said to be *simplified* when it is reduced (1) so that no radical occurs in a denominator, and (2) so that no factor which is a perfect square remains under the radical sign.

Such radical expressions may always be simplified by Principle XVIII.

WRITTEN EXERCISES

Given $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, compute each result in Examples 1 to 15, correct to two places of decimals, without further extraction of roots:

1. $\sqrt{80}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt{48}$.
5. $\sqrt{75}$.
6. $\sqrt{2 \cdot 3}$.
7. $\sqrt{72}$.
8. $\sqrt{98}$.
9. $\sqrt{363}$.
10. $\sqrt{125}$.
11. $\sqrt{27} + \sqrt{\frac{1}{3}}$
12. $\sqrt{45} + \sqrt{\frac{1}{5}}$.
13. $\sqrt{50} - \sqrt{\frac{1}{2}} + \sqrt{8}$.
14. $\sqrt{48} + \sqrt{75} - \sqrt{3}$.
15. $\sqrt{32} + \sqrt{72} - \sqrt{18}$.

Simplify the following:

16. $\sqrt{32\,a^2b}$.
17. $\sqrt{81\,x^3b^2}$.
18. $\sqrt{50\,a^5b^4c^2}$.
19. $\sqrt{45\,x^3y^5b^3}$.
20. $\sqrt{63\,bc^5d^4}$.
21. $\sqrt{900\,ab^4c^5}$.
22. $\sqrt{500\,x^7a^3b}$.
23. $\sqrt{3x^2 + 6xy + 3y^2}$.
24. $\sqrt{8x^2 - 12y^2}$.

EQUATIONS SOLVED BY SQUARE ROOTS

181. Since $2^2 = 4$ and also $(-2)^2 = 4$, it follows that the equation $x^2 = 4$ has two roots, namely $x = 2$ and $x = -2$. These are usually written $x = \pm 2$.

This solution is obtained by taking the square root of both sides, that is, by dividing both sides by the same number.

This operation may now be added to those enumerated in Principle VI for the solution of equations.

ORAL EXERCISES

Find all roots of the following equations:

1. $x^2 = 9$.
2. $x^2 = 16$.
3. $x^2 = 25$.
4. $x^2 = 36$.
5. $x^2 = 49$.
6. $x^2 = 64$.
7. $x^2 = 81$.
8. $x^2 = 100$.
9. $x^2 = 8$.
10. $x^2 = 12$.
11. $x^2 = 9a^2$.
12. $x^2 = 8a^2$.
13. $x^2 = 16a^2$.
14. $x^2 = 49b^2$.
15. $x^2 = 25a^2$.
16. $x^2 = 81a^2b^2$.
17. $x^2 = 49a^4b^2$.
18. $x^2 = 9a^2b$.
19. $x^2 = 25(a + b)^2$.
20. $x^2 = 50(a - b)^2$.
21. $x^2 = 64am^6$.
22. $x^2 = 36r^2s^4$.
23. $x^2 = 81s^8r^6$.
24. $x^2 = 50$.
25. $x^2 = 72$.
26. $x^2 = 98$.
27. $x^2 = 32$.
28. $x^2 = 49a^5$.
29. $x^2 = 36a^3b^5$.
30. $x^2 = 200a^2$.

WRITTEN EXERCISES

1. Find approximately to two decimal places the sides of a square whose area is 120.

2. Approximate to two decimals the side of a square having an area equal to that of a rectangle whose sides are 15 and 20.

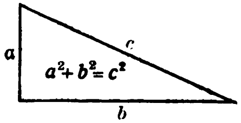
3. How many rods of fence are required to fence a square piece of land containing 50 acres, each acre containing 160 square rods?

4. A square checkerboard has an area of 324 square inches. What are its dimensions?

APPLICATIONS OF SQUARE ROOT

182. The Theorem of Pythagoras. Some of the most interesting and useful applications of the square root process are concerned with the sides and areas of triangles.

The fact that the sum of the squares on the two sides of a right triangle equals the square on the hypotenuse was used in Chapter VIII. See page 151.



If a and b are the lengths of the sides, and c the length of the hypotenuse, all measured in the same unit, the theorem of Pythagoras says:

$$c^2 = a^2 + b^2. \tag{1}$$

Hence, by S,

$$a^2 = c^2 - b^2, \tag{2}$$

and

$$b^2 = c^2 - a^2. \tag{3}$$

Taking the square root of both sides in each equation, we have

$$c = \sqrt{a^2 + b^2}. \tag{4}$$

$$a = \sqrt{c^2 - b^2}. \tag{5}$$

$$b = \sqrt{c^2 - a^2}. \tag{6}$$

The negative root is omitted, since the side of a triangle cannot be a negative number. By these formulas, if any two sides of a right triangle are given, the other may be found.

E.g. if $a = 4$, $b = 3$, then, by Equation (4) above,
$c = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$

Again if $c = 5$, $b = 3$, then, by Equation (5) above,
$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4;$

and if $c = 5$, $a = 4$, then, by Equation (6),
$b = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3.$

Example 1. If the two sides of a right triangle are 8 and 12, find the hypotenuse correct to two decimal places.

Solution. We have $c = \sqrt{a^2 + b^2} = \sqrt{64 + 144} = \sqrt{208}$,
$\sqrt{208} = \sqrt{16 \cdot 13} = \sqrt{16} \cdot \sqrt{13} = 4\sqrt{13} = 4(3.605) = 14.420.$

Example 2. If the hypotenuse of a right triangle is 10 and one side is 8, find the other side.

Solution. We have $b = \sqrt{c^2 - a^2} = \sqrt{100 - 64} = \sqrt{36} = 6.$

PROBLEMS

In solving the following problems, simplify each expression under the radical sign before extracting the root. Find all results correct to two decimal places.

1. The sides about the right angle of a right triangle are each 15 inches. Find the hypotenuse.

2. The hypotenuse of a right triangle is 9 inches and one of the sides is 6 inches. Find the other side.

Hint. If $x =$ the length of the required side, then $x^2 = a^2 - b^2 = 81 - 36$.

3. The hypotenuse of a right triangle is 25 feet and one of the other sides is 15 feet. Find the other side.

4. The hypotenuse of a right triangle is 12 inches and the other two sides are equal. Find their length.

Solution. Let s be the length of one of the equal sides.

Then,
$$s^2 + s^2 = 144.$$
$$2\,s^2 = 144.$$
$$s^2 = 72.$$
$$s = \sqrt{72} = 6\sqrt{2} = 6 \times 1.414 = 8.484.$$

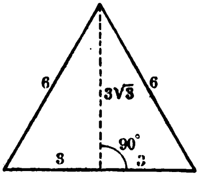
5. The hypotenuse of a right triangle is 30 feet and the other sides are equal. Find their length.

6. The hypotenuse of a right triangle is c and the sides are equal. Find their length. Solve Examples 4 and 5 by means of the formula here obtained.

7. The diagonal of a square is 8 feet. Find its area.

8. The side of an equilateral triangle is 6 inches. Find the altitude.

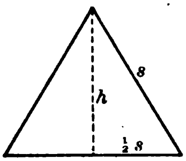
A line drawn from a vertex of an equilateral triangle perpendicular to the base meets the base at its middle point. Hence this problem becomes: The hypotenuse of a right triangle is 6 and one side is 3. Find the remaining side.



9. The side of an equilateral triangle is 10. Find the altitude.

10. The side of an equilateral triangle is s. Find the altitude.

This is equivalent to finding a side of a right triangle whose hypotenuse is s, the other side being $\frac{s}{2}$. Let h equal the altitude.



Then
$$\begin{aligned} h &= \sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{s^2 - \frac{s^2}{4}} \\ &= \sqrt{\frac{4s^2 - s^2}{4}} = \sqrt{\frac{3s^2}{4}} = \sqrt{\frac{s^2}{4} \cdot 3} \\ &= \sqrt{\frac{s^2}{4}} \cdot \sqrt{3} = \frac{s}{2}\sqrt{3}. \end{aligned}$$

Hence
$$h = \frac{s}{2}\sqrt{3}.$$

This formula gives the altitude of any equilateral triangle in terms of its side; namely, *the altitude of an equilateral triangle is equal to one half of the side multiplied by* $\sqrt{3}$.

By means of this formula solve Examples 8 and 9.

11. Find the altitude of an equilateral triangle whose side is $4\frac{1}{2}$. Substitute in the formula under Example 10.

12. Find the area of an equilateral triangle whose side is 5.

Since the area of a triangle is $\frac{1}{2}$ the product of the base and altitude, we first find the altitude by means of the formula under Example 10, and then multiply by $\frac{1}{2}$ the base.

13. Find the area of the equilateral triangle whose side is s. Show the result to be $\frac{s^2}{4}\sqrt{3}$.

Solution

From Example 10, the altitude is
$$h = \frac{s}{2}\sqrt{3}.$$

The area of a triangle is equal to one half the product of the base and the altitude.

Hence
$$\text{Area} = \frac{s}{2} \cdot \frac{s}{2}\sqrt{3} = \frac{s^2}{4}\sqrt{3}.$$

That is, *the area of an equilateral triangle is equal to one fourth of the square of its side multiplied by* $\sqrt{3}$.

14. If the area of an equilateral triangle is 16 square inches, find the length of the side.

Let s equal the length of the side. Then by the formula derived in Example 13, we have $16 = \frac{s^2}{4}\sqrt{3}$.

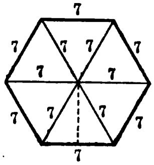
Hence, $s^2 = \frac{64}{\sqrt{3}} = \frac{64}{3}\sqrt{3} = 21.33 \times 1.732$.

15. The area of an equilateral triangle is 50 square inches. Find its side and altitude.

16. The area of an equilateral triangle is a square inches. Find the side.

Solve the equation $a = \frac{s^2}{4}\sqrt{3}$ for s, and simplify the expression, finding $s^2 = \frac{4a}{\sqrt{3}}$, and $s = \sqrt{\frac{4a\sqrt{3}}{3}} = \frac{2}{3}\sqrt{3a\sqrt{3}}$.

17. The area of an equilateral triangle is 240 square inches. Find its side. (Substitute in the formula obtained in Example 16.)



18. Find the area of a regular hexagon whose side is 7.

A regular hexagon is composed of 6 equal equilateral triangles, whose sides are each equal to the side of the hexagon (see figure). Hence this problem may be solved by finding the area of an equilateral triangle whose side is 7, and multiplying the result by 6.

19. Find the area of a regular hexagon whose side is s. Solve Example 18 by substituting in the formula obtained here.

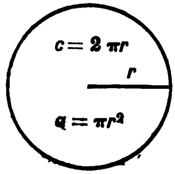
20. The area of a regular hexagon is 108 square inches. Find its side.

If the area of the hexagon is 108 square inches, the area of one of the equilateral triangles is 18 square inches.

21. The area of a regular hexagon is a square inches. Find its side. Solve Example 20 by substituting in the formula obtained here. *Ans.* $s = \frac{1}{3}\sqrt{2a\sqrt{3}}$.

22. Find the radius of a circle whose area is 9 square inches.

The area of a circle is found by squaring the radius and multiplying by 3.1416. The number 3.1416 is approximately the quotient obtained by dividing the length of the circumference by the diameter of the circle. This quotient is represented by the Greek letter π (pronounced *pi*). In this chapter we use $3\frac{1}{7}$ as an approximation to π. This differs from the real value of π by less than .0013, and hence is accurate enough for most purposes. If a represents the area of a circle, the above rule may be written



$$a = \pi r^2.$$

Hence, if $a = 9$, $\quad r^2 = \frac{9}{\pi} = \frac{9}{3\frac{1}{7}} = \frac{63}{22} = 2.863,$

and $\quad r = \sqrt{2.863}.$

23. Find the radius of a circle whose area is 68 square feet.

REVIEW QUESTIONS

1. What is meant by *square root*?

2. State in words the principle

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}.$$

3. How is this principle used to simplify radicals? Show by use of this principle how to find $\sqrt{28}$, having given $\sqrt{7} = 2.696$.

4. Why is $\frac{1}{3}\sqrt{3}$ considered simpler than $\sqrt{\frac{1}{3}}$ or $\frac{1}{\sqrt{3}}$? Show how the value of the following may be approximated by finding only one square root.

$$5\sqrt{20} + 2\sqrt{45} - 3\sqrt{80} + 2\sqrt{\tfrac{1}{5}}.$$

5. Write the square of $a + b + c + d$ in such a form as to derive from it the rule for finding the square root of a polynomial.

6. In finding the square root of an arithmetic number, how is it divided into groups (1) in case of a whole number, (2) in case of a decimal?

CHAPTER XVII

FURTHER OPERATIONS ON RADICALS

183. Higher Roots. By means of an index figure the **radical sign** is made to indicate other roots than square roots.

Thus, the *cube root of* 8, or one of its three equal factors, is written $\sqrt[3]{8} = 2$. The *fourth root* of 16 is written $\sqrt[4]{16} = 2$.

Radical Expressions. Any expression which contains an indicated root is called a *radical expression.*

The expression under the radical sign is called the **radicand.**

Rational Numbers. Integers and fractions whose terms are integers are called *rational numbers.*

E.g. $2 + \sqrt[3]{5}$ is a radical expression. 5, $\frac{2}{3}$, $\frac{75}{100}$, are rational numbers.

A surd is an indicated root of a rational number, which is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}$, $\sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9} = 3$. $\sqrt{2 + \sqrt{2}}$ is not a surd since $2 + \sqrt{2}$ is not a rational number.

Order of a Surd. The *order of a surd* is indicated by the **index** of the root.

E.g. $\sqrt[3]{4}$ is a surd of the *third order*, or of index three; $\sqrt[5]{3}$ is a surd of the fifth order or of index five.

Quadratic Surd. Surd expressions containing no indicated roots except square roots are called **quadratic surds.**

E.g. $\sqrt{7}$, $\sqrt{2} + \sqrt{3}$, $3 + \sqrt{5}$, $\dfrac{3}{\sqrt{7} - \sqrt{5}}$, are quadratic surds.

Mixed Surd. Entire Surd. The product of a surd and a rational factor is called a *mixed surd.* A surd which has no rational factor is called an *entire surd.*

E.g. $\sqrt{2}$ is an entire surd; $3\sqrt{2}$ is a mixed surd.

REDUCTION OF SURDS

184. By **reduction of a surd** we mean the *changing of its form without changing its value.*

185. Reduction of a Surd to its Simplest Form. This kind of reduction was used throughout the preceding chapter and is based on Principle XVIII; namely; $\sqrt{a\cdot b}=\sqrt{a}\cdot\sqrt{b}$.

E.g. $\sqrt{80}=\sqrt{16\cdot 5}=\sqrt{16}\cdot\sqrt{5}=4\sqrt{5}$,

and $\sqrt{\frac{1}{3}}=\sqrt{\frac{3}{9}}=\sqrt{\frac{1}{9}\cdot 3}=\sqrt{\frac{1}{9}}\cdot\sqrt{3}=\frac{1}{3}\sqrt{3}$.

186. Reduction of a Mixed Surd to an Entire Surd. It is sometimes desired to place the rational coefficient of a mixed surd under the radical sign. This is called *reducing a mixed surd to an entire surd.* This is also based on Principle XVIII.

Example. Reduce $3\sqrt{2}$ to an entire surd.

Since $3=\sqrt{9}$ we have $3\sqrt{2}=\sqrt{9}\cdot\sqrt{2}=\sqrt{9\cdot 2}=\sqrt{18}$.

In this form, Principle XVIII may be stated thus:

The product of the square roots of two numbers is equal to the square root of the product of the numbers.

ORAL EXERCISES

Reduce each of the following to entire surds.

1. $2\sqrt{2}$.
2. $2\sqrt{5}$.
3. $3\sqrt{3}$.
4. $3\sqrt{10}$.
5. $a\sqrt{b}$.
6. $a\sqrt{ab}$.
7. $x\sqrt{xy}$.
8. $ab\sqrt{ab}$.
9. $xy\sqrt{x}$.
10. $a^2\sqrt{b}$.
11. $b^2\sqrt{a^3}$.
12. $c^2\sqrt{2}$.
13. $(a+b)\sqrt{c}$.
14. $(a-b)\sqrt{c}$.
15. $(a+b)\sqrt{a-b}$.
16. $a\sqrt{a+b}$.
17. $a\sqrt{a-b}$.
18. $(a+b)\sqrt{a+b}$.
19. $(a-b)\sqrt{a-b}$.
20. $(a-b)\sqrt{\frac{1}{a-b}}$.
21. $3x\sqrt{\frac{1}{3x}}$.
22. $(x+y)\sqrt{\frac{1}{x+y}}$.
23. $x^2\sqrt{\frac{1}{x^3}}$.
24. $ab^2\sqrt{\frac{1}{ab}}$.

187. Simplifying a Surd of any Order. Reductions of surds of any order may be made by means of the following:

(1) *The nth root of the product of two numbers is equal to the product of the nth roots of the numbers.*

(2) *The product of the nth roots of two numbers is equal to the nth root of the product of the numbers. In symbols:*

$$(1)\ \sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b};\qquad (2)\ \sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}.$$

Example 1. Simplify $\sqrt[3]{x^4y^5}$.

$$\sqrt[3]{x^4y^5}=\sqrt[3]{x^3y^3\cdot xy^2}=\sqrt[3]{x^3y^3}\cdot\sqrt[3]{xy^2}=xy\sqrt[3]{xy^2}.$$

Example 2. **Reduce $2\sqrt[5]{a^2}$ to an entire surd.**

Since 2 may be written $\sqrt[5]{2^5}$, we have

$$2\sqrt[5]{a^2}=\sqrt[5]{2^5}\cdot\sqrt[5]{a^2}=\sqrt[5]{32\,a^2}.$$

Example 3. Simplify $\sqrt[3]{\frac{x^4}{3}}$

$$\sqrt[3]{\frac{x^4}{3}}=\sqrt[3]{\frac{9x^4}{27}}=\sqrt[3]{\frac{x^3}{27}\cdot 9x}=\sqrt[3]{\frac{x^3}{27}}\cdot\sqrt[3]{9x}=\frac{x}{3}\sqrt[3]{9x}.$$

188. These examples all involve the principle that *the nth root of the nth power of any number is the number itself.*

In symbols, $\sqrt[n]{a^n}=a.$

189. Steps in Simplifying a Surd. In simplifying a surd of any order, the first step is *to factor the radicand so that one factor shall be a perfect power of the same degree as the root indicated.*

Thus, in an indicated cube root we find the greatest factor which is a perfect cube and write its cube root before the radical sign, leaving the other factor under the radical sign.

E.g. $\sqrt[3]{16}=\sqrt[3]{8}\cdot\sqrt[3]{2}=2\sqrt[3]{2}.$

In an indicated fourth root we find the greatest factor which is a perfect fourth power and write its fourth root before the radical sign.

E.g. $\sqrt[4]{32}=\sqrt[4]{16}\cdot\sqrt[4]{2}=2\sqrt[4]{2}.$

ORAL EXERCISES

Simplify the following:

1. $\sqrt[3]{16}$.
2. $\sqrt[3]{32}$.
3. $\sqrt[3]{40}$.
4. $\sqrt[3]{48}$.
5. $\sqrt[3]{x^3y^2}$.
6. $\sqrt[3]{x^6y^4}$.
7. $\sqrt[3]{ab^4}$.
8. $\sqrt[4]{32}$.
9. $\sqrt[4]{48}$.
10. $\sqrt[4]{64}$.
11. $\sqrt[5]{32}$.
12. $\sqrt[5]{64}$.
13. $\sqrt[5]{a^5b}$.
14. $\sqrt[4]{a^4b^3}$.
15. $\sqrt[4]{a^6b^5}$.
16. $\sqrt[4]{a^8b^4}$.
17. $\sqrt[6]{x^6y^6}$.
18. $\sqrt[6]{x^{12}y}$.
19. $\sqrt[6]{x^{24}y}$.
20. $\sqrt[7]{x^5y^8}$.

WRITTEN EXERCISES

Simplify the following:

1. $\sqrt[3]{a^3(a-b)^4}$.
2. $\sqrt[3]{250}$.
3. $\sqrt{147\,a^{12}b^{10}c^8}$.
4. $\sqrt{72\,x^4y^6z^8}$.
5. $\sqrt{243\,a^5x^{10}}$.
6. $\sqrt[3]{216\,x^3y^3(x+y)^3}$.
7. $\sqrt{a^3+5\,a^2}$.
8. $\sqrt[3]{125(x+y)^4}$.
9. $\sqrt{18\,a-9}$.
10. $\sqrt{98\,(x+y)^2(x-y)^2}$.
11. $\sqrt[4]{625\,x^5y\,(x-y)^4}$.
12. $\sqrt[3]{x^6-2\,x^3}$.
13. $\sqrt[3]{x^4+3\,x^3b+3\,x^2b^2+xb^3}$.
14. $\sqrt{27\,(x+y)(a-b)^3}$.
15. $\sqrt[5]{64\,b^6c^8d^{10}}$.
16. $\sqrt{(x^3+2\,x^2y+xy^2)(x^2+bx)}$.

ORAL EXERCISES

Reduce the following to entire surds:

1. $2\sqrt[3]{x}$.
2. $3\sqrt[3]{2}$.
3 $2\sqrt[4]{x}$.
4. $2\sqrt[4]{2}$.
5. $a\sqrt[3]{b}$.
6. $a\sqrt[4]{b}$.
7. $2\,a\sqrt[3]{x}$.
8. $a^2\sqrt[3]{ab}$.
9. $2\,a\sqrt[4]{x}$.
10. $2\sqrt[5]{2}$.
11. $2\sqrt[6]{2}$.
12. $ab\sqrt[3]{2}$.
13. $ab\sqrt[3]{a}$.
14. $xy^2\sqrt[3]{x}$.
15. $a^2b^2\sqrt[3]{ab}$.
16. $abx^2\sqrt[3]{2\,ab}$.
17. $2\,a^2m\sqrt[3]{3\,am^2}$.
18. $a^2m^3n\sqrt[3]{2\,a^2mn^2}$.

WRITTEN EXERCISES

Reduce the following to entire surds:

1. $2\sqrt[3]{4}$.
2. $(a+b)\sqrt{a+b}$.
3. $3\sqrt[3]{2}$.
4. $2x^3\sqrt{xy}$.
5. $2x^2\sqrt{3xyz}$.
6. $2a^2\sqrt[3]{a^2}$.
7. $a^2\sqrt[3]{b}$.
8. $2mn\sqrt{6mn}$.
9. $a^2b\sqrt[3]{a^2b}$.
10. $a\sqrt{a^2+b^2}$.
11. $\frac{2}{3}x\sqrt[3]{27x^2}$.
12. $\frac{1}{x+y}\sqrt{(x^2+y^2+2xy)xy}$.
13. $ab\sqrt{\frac{a+b}{ab}}$.
14. $ab\sqrt[5]{2ab}$.
15. $4x\sqrt[3]{2x^2y}$.
16. $-2y\sqrt[3]{6y^2}$.
17. $a^2b^3\sqrt[5]{x^3y^2}$.
18. $3xy^2\sqrt[4]{x^4y}$.
19. $-2a\sqrt[4]{a^2b}$.
20. $2xyz\sqrt[5]{xy^2z^3}$.
21. $(a+b)\sqrt{a-b}$.
22. $-abc\sqrt[5]{2abc}$.
23. $x^2y^2z^2\sqrt[5]{x^2y^2z^2}$.
24. $x\sqrt[3]{x^2+y^2}$.
25. $(x-y)\sqrt[3]{x+y}$.
26. $ab\sqrt[7]{ab}$.
27. $a(x+y)\sqrt{\frac{b}{a^2(x+y)^2}}$.
28. $\frac{xy}{x+y}\sqrt{2+\frac{x^2+y^2}{xy}}$.
29. $-abc\sqrt[9]{abc}$.

190. Fractional Exponents. Thus far a fraction has never been used as an exponent, and evidently it could not be so used without extending the meaning of the word *exponent.*

Thus, a^3 means $a \cdot a \cdot a$, but $a^{\frac{1}{2}}$ evidently cannot mean that a is to be multiplied by itself one half of a time.

We shall agree that the meaning attached to *fractional* exponents must be such as to make them obey the laws which govern *integral* exponents.

E.g. Just as $a^2 \cdot a^3 = a^{2+3} = a^5$, so we shall agree that $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$ must equal $a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

Since we agree that $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a$, we see that $a^{\frac{1}{2}}$ is *one of the two equal factors of* a. That is, $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$. See § 168.

Similarly $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. Hence $a^{\frac{1}{3}}$ is *one of the three equal factors of* a. That is, $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$. See § 183.

Again $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^{\frac{6}{3}} = a^2$.

Hence, $a^{\frac{2}{3}}$ is *one of the three equal factors of* a^2. That is $\boldsymbol{a^{\frac{2}{3}} = \sqrt[3]{a^2}}$.

Definition of Fractional Exponent. From these examples we see that

The numerator of a fractional exponent indicates the power of the expression over which the exponent stands, and the denominator indicates what root of this power is to be taken.

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$, $a^{\frac{2}{5}} = \sqrt[5]{a^2}$, $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, etc.

It is also true that $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$, that is, either the power or the root may be taken first.

E.g. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$; and $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$.

ORAL EXERCISES

Give the equivalents of the following, using fractional exponents.

1. $\sqrt{2}$.
2. $\sqrt{3}$.
3. $\sqrt[3]{2}$.
4. $\sqrt[3]{3}$.
5. $\sqrt{2^3}$.
6. $\sqrt{x^5}$.
7. $\sqrt[3]{x^2}$.
8. $\sqrt[3]{x^4}$.
9. $\sqrt[3]{x^5}$.
10. $\sqrt[3]{x^7}$.
11. $\sqrt{a^3bc}$.
12. $\sqrt[3]{a^2b^4}$.
13. $\sqrt[3]{m^2n^4}$.
14. $\sqrt{nm^3y^5}$.
15. $\sqrt[4]{xy^3z^5}$.
16. $\sqrt[5]{ab^2c^3}$.
17. $\sqrt[5]{x^2y^3z^4}$.
18. $\sqrt[5]{x^3y^6z^7}$.
19. $\sqrt[7]{a^2b^4c^6}$.
20. $\sqrt[9]{m^2n^5p^7}$.

Operations with Fractional Exponents. If a fractional exponent is reduced to higher or lower terms, the value of the whole expression is not changed.

Thus, $4^{\frac{1}{2}} = 4^{\frac{3}{6}}$, since $4^{\frac{1}{2}} = \sqrt{4} = 2$, and $4^{\frac{3}{6}} = \sqrt[6]{4^3} = \sqrt[6]{64} = 2$.

191. Reduction of Surds to a Common Index. In reducing surds of different index to equivalent surds with a common index, fractional exponents are used as in the following examples:

1. Reduce $\sqrt[3]{5}$ and $\sqrt{3}$ to equivalent surds with the same index.

Solution. $\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = \sqrt[6]{5^2} = \sqrt[6]{25}.$

$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = \sqrt[6]{3^3} = \sqrt[6]{27}.$

2. Reduce $\sqrt[5]{a}$ and $\sqrt[4]{b}$ to surds with the same index.

Solution. $\sqrt[5]{a} = a^{\frac{1}{5}} = a^{\frac{4}{20}} = \sqrt[20]{a^4}.$

$\sqrt[4]{b} = b^{\frac{1}{4}} = b^{\frac{5}{20}} = \sqrt[20]{b^5}.$

3. Reduce $\sqrt[3]{x^2}$ and $\sqrt{x^3}$ to surds with the same index.

Solution. $\sqrt[3]{x^2} = x^{\frac{2}{3}} = x^{\frac{4}{6}} = \sqrt[6]{x^4}.$

$\sqrt{x^3} = x^{\frac{3}{2}} = x^{\frac{9}{6}} = \sqrt[6]{x^9}.$

WRITTEN EXERCISES

Reduce to equivalent surds of the same order:

1. $\sqrt{a}, \sqrt[3]{b}.$
2. $\sqrt[3]{a}, \sqrt[4]{b}.$
3. $\sqrt{2}, \sqrt[3]{2}.$
4. $\sqrt[3]{3}, \sqrt{2}.$
5. $\sqrt[3]{3}, \sqrt[4]{3}.$
6. $\sqrt{3}, \sqrt[3]{3}.$
7. $\sqrt{x}, \sqrt[4]{x}.$
8. $\sqrt{ab}, \sqrt[3]{ab}.$
9. $\sqrt[3]{x^2y^2}, \sqrt[4]{xy}.$
10. $\sqrt{5}, \sqrt[3]{25}.$
11. $a^{\frac{1}{3}}, \sqrt{a}, \sqrt[4]{a}.$
12. $x^{\frac{1}{2}}, x^{\frac{1}{3}}, x^{\frac{1}{6}}.$
13. $2\sqrt{a}, 3\sqrt[3]{a}, 3\sqrt[4]{a}.$
14. $3a^{\frac{2}{3}}, 2a^{\frac{3}{4}}.$
15. $2x^{\frac{1}{3}}, 2x^{\frac{1}{4}}, 2x^{\frac{1}{12}}.$

ADDITION AND SUBTRACTION OF SURDS

192. Similar surds are those which have the same surd factors, that is, the same index and same radicand.

E.g. $4\sqrt[3]{a^2}$ and $-2\sqrt[3]{a^2}$ are similar, while $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are not similar.

Similar surds and those which can be reduced to this form may be combined into a single surd. For example:

1. $4\sqrt[3]{a^2} + 2\sqrt[3]{a^2} = 6\sqrt[3]{a^2}.$
2. $\sqrt{32} + \sqrt{72} = \sqrt{16 \cdot 2} + \sqrt{36 \cdot 2} = 4\sqrt{2} + 6\sqrt{2} = 10\sqrt{2}.$
3. $\sqrt{\frac{1}{3}} - \sqrt{\frac{3}{4}} = \sqrt{\frac{1}{9} \cdot 3} - \sqrt{\frac{1}{4} \cdot 3} = \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{3} = -\frac{1}{6}\sqrt{3}.$

ORAL EXERCISES

Combine each of the following into a single surd:

1. $3\sqrt{5}-2\sqrt{5}$.
2. $5\sqrt[3]{2}+3\sqrt[3]{2}$.
3. $2\sqrt{x}+3\sqrt{x}-\sqrt{x}$.
4. $a\sqrt{b}-c\sqrt{b}$.
5. $2\sqrt[3]{2}+7\sqrt[3]{2}-\sqrt[3]{2}$.
6. $4\sqrt[3]{a^2}+5\sqrt[3]{a^2}-3\sqrt[3]{a^2}$.
7. $12\sqrt[4]{x^3}+8\sqrt[4]{x^3}-5\sqrt[4]{x^3}$.
8. $a\sqrt[3]{b^2}+c\sqrt[3]{b^2}-d\sqrt[3]{b^2}$.

WRITTEN EXERCISES

Simplify each of the following as far as possible without approximating roots.

1. $\sqrt{27}+2\sqrt{48}-3\sqrt{75}$.
2. $\sqrt{20}+\sqrt{125}-\sqrt{180}$.
3. $3\sqrt{432}-4\sqrt{3}+\sqrt{147}$.
4. $3\sqrt{2450}-25\sqrt{2}+4\sqrt{13122}$.
5. $3y^2\sqrt{x^3z}+2\sqrt{x^5z^3}-yz^4\sqrt{\frac{xz}{y^2}}$.
6. $\sqrt{4x^3y}+\sqrt{25xy^3}-x\sqrt{xy}$.
7. $\sqrt{ax^2-bx^2}+\sqrt{4ar^2s^2-4br^2s^2}$.
8. $4\sqrt{\frac{3}{4}}-\frac{3}{4}\sqrt{\frac{3}{16}}-3\sqrt{27}$.
9. $2\sqrt{\frac{5}{8}}+\sqrt{60}+\sqrt{\frac{3}{5}}$.
10. $5\sqrt{3}-2\sqrt{48}+7\sqrt{108}$.
11. $\sqrt{a^3-a^2b}-\sqrt{ab^2-b^3}-\sqrt{(a+b)(a^2-b^2)}$.
12. $\sqrt{a}+3\sqrt{2a}-2\sqrt{3a}+\sqrt{4a}-\sqrt{8a}+\sqrt{12a}$.
13. $\sqrt{x^3+2x^2y+xy^2}-\sqrt{x^3-2x^2y+xy^2}-\sqrt{4xy^2}$.
14. $\sqrt{r-s}+\sqrt{16r-16s}+\sqrt{rt^2-st^2}-\sqrt{9(r-s)}$.
15. $\sqrt{(m-n)^2a}+\sqrt{(m+n)^2a}-\sqrt{am^2}+\sqrt{a(1-m)^2}-\sqrt{a}$.
16. $\sqrt{32x^2y^4}+\sqrt{162x^2y^4}-\sqrt{512x^2y^4}+\sqrt{1250x^2y^4}$.
17. $\sqrt[3]{16}+\sqrt[3]{54}-\sqrt[3]{250}+\sqrt[3]{128}$.
18. $\sqrt[3]{54}+\sqrt[3]{128}+\sqrt{128}+\sqrt{200}$.
19. $\sqrt[4]{32}+\sqrt[4]{162}+\sqrt[4]{512}-\sqrt[4]{1250}$.

MULTIPLICATION OF SURDS

193. Surds with the same index are multiplied according to § 187.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

E.g. $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{2 \cdot 4} = \sqrt[3]{8} = 2.$

Surds of different index but with the same radicand are best multiplied by use of fractional exponents (§ 190).

E.g. $\sqrt[3]{2} \cdot \sqrt{2} = 2^{\frac{1}{3}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{3}+\frac{1}{2}} = 2^{\frac{5}{6}} = \sqrt[6]{2^5}.$

Surds of different index and having different radicands must be reduced to the same order before multiplying.

E.g. $\sqrt[3]{2} \cdot \sqrt{3} = 2^{\frac{1}{3}} \cdot 3^{\frac{1}{2}} = 2^{\frac{2}{6}} \cdot 3^{\frac{3}{6}} = \sqrt[6]{2^2} \cdot \sqrt[6]{3^3} = \sqrt[6]{2^2 \cdot 3^3} = \sqrt[6]{4 \cdot 27} = \sqrt[6]{108}.$

ORAL EXERCISES

Multiply and simplify the products:

1. $\sqrt{2} \cdot \sqrt{3}$.
2. $\sqrt[5]{2} \cdot \sqrt[5]{3}$.
3. $\sqrt{7} \cdot \sqrt{3}$.
4. $\sqrt[3]{3} \cdot \sqrt[3]{9}$.
5. $\sqrt{2} \cdot \sqrt{8}$.
6. $\sqrt[5]{2} \cdot \sqrt[5]{16}$.
7. $\sqrt{5} \cdot \sqrt{7}$.
8. $\sqrt{6} \cdot \sqrt{12}$.
9. $\sqrt{18} \cdot \sqrt{8}$.
10. $\sqrt{x^3} \cdot \sqrt{x^5}$.
11. $\sqrt{a^2b} \cdot \sqrt{ab^2}$.
12. $\sqrt{rs^3t} \cdot \sqrt{r^3s^3t^3}$.
13. $\sqrt{7} \cdot \sqrt{\frac{1}{7}}$.
14. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{3}}$.
15. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{6}{5}}$.
16. $2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}$.
17. $3^{\frac{1}{2}} \cdot 3^{\frac{1}{4}}$.
18. $4^{\frac{1}{3}} \cdot 4^{\frac{1}{2}}$.

WRITTEN EXERCISES

Multiply and simplify the products.

1. $\sqrt{6} \cdot \sqrt{8}$.
2. $\sqrt{7} \cdot \sqrt{\frac{1}{14}}$.
3. $\sqrt{a^2b} \cdot \sqrt{ab^2}$.
4. $\sqrt{20} \cdot \sqrt{50}$.
5. $\sqrt{5} \cdot \sqrt{15}$.
6. $\sqrt{a} \cdot \sqrt[3]{b}$.
7. $\sqrt{6} \cdot \sqrt[3]{12}$.
8. $2\sqrt{3} \cdot 3\sqrt[3]{2}$.
9. $2\sqrt[3]{16} \cdot \sqrt{4}$.
10. $2\sqrt{12} \cdot \sqrt[3]{4}$.
11. $50^{\frac{1}{3}} \cdot 20^{\frac{1}{3}}$.
12. $2\sqrt{7} \cdot 7\sqrt{2}$.

13. $18^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$.

14. $100^{\frac{1}{3}} \cdot 40^{\frac{1}{3}}$.

15. $3^{\frac{1}{2}} \cdot 27^{\frac{1}{2}}$.

16. $\sqrt[3]{\frac{2x}{y}} \cdot \sqrt[2]{\frac{y^3}{2x^2}}$.

17. $\left(\frac{9}{x}\right)^{\frac{1}{2}}\left(\frac{4x}{9}\right)^{\frac{1}{2}}$.

18. $\left(\frac{16}{98}\right)^{\frac{1}{2}} \cdot (147)^{\frac{1}{2}}$.

19. $\sqrt{ab} \cdot \sqrt[4]{a^2b} \cdot \sqrt[8]{ab^4}$.

20. $(3abc)^{\frac{1}{2}} \cdot (bc)^{\frac{1}{3}} \cdot (a^2bc)^{\frac{1}{4}}$.

194. Binomial Quadratic Surds are multiplied as in the following example:

$$\begin{array}{l} 3\sqrt{2} - 4\sqrt{5} \\ \underline{2\sqrt{2} + 3\sqrt{5}} \\ 6 \cdot 2 - 8\sqrt{10} \\ \underline{\qquad + 9\sqrt{10} - 12 \cdot 5} \\ 12 + \sqrt{10} - 60 = \sqrt{10} - 48. \end{array}$$

WRITTEN EXERCISES

1. $(\sqrt{2} - \sqrt{3})^2$.

2. $(1 + \sqrt{2})^2$.

3. $(2 - \sqrt{3})^2$.

4. $(1 + \sqrt{2})(1 - \sqrt{2})$.

5. $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$.

6. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

7. $\sqrt{2}(2\sqrt{3} + 3\sqrt{8} - 5\sqrt{6})$.

8. $\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c})$.

9. $(3\sqrt{5} - 2\sqrt{3})(3\sqrt{5} + 2\sqrt{3})$.

10. $(1 + \sqrt{2})(1 - \sqrt{3} + \sqrt{5})$.

11. $\sqrt{3 + \sqrt{5}} \times \sqrt{3 - \sqrt{5}} = \sqrt{(3 + \sqrt{5})(3 - \sqrt{5})}$.

12. $\sqrt{4 - 2\sqrt{3}} \times \sqrt{4 + 2\sqrt{3}}$.

13. $\sqrt{8 + 3\sqrt{2}} \times \sqrt{8 - 3\sqrt{2}}$.

14. $(7\sqrt{5} - 2\sqrt{3})(7\sqrt{5} + 2\sqrt{3})$.

15. $(a + b)(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$.

16. $(3x + x\sqrt{3})(3x - x\sqrt{3})$.

17. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{3})$.

195. **Powers and Roots of Monomial Surds** are best found by use of fractional exponents, as in the following examples:

1. $(\sqrt{5})^3 = (5^{\frac{1}{2}})^3 = 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5 \cdot 5^{\frac{1}{2}} = 5\sqrt{5}.$
2. $(2^3\sqrt[4]{3})^2 = (2^3 \cdot \sqrt[4]{3})(2^3 \cdot \sqrt[4]{3}) = 2^3 \cdot 2^3 \cdot 3^{\frac{1}{4}} \cdot 3^{\frac{1}{4}}$
   $= 2^6 \cdot 3^{\frac{2}{4}} = 2^6 \cdot 3^{\frac{1}{2}} = 64\sqrt{3}.$
3. $(\sqrt[5]{3}\sqrt[3]{2})^2 = (3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}})(3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}}) = 3^{\frac{1}{5}} \cdot 3^{\frac{1}{5}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 3^{\frac{2}{5}} \cdot 2^{\frac{2}{3}} = \sqrt[5]{3^2}\sqrt[3]{2^2}.$

In general, $(a^m \cdot b^n)^k = a^{km}b^{kn}.$

In the above formula k is an *integer*, but the formula is also true when k is a *fraction*.

Thus $(\sqrt{64})^{\frac{1}{3}} = \{(64)^{\frac{1}{2}}\}^{\frac{1}{3}} = 64^{\frac{1}{2} \cdot \frac{1}{3}} = 64^{\frac{1}{6}} = \sqrt[6]{64} = 2,$ which we know to be the value of $(\sqrt{64})^{\frac{1}{3}}$,

since $(\sqrt{64})^{\frac{1}{3}} = \sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2.$

In general, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$

This latter formula is sometimes useful in simplifying a radical.

E.g. (1) $\sqrt[6]{125} = \sqrt{\sqrt[3]{125}} = \sqrt{5}.$

(2) $\sqrt[4]{64} = \sqrt{\sqrt{64}} = \sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}.$

ORAL EXERCISES

Perform the following indicated operations:

1. $(x^{\frac{1}{2}})^2.$
2. $(x^{\frac{1}{2}})^4.$
3. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^2.$
4. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^4.$
5. $(a^{\frac{1}{3}})^3.$
6. $(a^{\frac{1}{3}})^6.$
7. $(a^{\frac{1}{3}}b^{\frac{1}{3}})^3.$
8. $(a^{\frac{1}{3}}b^{\frac{1}{3}})^6.$
9. $(4^{\frac{1}{3}})^3.$
10. $(x^{\frac{2}{3}})^3.$
11. $(x^{\frac{2}{3}})^4.$
12. $(x^{\frac{2}{3}})^2.$
13. $(x^{\frac{2}{3}})^6.$
14. $(x^{\frac{1}{4}})^2.$
15. $(x^{\frac{1}{4}})^3.$
16. $(x^{\frac{1}{4}})^4.$
17. $(x^{\frac{1}{4}})^8.$
18. $(x^{\frac{1}{4}})^6.$
19. $(x^{\frac{1}{2}}y^{\frac{1}{3}})^2.$
20. $(x^{\frac{1}{3}}y^{\frac{1}{2}})^6.$
21. $(x^{\frac{2}{3}}y^{\frac{1}{3}})^3.$
22. $(x^{\frac{2}{3}}y^{\frac{1}{3}})^6.$
23. $(a^{\frac{1}{2}}b^{\frac{1}{4}})^4.$
24. $(a^{\frac{1}{4}}b^{\frac{1}{2}})^2.$
25. $(a^{\frac{1}{2}}b^{\frac{1}{5}})^{10}.$
26. $(a^{\frac{1}{4}}b^{\frac{1}{3}})^6.$
27. $(a^{\frac{1}{3}}b^{\frac{1}{4}})^{12}.$

ORAL EXERCISES

1. Find the square of $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[5]{3}$, $\sqrt[6]{x}$.
2. Find the cube of $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{2}$, $\sqrt[4]{x}$.
3. Find the 4th power of $\sqrt{3}$, $\sqrt[3]{3}$, $\sqrt[4]{3}$, $\sqrt[5]{x}$.
4. Find the 5th power of $\sqrt{2}$, $\sqrt[3]{2a}$, $\sqrt{3x}$, $\sqrt[5]{4b}$.
5. Express as a single root $\sqrt{\sqrt{2}}$, $\sqrt[3]{\sqrt{5}}$, $\sqrt{\sqrt[3]{5}}$.
6. Express as a square root $\sqrt[6]{8}$, $\sqrt[4]{36}$, $\sqrt[8]{16}$.
7. Express as a single root $\sqrt[3]{\sqrt[3]{4}}$, $\sqrt[3]{\sqrt[4]{3}}$, $\sqrt[3]{\sqrt[5]{5}}$.
8. Express as a cube root $\sqrt[6]{4}$, $\sqrt[9]{8}$, $\sqrt[12]{16}$.
9. A cube root of a fifth root is what root?
10. A sixth root of a cube is what root?

WRITTEN EXERCISES

Perform the following indicated operations:

1. $(4a^2b)^2$.
2. $(3a^3b^2c)^3$.
3. $(2\sqrt[3]{3a})^3$.
4. $(\sqrt[3]{x^2})^2$.
5. $(\sqrt[3]{a^2})^4$.
6. $(\sqrt[5]{a})^4$.
7. $(\sqrt[4]{a^3})^5$.
8. $(\sqrt[3]{a^2b^2c^2})^4$.
9. $(\sqrt[3]{bc^3d^5})^3$.
10. $(\sqrt[3]{-a^3b^5})^5$.
11. $(\sqrt{a^2b^3c^5})^3$.
12. $[(a)^{\frac{1}{2}}]^{\frac{1}{3}}$.
13. $(\sqrt{a+b})^2$.
14. $[(2x)^{\frac{1}{3}}]^4$.
15. $[(a^2)^3]^{\frac{1}{4}}$.
16. $[(ax^2)^{\frac{2}{3}}]^4$.
17. $\sqrt[3]{\sqrt[3]{x^6}}$.
18. $[(y^3)^{\frac{1}{2}}]^{\frac{1}{3}}$.
19. $[\sqrt{(x+y)^3}]^3$.
20. $[\sqrt[3]{(a-b)^2}]^4$.
21. $[x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}}]^{12}$.
22. $[a^{\frac{2}{3}}b^{\frac{4}{5}}]^{15}$.
23. $[(\sqrt{x+y})^3]^2$.
24. $\sqrt[4]{\sqrt[3]{16}}$.
25. $\sqrt[3]{\sqrt[2]{64}}$.
26. $\sqrt{\sqrt[3]{64}}$.
27. $\sqrt{\sqrt[4]{256}}$.
28. $\sqrt[4]{\sqrt{256}}$.
29. $\sqrt{\sqrt[3]{729}}$.
30. $\sqrt[3]{\sqrt{729}}$.
31. $\sqrt[5]{\sqrt{32}}$.
32. $(\sqrt[4]{8})^2 \cdot (\sqrt[3]{4})$.
33. $\sqrt{\sqrt[4]{9}}$.
34. $[(27)^{\frac{1}{2}}]^{\frac{1}{3}}$.
35. $(\sqrt{12})^3 \cdot (\sqrt[3]{16})^2$.
36. $[\{\sqrt[3]{xy(a-b)}\}^{\frac{1}{2}}]^6$.

DIVISION OF SURDS*

196. **When a divisor is a surd,** it is convenient to indicate the division in the form of a fraction and then to reduce the denominator to the *rational form* as in the following example:

$$\frac{\sqrt{5}}{\sqrt{5}-\sqrt{2}}=\frac{\sqrt{5}(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}=\frac{5+\sqrt{10}}{5-2}=\frac{5+\sqrt{10}}{3}.$$

Explanation. We see that if we multiply the denominator by $\sqrt{5}+\sqrt{2}$ we shall have the product of the sum and difference of $\sqrt{5}$ and $\sqrt{2}$, which is equal to the difference of their squares.

That is, $(\sqrt{5})^2-(\sqrt{2})^2=5-2=3$, which is *rational.*

If now we multiply the numerator also by $\sqrt{5}+\sqrt{2}$, the value of the fraction will not be changed. The numerator then becomes

$$\sqrt{5}(\sqrt{5}+\sqrt{2})=(\sqrt{5})^2+\sqrt{5}\cdot\sqrt{2}=5+\sqrt{10}.$$

The success of this proceeding depends upon choosing $\sqrt{5}+\sqrt{2}$ with which to multiply the denominator so as to obtain a *rational product.*

197. **Rationalizing the Denominator.** This is called *rationalizing the denominator,* and the factor by which we multiply is called the *rationalizing factor.* See § 178.

If we wish to compute the approximate value of $\sqrt{5}\div(\sqrt{5}-\sqrt{2})$ above, it obviously requires less numerical work to use the form $(5+\sqrt{10})\div 3$ with rational denominator, instead of $\sqrt{5}\div(\sqrt{5}-\sqrt{2})$, which involves the extraction of two square roots and a *long* division, while the former requires the extraction of only one root and a *short* division.

To rationalize the denominator of a fraction, it is necessary first to find an expression which, multiplied by the denominator of the fraction, gives a rational product, and then to multiply both terms of the fraction by this expression.

In the case of a *monomial* or *binomial quadratic surd,* the rationalizing factor may be found at sight.

Thus, if the denominator is of the form $\sqrt{x}$ or $a\sqrt{x}$, then $\sqrt{x}$ is the rationalizing factor, since $\sqrt{x}\cdot\sqrt{x}=x$.

If the denominator is of the form $\sqrt{x}+\sqrt{y}$, then $\sqrt{x}-\sqrt{y}$ is the rationalizing factor, since $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$.

---

* Articles 196 and 197 may be omitted without destroying the continuity.

ORAL EXERCISES

Give a rationalizing factor of each of the following:

(1) $\sqrt{3x}$, (2) $\sqrt{2x^3b}$, (3) $\sqrt{8b^3c^5}$, (4) $\sqrt{a}+\sqrt{b}$, (5) $\sqrt{a}-\sqrt{b}$, (6) $\sqrt{3a}+\sqrt{2b^3}$, (7) $\sqrt{7}-\sqrt{27}$,

ORAL EXERCISES

Rationalize the denominators of each of the following:

1. $\dfrac{1}{\sqrt{x}}$.
2. $\dfrac{1}{\sqrt{x}+1}$.
3. $\dfrac{1}{\sqrt{2}-1}$.
4. $\dfrac{1}{\sqrt{5}-\sqrt{3}}$.
5. $\dfrac{1}{\sqrt{3}+\sqrt{5}}$.
6. $\dfrac{2}{\sqrt{7}-\sqrt{3}}$.
7. $\dfrac{2}{\sqrt{a}+1}$.
8. $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.
9. $\dfrac{1}{\sqrt{a}-\sqrt{b}}$.
10. $\dfrac{1}{\sqrt{xy}-\sqrt{yz}}$.
11. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$.
12. $\dfrac{\sqrt{8}-\sqrt{3}}{\sqrt{3}}$.

WRITTEN EXERCISES

Rationalize the denominators of each of the following:

1. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{xy}}$.
2. $\dfrac{3\sqrt{6}+9\sqrt{2}}{2\sqrt{2}}$.
3. $\dfrac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
4. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.
5. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.
6. $\dfrac{1}{a\sqrt{x}+b\sqrt{y}}$.
7. $\dfrac{a+b}{a\sqrt{x}-b\sqrt{y}}$.
8. $\dfrac{2+\sqrt{3}}{3-\sqrt{2}}$.
9. $\dfrac{5+\sqrt{6}}{4-\sqrt{3}}$.
10. $\dfrac{2-\sqrt{5}}{3+\sqrt{3}}$.
11. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.
12. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$.
13. $\dfrac{2\sqrt{5}+1}{3\sqrt{5}-\sqrt{3}}$.
14. $\dfrac{6\sqrt{3}+4\sqrt{2}}{6\sqrt{3}-4\sqrt{2}}$.
15. $\dfrac{\sqrt{x-3}-3}{\sqrt{x-3}+3}$.
16. $\dfrac{x-\sqrt{a+b}}{x+\sqrt{a+b}}$.
17. $\dfrac{1+\sqrt{2}+\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}$.
18. $\dfrac{5+\sqrt{2}-\sqrt{3}}{5+\sqrt{2}+\sqrt{3}}$.

EQUATIONS INVOLVING RADICALS

198. Illustrative Example. Solve the equation:

$$\sqrt{x-5}+\sqrt{x+1}=3 \qquad (1)$$

By $S|\sqrt{x+1}$ $\qquad \sqrt{x-5}=3-\sqrt{x+1} \qquad (2)$

Squaring both sides $\qquad x-5=9-6\sqrt{x+1}+x+1. \qquad (3)$

Transposing, $\qquad 6\sqrt{x+1}=15$

or $\qquad 2\sqrt{x+1}=5 \qquad (4)$

Squaring both sides $\qquad 4(x+1)=25 \qquad (5)$

Hence $\qquad x=5\frac{1}{4} \qquad (6)$

Check. Substitute $\qquad x=5\frac{1}{4}$ in (1)

If two radicals are involved, or one radical and one or more rational terms, it is best to get a radical alone on one side of the equation before squaring as in Equations (2) and (4). Note that squaring both members of an equation is equivalent to multiplying both sides by the same number.

WRITTEN EXERCISES

Solve the following equations and check each result.

1. $\sqrt{x-5}=3.$
2. $x-2=\sqrt{x^2-4}.$
3. $\sqrt{x+6}=3\sqrt{x-2}.$
4. $\sqrt{x^2-2x+8}=x-4.$
5. $\sqrt{x^2+7x-4}=\sqrt{x^2+8x-5}.$
6. $\sqrt{x^2+5}+x=5.$
7. $\sqrt{4x^2-7}+2x=7.$
8. $\sqrt{x+1}-\sqrt{x-7}=2.$
9. $\sqrt{y+4}+\sqrt{y-1}=5.$
10. $\sqrt{x}+2=\sqrt{x+16}.$
11. $5-\sqrt{x}=\sqrt{x+5}.$
12. $\sqrt{x+7}=1+\sqrt{x+2}.$
13. $\dfrac{\sqrt{x}-2}{\sqrt{x}+1}=\dfrac{\sqrt{x}+1}{\sqrt{x}+13}.$
14. $\dfrac{\sqrt{x-2}}{\sqrt{x-4}}=\dfrac{\sqrt{x+1}}{\sqrt{x-3}}.$
15. $\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=\dfrac{4}{x-1}.$
16. $\dfrac{2-\sqrt{x}}{1+\sqrt{x}}=\dfrac{3+\sqrt{x}}{3-\sqrt{x}}.$

Suggestions. In Example 13 clear of fractions first; in 14 square both members and then clear of fractions; in 15 rationalize the denominators.

REVIEW QUESTIONS

1. Give examples of rational expressions, of surds, of quadratic surds.

2. What three reductions of surds are considered in this chapter?

3. How is Principle XVIII used to reduce a mixed surd to an entire surd?

4. State in words the extension of Principle XVIII which in symbols is:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

5. How is a fractional exponent defined? Show by an example such as $16^{\frac{3}{4}} \times 16^{\frac{1}{2}}$ that the law of exponents,

$$x^a \times x^b = x^{a+b}$$

holds for fractional exponents.

6. How are fractional exponents used in reducing surds with different indices to equivalent surds with the same index?

7. What are similar surds? When can two or more surds be combined into a single surd by addition or subtraction?

8. By what principle are surds of the same index multiplied? Surds of different index but with the same radicand? Surds of different index and having different radicands?

9. State in words the principle:

$$(a^m \cdot a^n)^k = a^{km} \cdot b^{kn}.$$

10.* How do we divide by a binomial quadratic surd?

11.* Give examples to show how a fraction with a binomial surd denominator may be changed into an equivalent fraction having a rational denominator.

12. In solving an equation containing radicals, how are they removed (1) when only one radical expression is involved; (2) when two such expressions are involved?

CHAPTER XVIII

QUADRATIC EQUATIONS

199. Equations of the form $x^2+ax+b=0$ have already been solved in cases where the left members could be factored by inspection. See § 116. However, in a case like $x^2+5x+3=0$, the factors of the left member cannot be found by any method thus far studied. It is therefore necessary to consider other methods for solving equations of this type.

200. Completing the Square. As a preliminary step we consider again the properties of a trinomial square. See § 97.

From $(x+a)^2=x^2+2ax+a^2$ we see that the third term is the square of half the coefficient of x in the second term.

Hence, if we had given only the two terms x^2+2ax, we would have to add a^2 in order to *complete the square.*

E.g. To find the term to be added to x^2+6x in order to *complete the square*, we take the square of half the coefficient of x, that is $3^2=9$. Thus, x^2+6x+9 is *a complete square.*

In general, to complete the square in x^2+px, we add $\left(\frac{1}{2}p\right)^2=\frac{p^2}{4}$. Thus $x^2+px+\frac{p^2}{4}$ is a complete square.

ORAL EXERCISES

Complete the trinomial square in each of the following:

1. x^2+2x.
2. x^2+4x.
3. x^2+6x.
4. x^2+8x.
5. x^2+3x.
6. x^2+5x.
7. $(3x)^2+2(3x)$.
8. $(2x)^2+4(2x)$.
9. $16x^2+2(4x)$.

10. How do you complete the square in a^2-2ab? Is the rule different in this case?

11. Complete the square in each of the above exercises, after replacing the sign $+$ by $-$.

201. Solution of a Quadratic by Completing the Square.

Example 1. Solve the equation: $x^2 + 6x + 5 = 0$. (1)

Transposing in (1), $x^2 + 6x = -5$. (2)

To complete the square, we add $3^2 = 9$ to both members.

Thus we have $x^2 + 6x + 3^2 = 3^2 - 5 = 4$. (3)

Taking square roots of both sides, $x + 3 = \pm\sqrt{4} = \pm 2$.

Hence $x = -3 + 2 = -1$,

and $x = -3 - 2 = -5$.

Example 2. Solve the equation:

$$x^2 - 12x + 42 = 56. \qquad (1)$$

Transposing, $x^2 - 12x = 14$. (2)

Completing the square by adding $(\frac{12}{2})^2 = 6^2 = 36$ to both sides,

$$x^2 - 12x + 36 = 14 + 36 = 50. \qquad (3)$$

Taking square roots, $x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}$. (4)

Transposing, $x = 6 \pm 7.071$. (5)

Hence $x = 6 + 7.071 = 13.071$,

and also $x = 6 - 7.071 = -1.071$.

The steps involved in the above solutions are:

(1) *Write the equation in the form* $x^2 + px = q$.

(2) *Complete the square by adding* $(\frac{1}{2}p)^2$ *to each member.*

(3) *Take the square root of both members of this equation.*

(4) *Solve each of the first degree equations thus obtained.*

WRITTEN EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. (§ 180.) Use $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$.

1. $x^2 - 4x = 8$.
2. $x^2 = 3 - 6x$.
3. $4x = 16 - x^2$.
4. $x^2 + 6x = 9$.
5. $x^2 + 6x = 11$.
6. $x^2 - 12x = 12$.
7. $x^2 - 8x = -14$.
8. $x^2 = 2x + 1$.
9. $x^2 - 4x = 16$.
10. $x^2 = 23 + 4x$.
11. $8 = x^2 + 4x$.
12. $23 - 6x = x^2$.
13. $7 + 2x = x^2$.
14. $25 - x^2 = 5x$.
15. $x^2 + \frac{1}{3}x = 2$.

202. The Hindu Method of Completing the Square. In case the coefficient of x^2 is not unity, as in $3x^2 + 8x = 4$, both members may be divided by this coefficient, and the solution is then like that of Examples 1 and 2 on page 279.

However, the following method is sometimes desirable:

$$3x^2 + 8x = 4. \qquad (1)$$

Multiplying each member of equation (1) by $4 \cdot 3 = 12$, we get

$$36x^2 + 96x = 48. \qquad (2)$$

This can now be written in the form $x^2 + px = q$, namely, $(6x)^2 + 16(6x) = 48$, in which $p = 16$ and $6x$ is the unknown. Hence, we add $(\frac{16}{2})^2 = 8^2 = 64$ to complete the square, and get

$$(6x)^2 + 16(6x) + 64 = 48 + 64 = 112. \qquad (3)$$

Taking square roots, $6x + 8 = \pm\sqrt{112} = \pm 4\sqrt{7}. \qquad (4)$

Hence $6x = -8 \pm 4\sqrt{7},$

and $x = -\frac{4}{3} \pm \frac{2}{3}\sqrt{7}. \qquad (5)$

In this solution both sides were multiplied by 4 times the original coefficient of x^2, and then the number added to complete the square was found to be *the square of the original coefficient of x.*

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is equal to the square of the coefficient of x in the original equation.*

This is called the **Hindu method** of completing the square because it was first used by the Hindus.

NOTE. — Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

Hence, if the given equation is in the form $ax^2 + bx + c = 0$, we may complete the square *without dividing through by a*, by the following

Rule. 1. *Write the equation in the form* $ax^2 + bx = -c$.

2. *Multiply both sides by* $4a$.

3. *Now write the equation in the form*

$$(2ax)^2 + 2b(2ax) = -4ac.$$

4. *Complete the square by adding* b^2 *to both sides.*

5. *Solve as though* $2ax$ *were the unknown.*

EXERCISES

In the solution of the following equations the roots which contain surds may be left in simplified radical form.

1. $2x^2 + 3x = 2.$
2. $3x^2 + 5x = 2.$
3. $3x = 9 - 2x^2.$
4. $6x + 1 = -3x^2.$
5. $2x^2 = 5x + 3.$
6. $4x = 2x^2 - 1.$
7. $2x^2 - 3x = 14.$
8. $3x^2 = 9 + 2x.$
9. $4x^2 = 2x + 1.$
10. $6x - 1 = 3x^2.$
11. $2x^2 + 4x = 23.$
12. $3x^2 - 7 = 4x.$
13. $2x^2 - 5 = 3x.$
14. $4x^2 = 6x - 1.$
15. $2x = 1 - 5x^2.$
16. $3x - 20 = -2x^2.$
17. $2x + 3x^2 = 9.$
18. $4x^2 - 1 = 3x.$
19. $4x = 7 - 2x^2.$
20. $2x + 1 = 5x^2.$

CHECKING RESULTS IN QUADRATIC EQUATIONS

203. Illustrative Examples. 1. Solving $x^2 - 7x + 12 = 0$, we get $x = 4$ and $x = 3$.

What is the *sum* of these roots? How does this compare with the coefficient of x? What is the *product* of these roots? How does this compare with the known term of the equation?

2. Solving $x^2 - 6x + 4 = 0$, we get $x = 3 + \sqrt{5}$ and $x = 3 - \sqrt{5}$.

The sum of these roots, $(3 + \sqrt{5}) + (3 - \sqrt{5}) = 6$, is equal to the coefficient of x with the sign changed; and the product, $(3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 = 4$, is equal to the known term of the equation.

EXERCISES

Solve the following equations. In each case compare the product of the roots with the known term and the sum of the roots with the coefficient of x.

1. $x^2 - 5x + 3 = 0.$
2. $x^2 + 3x + 2 = 0.$
3. $x^2 + 9x + 8 = 0.$
4. $x^2 - 4x - 8 = 0.$
5. $x^2 + 6x - 3 = 0.$
6. $x^2 - 8x = 6.$

204. Relation of Roots and Coefficients. These exercises are illustrations of a **general rule** for all quadratics written in the form $x^2 + px + q = 0$, in which the coefficient of the squared term is $+1$; namely:

The sum of the roots is equal to the coefficient of x with its sign changed, i. e. $-p$; *and the product of the roots is equal to the known term,* q.

This may be used to check the results obtained in solving a quadratic. Note that *before applying the test* the equation must be in the form specified. See example on next page.

205. Solution of the Quadratic by Formula.

Solve the equation

$$ax^2 + bx + c = 0. \qquad (1)$$

Transposing, $ax^2 + bx = -c.$

Multiplying by $4a$, $4a^2x^2 + 4abx = -4ac. \qquad (2)$

Equation (2) may be written in the form

$$(2ax)^2 + 2b(2ax) = -4ac.$$

Completing the square as if $2ax$ were the unknown,

$$(2ax)^2 + 2b(2ax) + b^2 = b^2 - 4ac. \qquad (3)$$

Taking square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}. \qquad (4)$

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}.$

Dividing by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (5)$

Calling the two values of x in the result x_1 and x_2 we have,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Any quadratic equation may be reduced to the form of (1) by simplifying and collecting the coefficients of x^2 and x. Hence any quadratic equation may be solved by substituting in the formulas just obtained.

Example. Solve $2x^2 - 4x + 1 = 0.$

Substituting $a = 2,\ b = -4,\ c = 1$ in the formula,

we get
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2}.$$

From which
$$x_1 = \frac{2+\sqrt{2}}{2} \text{ and } x_2 = \frac{2-\sqrt{2}}{2}.$$

Check. Writing the equation in the form $x^2 + px + q = 0$, we have $x^2 - 2x + \frac{1}{2} = 0$, in which $p = -2$ and $q = \frac{1}{2}$.

Then
$$x_1 + x_2 = \frac{2+\sqrt{2}}{2} + \frac{2-\sqrt{2}}{2} = \frac{4}{2} = 2 = -p,$$

and
$$x_1 \cdot x_2 = \frac{2+\sqrt{2}}{2} \cdot \frac{2-\sqrt{2}}{2} = \frac{4-2}{4} = \frac{1}{2} = q.$$

EXERCISES

Solve the following equations by the formula and check all results by means of § 204.

1. $4x^2 + 1 = 8x.$
2. $2x^2 - 3x = 20.$
3. $2x^2 - 3 = -5x.$
4. $3x^2 + 4x = 8.$
5. $10 - 4x = 5x^2.$
6. $1 + 4x^2 = -6x.$
7. $5 - 3x = 2x^2.$
8. $7 + 4x = 2x^2.$
9. $6x^2 + 12x = 2.$
10. $6x^2 - 12x = -2.$
11. $6x^2 + 12x = -2.$
12. $6x^2 - 12x = 2.$
13. $3x^2 + 2x = 5.$
14. $2 + 3x = 2x^2.$
15. $8x + 1 = -4x^2.$
16. $8 + 4x = 3x^2.$
17. $10 + 4x = 5x^2.$
18. $2 + 5x = 3x^2.$
19. $3x + 14 = 2x^2.$
20. $3x^2 - 2x = 5.$
21. $2x^2 + 4x = 1.$
22. $4x^2 + 3x = 1.$
23. $2x^2 - 4x = 23.$
24. $2x^2 - 3x = -1.$
25. $3x + 9 = 2x^2.$
26. $3x^2 + 4x = 7.$
27. $2x - 1 = -4x^2.$
28. $5x^2 + 16x = -2.$
29. $6x^2 + 11x = 10.$
30. $5x^2 - 11x - 12 = 0.$

IMAGINARY NUMBERS *

206. There are quadratic equations which have no roots expressible in terms of the numbers of arithmetic or algebra thus far studied.

Example. Solve $x^2 - 4x = -8.$

Completing the square, $x^2 - 4x + 4 = -8 + 4 = -4.$

Taking the root and transposing, $x = 2 \pm \sqrt{-4}.$

$\sqrt{-4}$ is thus far unknown to us as a number symbol, but we will now enlarge our number system by including in it numbers of the type $\sqrt{-4}$.

207. Definition of Imaginaries. An even root of a negative number is called an *imaginary number.* All other numbers are called **real numbers.**

Using Principle XVIII we may reduce an imaginary number to the standard form $a\sqrt{-1}$, in which a is a real number.

E.g.
$$\sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4} \cdot \sqrt{-1} = 2\sqrt{-1}.$$
$$\sqrt{-5} = \sqrt{5 \times (-1)} = \sqrt{5} \cdot \sqrt{-1}.$$

208. The Fundamental Operations on Imaginary Numbers. In operating upon imaginary numbers, they should first be reduced to the standard form $a\sqrt{-1}$, and then treated as in the following examples:

(A) **Addition and Subtraction.**

(1)
$$\begin{aligned}\sqrt{-4} + \sqrt{-16} &= \sqrt{4}\sqrt{-1} + \sqrt{16}\sqrt{-1}\\ &= 2\sqrt{-1} + 4\sqrt{-1} = 6\sqrt{-1}.\end{aligned}$$

(2)
$$\begin{aligned}\sqrt{-9} - \sqrt{-5} &= \sqrt{9}\sqrt{-1} - \sqrt{5}\sqrt{-1}\\ &= 3\sqrt{-1} - \sqrt{5}\sqrt{-1} = (3 - \sqrt{5})\sqrt{-1}.\end{aligned}$$

Thus, in general, we have
$$a\sqrt{-1} + b\sqrt{-1} + c\sqrt{-1} = (a + b + c)\sqrt{-1}.$$

---

* Articles 206–209 may be omitted without destroying the continuity.

(B) **Multiplication** of imaginaries is based upon the principle that the *square of the square root of a number is the number itself.*

Thus, $$(\sqrt{-1})^2 = -1.$$

Applying this principle again, we have

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -1 \cdot \sqrt{-1} = -\sqrt{-1}.$$

And still again,

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \cdot (\sqrt{-1})^2 = (-1)(-1) = +1.$$

The results

$$(\sqrt{-1})^2 = -1;\ (\sqrt{-1})^3 = -\sqrt{-1};\ (\sqrt{-1})^4 = 1,$$

should be remembered. They are used in performing operations like the following:

(1) $$\sqrt{-2} \cdot \sqrt{-8} = \sqrt{2}\sqrt{-1} \cdot \sqrt{8}\sqrt{-1} = \sqrt{2}\sqrt{8}(\sqrt{-1})^2$$
$$\sqrt{16}(-1) = -\sqrt{16} = -4.$$

(2) $$\sqrt{-4} \cdot \sqrt{-9} \cdot \sqrt{-16} = \sqrt{4} \cdot \sqrt{9} \cdot \sqrt{16} \cdot (\sqrt{-1})^3$$
$$= 2 \cdot 3 \cdot 4(-\sqrt{-1}) = -24\sqrt{-1}.$$

(3) $$\sqrt{-4} \cdot \sqrt{-9} \cdot \sqrt{-16} \cdot \sqrt{-25} = 2 \cdot 3 \cdot 4 \cdot 5(\sqrt{-1})^4$$
$$= 120(+1) = 120.$$

In general, *two imaginary factors give a negative real product, three imaginary factors give a negative imaginary product, and four imaginary factors give a positive real product.*

(C) **Division** by an imaginary number is best indicated in the form of a fraction, as in the following examples:

(1) $$2 \div \sqrt{-3} = \frac{2}{\sqrt{-3}} = \frac{2\sqrt{-3}}{\sqrt{-3} \cdot \sqrt{-3}} = \frac{2\sqrt{-3}}{(\sqrt{-3})^2} = \frac{2\sqrt{-3}}{-3} = -\frac{2}{3}\sqrt{-3}.$$

(2) $$\sqrt{-9} \div \sqrt{-16} = \frac{\sqrt{9} \cdot \sqrt{-1}}{\sqrt{16} \cdot \sqrt{-1}} = \frac{3}{4} \cdot \frac{\sqrt{-1}}{\sqrt{-1}} = \frac{3}{4} \cdot 1 = \frac{3}{4}.$$

EXERCISES

Addition and Subtraction.

1. $\sqrt{-4} + \sqrt{-9} - \sqrt{-16}.$
2. $\sqrt{-25} - \sqrt{-9} - \sqrt{-4}.$
3. $\sqrt{-2} + \sqrt{-3} - \sqrt{-1}.$
4. $\sqrt{-4} + \sqrt{-3} - \sqrt{-16}.$
5. $\sqrt{-5} + 4\sqrt{-5} + 7\sqrt{-5}.$
6. $\sqrt{-3} - 3\sqrt{-3} - 4\sqrt{-3}.$

EXERCISES

Multiplication and Division.

1. $\sqrt{-1}\cdot\sqrt{-1}$.
2. $(\sqrt{-1})^3$.
3. $(\sqrt{-1})^4$.
4. $(\sqrt{-1})^5$.
5. $\sqrt{-2}\cdot\sqrt{-2}$.
6. $\sqrt{-2}\cdot\sqrt{-3}$.
7. $\sqrt{-16}\cdot\sqrt{-9}$.
8. $\sqrt{-2}\cdot\sqrt{-18}$.
9. $\sqrt{-16}\div\sqrt{-9}$.
10. $\sqrt{-4}\div\sqrt{-5}$.
11. $\sqrt{-3}\div(\sqrt{-1})^3$.
12. $(\sqrt{-5})^2\div(\sqrt{-2})^3$.
13. Multiply $2\sqrt{-2}-3\sqrt{-3}$ by $3\sqrt{-2}-2\sqrt{-3}$.

Solution.

$$\begin{array}{l}
2\sqrt{-2}-3\sqrt{-3}\\
\underline{3\sqrt{-2}-2\sqrt{-3}\qquad\qquad}\\
6(\sqrt{-2})^2-9\sqrt{2}\sqrt{3}(\sqrt{-1})^2\\
\qquad\qquad\underline{-4\sqrt{2}\sqrt{3}(\sqrt{-1})^2+6(\sqrt{-3})^2}\\
6(-2)-13\sqrt{6}(-1)+6(-3)
\end{array}$$

Simplifying, $\quad -20+13\sqrt{6}$.

14. $(2-\sqrt{-5})(2+\sqrt{-5})$.
15. $(\sqrt{-5}+\sqrt{-3})(\sqrt{-5}-\sqrt{-3})$.
16. $(2-3\sqrt{-7})(2-5\sqrt{-7})$.
17. $(3-\sqrt{-2})(3-4\sqrt{-2})$.
18. $(2+\sqrt{-2})^2$.
19. $(3-\sqrt{-5})^2$.
20. $(\sqrt{-3}+\sqrt{-2})^2$.
21. $(\sqrt{-2}-\sqrt{-8})^2$.

209. We may now show that the values of x found in the example of § 206 do actually satisfy the equation.

(1) Substituting $x=2+\sqrt{-4}$ in the equation $x^2-4x=-8$, we have $(2+\sqrt{-4})^2-4(2+\sqrt{-4})$, which should reduce to -8.

Squaring $2+\sqrt{-4}$, we have $2^2+2\cdot 2\sqrt{-4}+(\sqrt{-4})^2$.

Simplifying this, we get $4+4\sqrt{-4}-4=4\sqrt{-4}$.

Finally, subtracting $4(2+\sqrt{-4})$, we have $4\sqrt{-4}-8-4\sqrt{-4}=-8$.

Hence the left side of the equation reduces to -8 when $2+\sqrt{-4}$ is substituted for x, and the given equation is satisfied.

(2) In the same manner show that $2-\sqrt{-4}$ satisfies the equation.

Quadratics with Imaginary Roots.

Solve the following equations. Simplify each result.

1. $7 - 3x = -5x^2.$
2. $11x - 33 = 3x^2.$
3. $14x + 8 - x^2 = 52 + 3x^2.$
4. $12 - 15x = 36 + 6x^2.$
5. $5x + x^2 + 8 = 0.$
6. $5x^2 - 10x = -6.$
7. $x^2 + 8 + 3x = -x.$
8. $11x^2 - 49x + 57 = 0.$
9. $3x^2 + 18 - 12x = 5.$
10. $37 - 4x^2 - 12x = 79.$
11. $10x^2 + 46 + 7x = 44.$
12. $45 + 3x^2 - 2x = 0.$

FRACTIONAL EQUATIONS LEADING TO QUADRATICS

Solve the following equations and check each solution by substituting in the original equation, except when the answer is given:

1. $\frac{3x-1}{x+1} - \frac{4x+3}{x-1} + \frac{x^2}{x^2-1} = -\frac{27}{x^2-1} + 1.$

2. $\frac{3x+5}{x-9} + \frac{2x+1}{x+2} = \frac{x-1}{x^2-7x-18}.$ *Ans.* $x = \frac{2}{5}$, or 1.

3. $\frac{x-4}{2x-10} - \frac{3x-15}{2x-6} = -\frac{3x^2-114}{4x^2-32x+60}.$

4. $\frac{6(x+4)}{x+5} - \frac{3(2x-1)}{x+1} = \frac{7}{2}.$ *Ans.* $x = 1$, or $-6\frac{1}{7}$.

5. $\frac{3x-4}{x-4} + \frac{5x-7}{2x-2} = \frac{9x^2-38}{2x^2-10x+8}.$ *Ans.* $x = 18\frac{1}{2}$, or 2.

6. $\frac{x-1}{x-2} + \frac{x-2}{3-x} = \frac{x-3}{x-4} - \frac{x-5}{x-6}.$ *Ans.* $x = \pm 2\sqrt{3}.$

7. $\frac{x+2}{x-5} + \frac{3x-15}{x-3} = \frac{3x-21}{x-3}.$

8. $\frac{2x-3}{-4x} + \frac{3x+1}{x-2} = \frac{4x+17}{x-2}.$ *Ans.* $x = \frac{-19 \pm \sqrt{345}}{4}.$

9. $\frac{3x-2}{2x+3} = \frac{2x^2+15x+28}{2x^2+5x+3} + \frac{2x-1}{x+1}.$

10. $\frac{2x-3}{2x+2} - \frac{x-8}{5x+2} = \frac{x+2}{2x+2}.$

MISCELLANEOUS QUADRATICS

Solve as many as possible of the following equations by factoring. Otherwise use the formula of § 205. In the case of surd solutions, find the results correct to two places of decimals.

1. $x^2 + 11x = 210.$
2. $5x^2 - 3x = 4.$
3. $7x + 3x^2 - 18 = 0.$
4. $2 = 5x + 7x^2.$
5. $6x - 11x^2 = -7.$
6. $-51 + 42x - 3x^2 = 0.$
7. $3x^2 + 3x = 2x + 4.$
8.* $13 - 8x + 3x^2 = 0.$
9. $2x^2 + 11x = 32x - x^2 - 27.$
10. $176 + 3x - x^2 = 2x.$
11. $x^2 + 6x - 54 = 0.$
12. $5x^2 + 9x + 12 = 4x^2 + x.$
13. $2x^2 - 4x - 25 = 0.$
14. $7x^2 + 11x = 6.$
15. $2x^2 - 11x + 5 = 0.$
16. $2x^2 - 11x = 6.$
17. $25x - 95 = x^2.$
18. $11x^2 - 42x = 2.$
19. $x^2 - 8x - 4 = x - 22.$
20.* $8x^2 + 5x = -8.$
21. $2x^2 + 3x - 3 = 12x + 2.$
22. $3x^2 - 7x = 10.$
23. $17x + 31 + 2x^2 = 0$
24. $18 - 41x = 3 + x^2.$
25. $10x + 25 = 5 - 2x - x^2.$
26. $3x - 59 + x^2 = 0.$
27. $5x^2 + 7x - 6 = 0.$
28. $x^2 + 12 = 7x.$
29. $8x - 5x^2 = 2.$
30. $5x + 3x^2 - 22 = 0.$
31. $50 + 20x + x^2 = 5x.$
32.* $x^2 + x + 4 = 0.$
33. $20x + 2x^2 + 42 = 33x + x^2.$
34. $17x - 3x^2 = -6.$
35. $8x + 5x^2 = -2.$
36. $10 + 15x + x^2 = 26x.$
37. $3x^2 - 2x - 7 = 0.$
38. $5x^2 - 9x - 18 = 0.$
39. $7x - 7x^2 + 24 = 0.$
40.* $31 + 2x + x^2 = 0.$
41. $7x^2 + 7x - 5x^2 + 20 = x^2 - 2x + 2.$
42. $5x^2 + 3x - 7 = (x - 1)(x + 2).$
43. $(x - 3)^2 - (2x - 1)(2x + 1) + 7 = 0.$
44.* $3x + (3x - 2)^2 = 4x^2 - 1.$
45. $9x^2 - (2x - 1)^2 = (x + 3)^2.$
46. $7x^2 = 5x - (x - 2)^2 + 7.$

EQUATIONS IN THE QUADRATIC FORM

210. Special Cases.* Sometimes equations of a higher degree than the second may be solved by means of quadratics, as in the following examples:

Example 1. Solve $x^4 - x^2 - 12 = 0$.

Factoring, $(x^2 - 4)(x^2 + 3) = 0$.
Putting $x^2 - 4 = 0$, or $x^2 = 4$, we have $x = \pm 2$.
Putting $x^2 + 3 = 0$, or $x^2 = -3$, we have $x = \pm\sqrt{-3}$.
Checking $x = 2$, $2^4 - 2^2 - 12 = 16 - 4 - 12 = 0$.
Checking $x = \sqrt{-3}$, $(\sqrt{3})^4(\sqrt{-1})^4 - (\sqrt{3})^2(\sqrt{-1})^2 - 12$
$= 9(+1) - 3(-1) - 12 = 12 - 12 = 0$.
Let the student check for $x = -2$ and $x = -\sqrt{-3}$.

Example 2. Solve $(x^2 + 2)^2 - 7(x^2 + 2) + 12 = 0$.

Consider $x^2 + 2$ as the unknown and call it z.
Then the equation is $z^2 - 7z + 12 = 0$.
Factoring, $(z - 4)(z - 3) = 0$.
From $z - 4 = 0$ or $x^2 + 2 - 4 = 0$,
we get $x^2 = 2$ and $x = \pm\sqrt{2}$.
From $z - 3 = 0$ or $x^2 + 2 - 3 = 0$,
we get $x^2 = 1$ or $x = \pm 1$.
Check. Substitute in the original equation.

WRITTEN EXERCISES

1. $x^4 - 5x^2 + 6 = 0$.
2. $x^4 + 5x^2 - 24 = 0$.
3. $x^4 - 3x^2 - 70 = 0$.
4. $2x^4 - x^2 - 6 = 0$.
5. $3x^4 - 10x^2 + 8 = 0$.
6. $x^4 - 7x^2 + 12 = 0$.
7. $(x^2 - 5)^2 - 9(x^2 - 5) + 20 = 0$.
8. $(x^2 + x - 2)^2 + 3(x^2 + x - 2) - 10 = 0$.
9. $(x^2 - 5x + 4)^2 - (x^2 - 5x + 4) - 20 = 0$.
10. $(x^2 - 3x - 4)^2 + 7(x^2 - 3x - 4) - 8 = 0$.

---

* In case Articles 206–209 have been omitted, then the solutions which involve the use of imaginary numbers may be omitted here.

Equations Containing Radicals.

Example 3. Solve $x - 5 + 2\sqrt{x-5} = 8$.

Regard $\sqrt{x-5}$ as the unknown and call it z. Then $x-5$ is the square of z.

Then $z^2 = (\sqrt{x-5})^2 = x - 5.$
Hence the equation is $z^2 + 2z - 8 = 0.$
Factoring, $(z+4)(z-2) = 0.$
Hence $z = -4$ and $z = 2.$
Then $\sqrt{x-5} = -4$ and $\sqrt{x-5} = 2.$
Hence $x - 5 = 16$ and $x - 5 = 4.$
From which $x = 21$ and $x = 9.$

Note that $x = 21$ does not satisfy the original equation, but that $x = 9$ does.

Example 4. Solve $\sqrt{x} - 5\sqrt[4]{x} + 6 = 0$.

Call $\sqrt[4]{x}$ the unknown and represent it by z. Then the first term, $\sqrt{x}$, is the square of z.

Then $z^2 = (\sqrt[4]{x})^2 = (x^{\frac{1}{4}})^2 = x^{\frac{1}{2}} = \sqrt{x}.$
Hence the equation is $z^2 - 5z + 6 = 0.$
Factoring, $(z-3)(z-2) = 0.$
Hence $z = 3$ and $z = 2,$
from which $\sqrt[4]{x} = 3$ and $\sqrt[4]{x} = 2.$
Hence $x = 81$ and $x = 16.$

Check. Substitute each value of x in the given equation.

WRITTEN EXERCISES

Find all the roots and check in the first eight:

1. $x + 7\sqrt{x} - 30 = 0.$
2. $\sqrt{x} + 3\sqrt[4]{x} - 28 = 0.$
3. $x - 4\sqrt{x} - 12 = 0.$
4. $x + 5\sqrt{x} - 14 = 0.$
5. $\sqrt{x} - 3\sqrt[4]{x} - 10 = 0.$
6. $x + 3 - 4\sqrt{x+3} + 3 = 0.$
7. $x - 2 - 6\sqrt{x-2} - 7 = 0.$
8. $x + 5 - \sqrt{x+5} = 6.$
9. $x^2 + 3x + 2 + 3\sqrt{x^2 + 3x + 2} = 0.$
10. $x^2 - 7x + 10 + \sqrt{x^2 - 7x + 10} = 0.$
11.* $2x^2 + 3x + 6 - 2\sqrt{2x^2 + 3x + 9} = 0.$

HISTORICAL NOTES

Quadratic Equations. The solution of the quadratic was effected by the Greek mathematician Diophantus in much the same manner that we solve it. He did not, however, admit negative roots.

Alkarismi, an Arabian, divided all quadratic equations into five classes; namely, $ax^2 = bx$, $ax^2 = c$, $ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$ when a, b, c are all positive numbers. This is an interesting instance of the complexities which we encounter when we fail to recognize that letters may represent negative as well as positive numbers. Alkarismi did not admit negative roots, but he recognized the possibility of a quadratic equation having two positive roots. Bhaskara, a Hindu mathematician, gives $x = 50$ and $x = -5$ as roots of $x^2 - 45x = 250$ but remarks that "The second root is not to be taken, for it is inadequate; people do not approve of negative roots." It was only after the time of Descartes (see page 240) that negative roots came into perfectly good standing.

Imaginary Roots.* The imaginary roots of equations were entirely rejected by the Ancients. Cardan in Italy (1501–1576), and Harriot in England (1560–1621), mentioned them, but simply said that they indicated that the equations which gave rise to them represented impossible conditions. It was not until the first decade of the nineteenth century when Argand in Switzerland and Gauss in Germany (see next page) gave a geometrical interpretation of imaginaries that they came to be fully understood. In this respect imaginaries shared the fate of negative numbers, which were not fully understood until Descartes in France applied them to geometry.

PROBLEMS INVOLVING QUADRATICS

Find two solutions in each case, and determine whether both are applicable to the problem.

1. Find two consecutive integers whose product is 272.

2. Find two consecutive integers whose product is 182.

3. Find two consecutive integers whose product is 992.

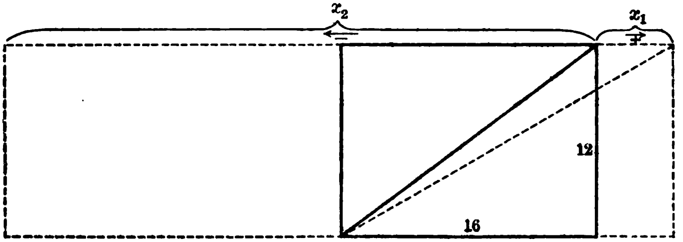
4. The sum of the squares of two consecutive integers is 145. Find the numbers.

5. Find a number such that four times the number added to 12 is equal to the square of the number.

6. Find a number such that nine times the number is equal to its square plus 18.

7. A picture is 12 inches wide and 17 inches long inside the frame. How wide is the frame if its diagonal is 25 inches?

8. A rectangle is 12 inches wide and 16 inches long. How much must be added to the length to increase the diagonal by 4 inches?



Let x = number of inches to be added to the length. The diagonal of the original rectangle is $\sqrt{12^2 + 16^2} = 20$. Hence the diagonal of the required rectangle is 24.

Then $$12^2 + (16 + x)^2 = 24^2,$$

or $$x^2 + 32x - 176 = 0.$$

Solving, $$x_1 = -16 + 12\sqrt{3} = 4.78,$$

and $$x_2 = -16 - 12\sqrt{3} = -36.78.$$

The negative solution obtained here may be taken to mean that if the rectangle is extended in the *opposite* direction from the fixed corner, we shall get a rectangle which has the required diagonal. See the figure.

9. How much must the *width* of the rectangle in Problem 8 be extended so as to increase the diagonal by 4 inches?

10. A rectangle is 21 inches long and 20 inches wide. The length of the rectangle is decreased twice as much as the width, thereby decreasing the length of the diagonal 4 inches. Find the dimensions of the new rectangle.

Karl Friedrich Gauss was born at Brunswick, Germany, in 1777 and died at Göttingen in 1855. Justly called the greatest mathematician of modern times, he was scarcely less famous as physicist and astronomer; in fact, he was director of the observatory and professor of astronomy at Göttingen during the last forty years of his life.

He wrote on almost every phase of mathematics, and in many subjects the modern development is largely due to his genius.

Gauss was the first to explain satisfactorily the nature of imaginary quantities and to put their use in solving equations on a systematic and scientific basis. He brought into general use the symbol i for the imaginary unit $\sqrt{-1}$.

11. The distance in feet which a body falls in t seconds is expressed by $d = vt + 16\,t^2$, where v is the velocity with which the body starts.

How long will it take a body to fall from the top of the Eiffel tower, a distance of 900 feet, if it starts with a velocity of 100 feet per second?

Suggestion. Solve the equation

$$16\,t^2 + 100\,t = 900.$$

12. How long will it take a body to fall 900 feet if it starts with a velocity of 50 feet per second?

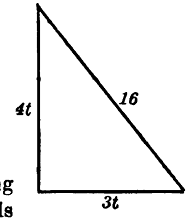
13. How long will it take a body to fall 900 feet if it starts at 10 feet per second?

14. How long will it take a bomb to fall from an aeroplane 900 feet high if it starts with no initial velocity?

15. A and B start from a certain cross roads at the same time, A going north 4 miles per hour and B going east 3 miles per hour. In how many hours will they be 16 miles apart, measuring in a straight line across country?

Solution. Let t equal the required number of hours.

Then $$(4\,t)^2 + (3\,t)^2 = 16^2 = 256.$$
$$16\,t^2 + 9\,t^2 = 256.$$
$$25\,t^2 = 256.$$
$$5\,t = \pm\,16.$$
$$t = \pm\,3\tfrac{1}{5}.$$



The solution $t = -\,3\tfrac{1}{5}$ may be interpreted as meaning that if the two men had been traveling along these roads before reaching the cross road, they would have been 16 miles apart $3\tfrac{1}{5}$ hours before meeting.

16. In the preceding problem if A goes 5 miles per hour and B goes 4 miles per hour, in how many hours will they be 24 miles apart? *Ans.* $3\tfrac{3}{4}$ hours.

17. The sides of a right triangle are 6 and 8 inches respectively. How much must be added to each side in order to make the hypotenuse 20 inches? It is understood that each side is increased by the same amount.

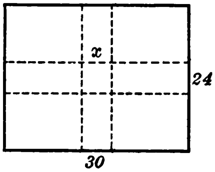
18. By what number must each side in Example 17 be *multiplied* in order to make the hypotenuse 20?

19. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

20. A fancy quilt is 72 inches long and 56 inches wide. It is decided to increase its area 10 square feet by adding a border. Find the width of the border.

21. A farmer starts cutting grain around a field 120 rods long and 70 rods wide. How wide a strip must be cut to make 12 acres?

22. In a rectangular table cover 24 by 30 inches there are two strips of drawn work of equal width running at right angles through the center of the piece. What is the width of these strips if the drawn work covers one tenth of the whole piece?

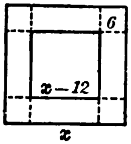


23. An athletic field is 800 feet long and 600 feet wide. The field is to be extended by the same amount in length and width so that the longest possible straight course (the diagonal) shall be 1100 feet. How much must the field be extended in each direction? *Ans.* 71.36 feet.

24. A farm is 320 rods long and 280 rods wide. There is a road running around the boundary of the farm and lying entirely within it. Find the width of the road if the area covered by it is 2384 square rods.

25. A rectangular park is 480 rods long and 360 rods wide. A walk is laid out completely around the park, and a drive is laid out through the length of the park parallel to the sides. What is the width of the walk if the drive is 3 times as wide as the walk and the combined area of the walk and the drive is 3110 square rods?

26. A square piece of tin is made into an open box, containing 864 cubic inches, by cutting out a 6-inch square from each corner of the tin and then turning up the sides. Find the dimensions of the original piece of tin.



27. A rectangular piece of tin is 8 inches longer than it is wide. By cutting out a 7-inch square from each corner and turning up the sides, an open box containing 1260 cubic inches is formed. Find the dimensions of the original piece of tin.

28. By cutting out a square 8 inches on a side from each corner of a sheet of metal and turning up the sides, we obtain an open box such that the area of the sides and ends is 4 times the area of the bottom. Find the dimensions of the original sheet if it is twice as long as it is wide.

Ans. 41.48 in. by 20.74 in.

29. An open box whose bottom is a square has a lateral area which is 400 square inches more than the area of the bottom. Find the other dimensions of the box if it is 10 inches high. (By lateral area is meant the sum of the areas of the four sides.)

30. A box whose bottom is 4 times as long as it is wide has a lateral area 600 square inches less than 4 times the area of the bottom. Find the dimensions of the bottom if the box is 6 inches high.

REVIEW QUESTIONS

1. Explain the method of solving a quadratic equation by *factoring.* Can you apply this to $(x+1)(x-2)=5$? Why?

2. Explain the method by *completing the square.* How can this be done so as to avoid fractions until the end of the solution?

3. Explain the solution of the quadratic by means of the *formula.* Why does solving the equation $ax^2+bx+c=0$ give a formula for *all* quadratic equations? Describe this *type form* of quadratic equation in words.

4.* How many roots has a quadratic equation? Find the roots of $x^2+2x+5=0$. What are these roots called? Solve $x^2+2x-5=0$. What are these roots called?

5. How are the roots of the equation $x^2+px+q=0$ related to p and q? How may this be used in checking the solutions?

Use $2x^2-4x+1=0$ to show how the *sum* of the roots and *product* of the roots may be found from the given equation.

6. When is an equation said to be in the *quadratic form?* Which of the following are in this form?

$$x^6-3x^3-4=0;$$
$$x^2-4-2\sqrt{x^2-4}-3=0;$$
$$(x^2+1)^2+3(x^2+1)+2=0;$$
$$x^2-3x+1+\sqrt{x^2-3x+4}+5=0.$$

How can the last one be put into the quadratic form?

7. Do both roots of a quadratic equation necessarily satisfy the conditions of the problem from which such an equation may be derived? In checking the solution of a problem is it sufficient to make the test alone in the equation derived from the problem?

CHAPTER XIX

SYSTEMS OF QUADRATICS

ONE QUADRATIC AND ONE LINEAR EQUATION

211. When two simultaneous equations are given, one quadratic and one linear, they may be solved by the process of substitution, which was used (§ 150) in the case of two linear equations.

Solutions by Factoring.

Example. Solve the equations:

$$\begin{cases} x^2 - y^2 = -16. & (1) \\ x - 3y = -12. & (2) \end{cases}$$

From (2) by *S*, $x = 3y - 12.$ (3)

Substituting (3) in (1), $(3y - 12)^2 - y^2 = -16.$ (4)

From (4) by *F*, $9y^2 - 72y + 144 - y^2 = -16.$ (5)

From (5) by *F*, *A*, $8y^2 - 72y + 160 = 0.$ (6)

By *D*, $y^2 - 9y + 20 = 0.$ (7)

Factoring, $(y - 5)(y - 4) = 0.$ (8)

Hence, $y = 5$, and $y = 4.$ (9)

Substitute $y = 5$ in (3) and find $x = 3.$

Substitute $y = 4$ in (3) and find $x = 0.$

Therefore (1) and (3) are satisfied by the two pairs of values,

$$x = 3,\ y = 5; \qquad \text{and } x = 0,\ y = 4.$$

Check by substituting these pairs of values in (1) and (2).

Whenever the quadratic in one unknown, resulting from the substitution, can be solved by *factoring*, this method should be used as in the above solution, beginning at equation (8).

When the solution by factoring is not possible, then the formula should be used, as in the example on page 299.

EXERCISES

In the manner just illustrated solve the following:

1. $\begin{cases} x + 2y = 8, \\ 5x^2 + 12y^2 = 128. \end{cases}$

2. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 1. \end{cases}$

3. $\begin{cases} 2x - y = 6, \\ 4x^2 + 5y^2 = 36. \end{cases}$

4. $\begin{cases} x + 3y = 6, \\ x^2 + 3y^2 = 12. \end{cases}$

5. $\begin{cases} x - 2y = -2, \\ x^2 - 6y^2 = 10. \end{cases}$

6. $\begin{cases} 8x - 16y = -120, \\ 7x^2 + 2y^2 = 585. \end{cases}$

7. $\begin{cases} 7x + 9y = 88, \\ 7x^2 + 9y^2 = 736. \end{cases}$

8. $\begin{cases} x - y = 6, \\ x^2 - 7y^2 = 36. \end{cases}$

9. $\begin{cases} 3x + 2y = 7, \\ 3x^2 + 8y^2 = 35. \end{cases}$

10. $\begin{cases} x - 8y = -11, \\ 3x^2 - 16y^2 = 11. \end{cases}$

11. $\begin{cases} x - y = -7, \\ 4x^2 + 3y^2 = 147. \end{cases}$

12. $\begin{cases} x - y = 2, \\ x^2 - 5y^2 = 4. \end{cases}$

13. $\begin{cases} x - y = 1, \\ 3x^2 - 2y^2 = -5. \end{cases}$

14. $\begin{cases} 5x - 7y = -28, \\ 15x^2 + 49y^2 = 784. \end{cases}$

15. $\begin{cases} 6x - 7y = 18, \\ 36x^2 - 7y^2 = 324. \end{cases}$

16. $\begin{cases} x - 9y = 2, \\ x^2 - 45y^2 = 4. \end{cases}$

17. $\begin{cases} x + y = 8, \\ 13x^2 + 3y^2 = 160. \end{cases}$

18. $\begin{cases} 2x - 5y = -16. \\ 4x^2 + 15y^2 = 256. \end{cases}$

19. $\begin{cases} 7x + 4y = 7, \\ 49x^2 - 8y^2 = 49. \end{cases}$

20. $\begin{cases} x - 3y = -12, \\ x^2 - y^2 = -16. \end{cases}$

21. $\begin{cases} x + 2y = 3, \\ 3x^2 + 4y^2 = 7. \end{cases}$

22. $\begin{cases} 3x - y = 5, \\ 3x^2 - y^2 = 11. \end{cases}$

23. $\begin{cases} x + 2y = 7, \\ 4x^2 - y^2 = 32. \end{cases}$

24. $\begin{cases} 5x + 2y = 13, \\ x + y^2 = 17. \end{cases}$

25. $\begin{cases} 4x - y = 3, \\ x^2 + 2y^2 = 54. \end{cases}$

26. $\begin{cases} x = y - 1, \\ 5x^2 + 2y = 53. \end{cases}$

27. $\begin{cases} 2y = 3x + 1, \\ 3x^2 = y^2 + 2. \end{cases}$

28. $\begin{cases} x = 3y + 3, \\ 2x^2 - 5y^2 = 67. \end{cases}$

212. Solutions by the Quadratic Formula.

Example. Solve
$$\begin{cases} x + y = 3, & (1) \\ 3x^2 - y^2 = 14. & (2) \end{cases}$$

From (1), $y = 3 - x.$ (3)

Substituting in (2) and reducing,
$$2x^2 + 6x - 23 = 0. \qquad (4)$$

Equation (4) is in the form $ax^2 + bx + c$, in which $a = 2, b = 6, c = -23$. Substituting in the formula, § 205,
$$x = \frac{-6 \pm \sqrt{36 - 4 \cdot 2(-23)}}{4} = \frac{-3 \pm \sqrt{55}}{2}. \qquad (5)$$

Hence $x_1 = 2.21$ and $x_2 = -5.21.$

Substituting these values of x in (1) we have as the approximate roots,
$$\left.\begin{matrix} x_1 = 2.21 \\ y_1 = 0.79 \end{matrix}\right\} \text{ and } \left.\begin{matrix} x_2 = -5.21 \\ y_2 = 8.21 \end{matrix}\right\}.$$

y_1 and y_2 are here used to designate those values of y which correspond to x_1 and x_2 respectively.

EXERCISES

In the above manner solve the following systems of equations, finding in each case two pairs of values.

1. $\begin{cases} x - y = 1, \\ x^2 + y^2 = 13. \end{cases}$

2. $\begin{cases} x + y = 9, \\ x^2 + y^2 = 41. \end{cases}$

3. $\begin{cases} x + y = 13, \\ xy = 42. \end{cases}$

4. $\begin{cases} 3x - y = 5, \\ x^2 + y^2 = 25. \end{cases}$

5. $\begin{cases} \frac{x^2}{9} - \frac{y^2}{4} = 3, \\ x - y = 4. \end{cases}$

6. $\begin{cases} x - y = 1, \\ \frac{x^2}{36} + \frac{y^2}{16} = \frac{1}{2}. \end{cases}$

7. $\begin{cases} 2x + y = 5, \\ 3x^2 - 5y^2 = 7. \end{cases}$

8. $\begin{cases} x - y = 3, \\ x^2 - 3y^2 = 13. \end{cases}$

9. $\begin{cases} 3x - 4y = 1, \\ x^2 - y^2 = 24. \end{cases}$

10. $\begin{cases} x + y = 4, \\ 2x^2 - 3xy + y^2 = 8. \end{cases}$

11. $\begin{cases} x - y = 1, \\ 4x^2 + 2xy - y^2 = 19. \end{cases}$

12. $\begin{cases} 5x + y = 12, \\ 2x^2 - 3xy + y^2 = 0. \end{cases}$

13. $\begin{cases} x + y = 9, \\ x^2 - 2y^2 = -7. \end{cases}$

213. **Special Case.** Certain systems involving quadratics may be solved by special devices.

Example 1. Solve $\begin{cases} x+y=5, & (1)\\ x^2+y^2=13. & (2)\end{cases}$

Solution. Squaring (1) $x^2+2xy+y^2=25$ (3)

Subtracting (2) from (3), $\underline{x^2 \qquad\quad + y^2 = 13}$

$2xy \qquad = 12$ (4)

Subtracting (4) from (2), $\underline{x^2 \qquad\quad + y^2 = 13}$

$x^2-2xy+y^2=1$

Extracting the square root, $x-y=\pm 1$ (4)

Solving the system $\begin{cases} x+y=5\\ x-y=1\end{cases}$ we find $\begin{cases} x=3\\ y=2.\end{cases}$

Solving the system $\begin{cases} x+y=5\\ x-y=-1\end{cases}$ we find $\begin{cases} x=2\\ y=3.\end{cases}$

Example 2. Solve $\begin{cases} xy=8. & (1)\\ x^2+y^2=20. & (2)\end{cases}$

Adding twice equation (1) to (2), $x^2+2xy+y^2=36.$ (3)

Taking square roots, $x+y=\pm 6.$ (4)

Subtracting twice equation (1) from (2), $x^2-2xy+y^2=4.$ (5)

Taking square roots, $x-y=\pm 2.$ (6)

Solving the systems

$$\begin{cases} x+y=6\\ x-y=2\end{cases};\quad \begin{cases} x+y=6\\ x-y=-2\end{cases};\quad \begin{cases} x+y=-6\\ x-y=2\end{cases};\quad \begin{cases} x+y=-6\\ x-y=-2\end{cases}.$$

we find four pairs of solutions.

WRITTEN EXERCISES

1. Solve by the above method Examples 1, 2, 3, in the preceding exercise.

Solve by the same method:

2. $\begin{cases} xy=12,\\ x-y=1.\end{cases}$

3. $\begin{cases} xy=12,\\ x^2+y^2=25.\end{cases}$

4. $\begin{cases} xy=6,\\ x^2-xy+y^2=7.\end{cases}$

5. $\begin{cases} xy=10,\\ x^2-xy+y^2=19.\end{cases}$

6. $\begin{cases} xy=-6,\\ x^2-3xy+y^2=31.\end{cases}$

7. $\begin{cases} xy=4,\\ 2x^2+5xy+2y^2=54.\end{cases}$

A SYSTEM OF TWO QUADRATICS*

214. Homogeneous Quadratics. A quadratic equation in x and y is *homogeneous* when every term is of the second degree in x and y, that is, when every term contains x^2, xy, or y^2.

E.g. $2x^2+3xy+y^2=0$ is homogeneous in x and y.

$x^2+3x+y^2=0$ is not homogeneous, because the term $3x$ is of the first degree; and $x^2+xy+y^2=5$ is not homogeneous because the term 5 contains neither x nor y.

If a quadratic equation which is homogeneous in x and y is divided through by x^2, every term will then contain $\frac{y}{x}$, and the equation may be solved for $\frac{y}{x}$ considered as the unknown, as in the following example:

Example. Solve for $\frac{y}{x}$ the equation $2x^2+3xy+y^2=0$.

Dividing by x^2, and writing $\frac{y^2}{x^2}$ first, we have

$$\frac{y^2}{x^2}+3\frac{y}{x}+2=0.$$

This is now a quadratic in which $\frac{y}{x}$ is the unknown.

Factoring, $$\left(\frac{y}{x}+2\right)\left(\frac{y}{x}+1\right)=0.$$

Hence, $$\frac{y}{x}=-2 \text{ and } \frac{y}{x}=-1.$$

Hence the homogeneous quadratic equation, $2x^2+3xy+y^2=0$, is equivalent to the two linear equations, $y=-2x$ and $y=-x$.

WRITTEN EXERCISES

Solve the following equations, regarding $\frac{y}{x}$ as the unknown.

1. $y^2-3xy+2x^2=0$.
2. $y^2+7xy-30x^2=0$.
3. $y^2-11xy+30x^2=0$.
4. $y^2-7xy-18x^2=0$.
5. $3x^2+11xy-20y^2=0$.
6. $x^2+8xy+16y^2=0$.
7. $2x^2+3xy-2y^2=0$.
8. $x^2+2xy-63y^2=0$.
9. $2x^2-14xy-60y^2=0$.
10. $24x^2-12xy-12y^2=0$.

---

* Articles 214–218 may be omitted without destroying the continuity.

215. Systems in which One Equation is Homogeneous. If one of two quadratic equations in x and y is homogeneous, the complete solution may be found as in the following:

Example. Solve
$$\begin{cases} 2x^2 + 3xy + y^2 = 0, & (1) \\ x^2 - 5x + 2y^2 = 4. & (2) \end{cases}$$

Since (1) is homogeneous, we may solve it for $\frac{y}{x}$ and find, as in the example solved in § 214,

$$\frac{y}{x} = -2 \text{ and } \frac{y}{x} = -1.$$

Hence, $$y = -2x \text{ and } y = -x.$$

First. Substituting $y = -2x$ in (2), $x^2 - 5x + 8x^2 = 4.$ (3)

Transposing, $9x^2 - 5x - 4 = 0.$

Factoring, $(x-1)(9x+4) = 0.$

Hence, $x = 1$ and $x = -\frac{4}{9}.$

Hence we have

$$\begin{cases} x = 1, \\ y = -2x = -2, \end{cases} \text{ and } \begin{cases} x = -\frac{4}{9}, \\ y = -2x = -2(-\frac{4}{9}) = \frac{8}{9}. \end{cases}$$

Second. Substituting $y = -x$ in (2), $x^2 - 5x + 2x^2 = 4.$ (4)

Transposing, $3x^2 - 5x - 4 = 0.$

This is in the form $ax^2 + bx + c = 0$, in which $a = 3$, $b = -5$, $c = -4$.

Solving by the formula, § 205, $x = \frac{5 + \sqrt{73}}{6}$ and $x = \frac{5 - \sqrt{73}}{6}.$

Hence we have

$$\begin{cases} x = \frac{5 + \sqrt{73}}{6}, \\ y = -x = \frac{-5 - \sqrt{73}}{6}. \end{cases} \qquad \begin{cases} x = \frac{5 - \sqrt{73}}{6}, \\ y = -x = \frac{-5 + \sqrt{73}}{6}. \end{cases}$$

216. The solutions must be carefully collected in pairs. For instance, when $y = -2x$ is substituted in equation (2) above we get equation (3), from which $x = 1$ and $x = -\frac{4}{9}$. But when $y = -x$ is substituted in (2) we get (4), from which $x = \frac{5 + \sqrt{73}}{6}$ and $x = \frac{5 - \sqrt{73}}{6}$.

Hence $y = -2x$ goes *only* with $x = 1$ and $x = -\frac{4}{9}$; while $y = -x$ goes *only* with $x = \frac{5 + \sqrt{73}}{6}$ and $x = \frac{5 - \sqrt{73}}{6}$.

WRITTEN EXERCISES

Solve the following systems and collect the solutions in pairs.

1. $\begin{cases} 3x^2 + xy = 85, \\ 5y^2 + 8xy - 4x^2 = 0. \end{cases}$

2. $\begin{cases} y^2 - xy + x^2 = 20, \\ 3y^2 - 13xy + 12x^2 = 0. \end{cases}$

3. $\begin{cases} y^2 + xy - x = 1, \\ 12x^2 - xy - 6y^2 = 0. \end{cases}$

4. $\begin{cases} 12x^2 + xy - y^2 = 8, \\ 40x^2 + 7xy - 3y^2 = 0. \end{cases}$

5. $\begin{cases} 9x^2 - 16y^2 = 0, \\ 2xy + 6x - 5y = 0. \end{cases}$

6. $\begin{cases} x^2 + 4xy + 5y^2 = 36, \\ x^2 + xy - 2y^2 = 0. \end{cases}$

217. Systems in which both equations are homogeneous except for the *numerical* terms.

Example. Solve
$$\begin{cases} x^2 + 3xy + 2y^2 = 15, & (1) \\ 4x^2 + 5y^2 = 24. & (2) \end{cases}$$

Solution. The plan is to eliminate the *numerical* terms, as follows:

Multiplying (1) by 8 and (2) by 5,
$$\begin{cases} 8x^2 + 24xy + 16y^2 = 120. & (3) \\ 20x^2 + 25y^2 = 120. & (4) \end{cases}$$

Subtracting (3) from (4), $12x^2 - 24xy + 9y^2 = 0.$ (5)

Since (5) is homogeneous, $3\frac{y^2}{x^2} - 8\frac{y}{x} + 4 = 0.$ (6)

Solving (6) for $\frac{y}{x}$, $\frac{y}{x} = 2$ or $\frac{y}{x} = \frac{2}{3}$,

from which $y = 2x$ or $y = \frac{2}{3}x$.

Substituting $y = 2x$ in (2), we have $4x^2 + 20x^2 = 24$.

Hence, $x = \pm 1$.

Substituting $y = \frac{2}{3}x$ in (2), we have $4x^2 + \frac{20}{9}x^2 = 24$,

from which $x = \frac{3}{7}\sqrt{7}$ and $x = -\frac{3}{7}\sqrt{7}$.

Collecting results in pairs, we have

$$\begin{cases} x = 1 \\ y = 2 \end{cases}; \quad \begin{cases} x = -1 \\ y = -2 \end{cases}; \quad \begin{cases} x = \frac{3}{7}\sqrt{21} \\ y = \frac{2}{7}\sqrt{21} \end{cases}; \quad \begin{cases} x = -\frac{3}{7}\sqrt{21} \\ y = -\frac{2}{7}\sqrt{21} \end{cases}.$$

WRITTEN EXERCISES

Solve and arrange the results in proper pairs:

1. $\begin{cases} 2xy - x^2 = 15, \\ x^2 - xy + y^2 = 21. \end{cases}$

2. $\begin{cases} x^2 - 2xy = -12, \\ 5y^2 + 3xy = 104. \end{cases}$

3. $\begin{cases} x^2 + xy = -28, \\ y^2 + xy = 44. \end{cases}$

4. $\begin{cases} y^2 + xy = 77, \\ x^2 - xy = -12. \end{cases}$

218. Other Special Cases. A few other special cases are illustrated in the following examples, some of which include equations of higher degree than the second.

Example 1. Solve $\begin{cases} x^3+y^3=9, & (1)\\ x\ \ +y\ \ =3. & (2)\end{cases}$

Cubing (2), $x^3+3x^2y+3xy^2+y^3=27.$ (3)

Subtracting (1) from (3), $3x^2y+3xy^2=18.$ (4)

Dividing by 3, $x^2y+xy^2=6.$ (5)

Multiplying (2) by xy, $x^2y+xy^2=3xy.$ (6)

Subtracting (5) from (6), $3xy=6.$

$xy=2.$ (7)

Solving (7) and (2) as in § 213, we get $\begin{cases}x=2,\\ y=1,\end{cases}$ and $\begin{cases}x=1,\\ y=2.\end{cases}$

Another method is to divide (1) by (2), obtaining

$$\begin{array}{lr} & x^2-\ \ xy+y^2=3\\ \text{Squaring (2),} & \underline{x^2+2xy+y^2=9}\\ \text{Subtracting,} & 3xy=6\end{array}$$

The remaining steps are the same as above.

Example 2. Solve $\begin{cases}x^4-y^4=65, & (1)\\ x^2-y^2=5. & (2)\end{cases}$

Dividing equation (1) by (2), $x^2+y^2=13.$ (3)

Solving equations (2) and (3), we find,

$$\begin{cases}x=3\\ y=2\end{cases};\quad \begin{cases}x=-3\\ y=-2\end{cases};\quad \begin{cases}x=3\\ y=-2\end{cases};\quad \begin{cases}x=-3,\\ y=2.\end{cases}$$

Example 3. Solve $\begin{cases}y^2+2xy=7x, & (1)\\ 2y^2-xy+x^2=8x. & (2)\end{cases}$

Multiplying (1) by 8, $8y^2+16xy=56x.$ (3)

Multiplying (2) by 7, $14y^2-7xy+7x^2=56x.$ (4)

Subtracting (3) from (4), $6y^2-23xy+7x^2=0.$ (5)

Factoring, $(3y-x)(2y-7x)=0.$ (6)

Hence $y=\frac{x}{3}$ and $y=\frac{7x}{2}.$ (7)

Solving $y=\frac{x}{3}$ with (1), we find $\begin{cases}x=0,\\ y=\frac{x}{3}=0,\end{cases}$ and $\begin{cases}x=9,\\ y=\frac{x}{3}=3.\end{cases}$

Solving $y=\frac{7x}{2}$ with (1), we find $\begin{cases}x=0,\\ y=\frac{7x}{2}=0,\end{cases}$ and $\begin{cases}x=\frac{4}{11},\\ y=\frac{7x}{2}=\frac{14}{11}.\end{cases}$

WRITTEN EXERCISES

Solve the following and arrange the results in pairs:

1. $\begin{cases} x^3 + y^3 = 243, \\ x + y = 9. \end{cases}$

2. $\begin{cases} x^4 - y^4 = 15, \\ x^2 + y^2 = 5. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 12, \\ x - y = 2. \end{cases}$

4. $\begin{cases} x^3 - y^3 = 26, \\ x - y = 2. \end{cases}$

5. $\begin{cases} x^4 - y^4 = 63, \\ x^2 - y^2 = 3. \end{cases}$

6. $\begin{cases} 2x^2 + 4xy - 5y = 0, \\ 2x^2 - xy + y^2 - 2y = 0. \end{cases}$

7. $\begin{cases} x^3 - y^3 = 54, \\ x - y = 6. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 243, \\ x^2y + xy^2 = 162. \end{cases}$

Suggestion for Example 8. Multiply the second equation by 3 and add to the first, thus obtaining a perfect cube.

PROBLEMS INVOLVING SYSTEMS OF QUADRATICS

In each case find all the solutions and determine whether all are applicable to the problem.

1. Find two numbers whose sum is 25 and whose product is 156.

2. The sum of two numbers is 35 and their product is 300. Find the numbers.

3. The sum of two numbers is -15 and their product is -700. Find the numbers.

4. The difference of two numbers is 13 and their product is 510. Find the numbers.

5. The sum of two numbers is 18 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.

6. The difference of two numbers is 6 and the sum of their reciprocals is $\frac{5}{36}$. Find the numbers.

7. A rectangular field is 20 rods longer than it is wide. Find its dimensions if its area is 2400 square rods.

8. A rectangular field is 20 rods longer than it is wide. Find its dimensions if its area is 50 acres.

9. The sum of the sides of a right triangle is 14 and the length of the hypotenuse is 10. Find the length of each side.

10. The length of a fence around a rectangular athletic field is 1400 feet, and the longest straight track possible on the field is 500 feet. Find the dimensions of the field.

Suggestion. Using 100 feet for the unit of measure, the equations are

$$\begin{cases} x + y = 7, \\ x^2 + y^2 = 25. \end{cases}$$

11. The difference between the sides of a right triangle is 8 and the hypotenuse is 40. Find the lengths of the sides.

12. A room is 5 feet longer than it is wide, and the distance between two opposite corners is 25 feet. Find the length and width of the room.

13. One side of a right triangle is 8 feet, and the hypotenuse is 2 feet more than twice the other side. Find the length of the hypotenuse and of the remaining side.

14. The area of a window is 2016 square inches and the perimeter of the frame is 180 inches. Find the dimensions of the window.

15. The area of a rectangular city block, including the sidewalk, is 19,200 square yards. The length of the sidewalk around the block when measured on the side next the street is 560 yards. Find the dimensions of the block.

16. A farmer starts to plow around a rectangular field which contains 48 acres. The length of the first furrow around the field is 376 rods. Find the dimensions of the field.

17. A rectangular blackboard contains 38 square feet and its perimeter is 27 feet. Find the dimensions of the board.

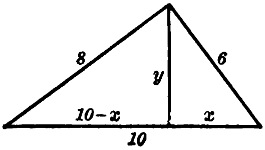
18. The sum of the squares of two numbers plus five times their product is equal to 445; and the sum of their squares minus their product is equal to 67. Find the numbers.

19. The diagonal of a rectangular mirror inside the frame is 10 inches. The square of the diagonal of the frame is 244 inches. Find the dimensions of the mirror if the frame is 2 inches wide.

20. Find the altitude of a right triangle whose sides are 6, 8, and 10, taking the hypotenuse 10 as the base. Also find the area of this triangle.

Suggestion. Using the figure, deduce the following equations:

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$



21. Find the altitude of a right triangle whose sides are 3, 4, and 5, taking the hypotenuse 5 as the base.

22. Find the altitude of a triangle whose sides are 10, 8, and 14, taking the side 14 as the base. Also find the area.

23. The sum of the squares of two numbers minus four times their product is -23; and the sum of their squares plus three times their product is 61. Find the numbers.

REVIEW QUESTIONS

1. Explain the method of solving a system of two equations in which one is linear and one quadratic. How many solutions are there? Explain how to solve the special case

$$\begin{cases} x + y = 5 \\ x^2 + y^2 = 13 \end{cases} ;\ \text{also the special case} \begin{cases} xy = 16 \\ x + y = 10. \end{cases}$$

2.* When is a quadratic homogeneous in two unknowns? How can a system of two quadratics be solved when at least one of them is homogeneous? How many solutions are there?

3.* How may a system of two quadratics be solved if they are both homogeneous except for the *numerical* terms?

4.* What other special methods of solving systems of equations were used in the exercises on p. 304?

CHAPTER XX

THE BINOMIAL FORMULA*

219. We have already considered the square and the cube of a binomial, namely,

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

If we multiply $(a+b)^3$ by $a+b$, we have

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In like manner,

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The right members of these equations are called the **expansions** of the left members. By studying these expansions we see that they may be written down according to the following rule:

(1) *In the expansion of a binomial there is one more term than the number of units in the index of the power.*

(2) *The first letter of the binomial is a factor of all terms in the expansion except the* **last**. *Its exponent in the first term is equal to the exponent of the binomial, and it decreases by one in each succeeding term.*

(3) *The second letter is a factor of all terms in the expansion except the* **first**. *Its exponent in the second term is one, and in each succeeding term it increases by one.*

(4) *The coefficient of the first term is one. The coefficient of the second term is the same as the index of the power. The coefficient of each succeeding term is found by multiplying the coefficient of the preceding term by the exponent of the first letter in that term and dividing by the exponent of the second letter increased by one.*

---

* Articles 219–222 may be omitted without destroying the continuity.

220. According to this rule we may write the expansions on page 308 in the following forms:

$$(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{2}b^2.$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{2}ab^2 + \frac{3\cdot 2\cdot 1}{2\cdot 3}b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{2}a^2b^2 + \frac{4\cdot 3\cdot 2}{2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4}b^4.$$

$$(a+b)^5 = a^5 + 5a^4b + \frac{5\cdot 4}{2}a^3b^2 + \frac{5\cdot 4\cdot 3}{2\cdot 3}a^2b^3 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3\cdot 4}ab^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 4\cdot 5}b^5.$$

ORAL EXERCISES

1. How many terms are there in the expansion of $(a+b)^5$?

2. In the expansion of $(a+b)^5$ is there any term in which a does not occur? Is there any term in which b does not occur?

3. In the expansion of $(a+b)^5$ what is the exponent of a in the first term? in the second? in the third? and so on.

4. In the expansion of $(a+b)^5$ what is the exponent of b in the second term? in the third? in the fourth? and so on.

5. In the expansion of $(a+b)^5$ state how each coefficient is obtained.

6. Ask and answer questions similar to those in Exercises 1 to 5 about the expansion of $(a+b)^6$ and $(a+b)^7$.

WRITTEN EXERCISES

Expand each of the following by the binomial formula:

1. $(x+y)^3$.
2. $(x+y)^4$.
3. $(x+y)^5$.
4. $(x+1)^3$.
5. $(x+1)^4$.
6. $(1+x)^3$.
7. $(1+x)^4$.
8. $(1+x)^5$.
9. $(a-b)^3$.
10. $(a-b)^4$.
11. $(a-b)^5$.
12. $(x-1)^4$.
13. $(x-1)^5$.
14. $(1-x)^4$.
15. $(1-x)^5$.

Suggestion for Examples 9–15: Write $a - b = a + (-b)$ and note that *even* powers of $-b$ are *positive*, while *odd* powers are *negative*.

221. Mathematical Induction. We have found by actual multiplication that the rule of § 219 holds for powers up to the fifth, and we wish to find out without continuing the multiplication whether it holds for higher powers. This is done by the process called **mathematical induction**, which is as follows:

(1) We write out the kth power according to the rule, thus:

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-3}b^3 +, \text{ etc.}$$

(2) We find $(a+b)^{k+1}$ by multiplying the above result by $a+b$ as follows:

Multiplying by a,

$$a^{k+1} + ka^kb + \frac{k(k-1)}{2}a^{k-1}b^2 + \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-2}b^3 + \cdots$$

Multiplying by b, $\quad a^kb + \quad ka^{k-1}b^2 + \quad \frac{k(k-1)}{2}a^{k-2}b^3 + \cdots$

Adding terms, $a^{k+1} + (k+1)a^kb + \left(\frac{k(k-1)}{2} + k\right)a^{k-1}b^2 +$

$$\left(\frac{k(k-1)(k-2)}{2\cdot 3} + \frac{k(k-1)}{2}\right)a^{k-2}b^3 + \cdots$$

To add $\frac{k(k-1)}{2} + k$, we use Principle I, thus

$$\frac{k(k-1)}{2} + k = k\left[\frac{k-1}{2} + 1\right] = \frac{k(k+1)}{2} = \frac{(k+1)k}{2}.$$

Similarly $\frac{k(k-1)(k-2)}{2\cdot 3} + \frac{k(k-1)}{2} = \frac{k(k-1)}{2}\left[\frac{k-2}{3} + 1\right]$

$$= \frac{k(k-1)}{2}\left[\frac{k-2+3}{3}\right] = \frac{k(k-1)(k+1)}{2\cdot 3} = \frac{(k+1)(k)(k-1)}{2\cdot 3}.$$

Hence, we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^kb + \frac{(k+1)(k)}{2}a^{k-1}b^2 + \frac{(k+1)(k)(k-1)}{2\cdot 3}a^{k-2}b^3 + \cdots$$

We thus see that, if the kth power of $a+b$ follows the rule, then the $(k+1)$st power also follows the rule, since the expansion of $(a+b)^{k+1}$, as just found, has $k+1$ everywhere in place of k in the expansion of $(a+b)^k$; that is, both proceed according to the same law.

Hence, if the rule for the binomial formula holds for any given exponent, it also holds for the next higher exponent.

But we have found by actual multiplication that the rule holds for the third power. Hence we know by the above argument that it holds for the fourth power, even without multiplying it out.

But the argument shows that if the rule holds for the fourth power, then it does for the fifth, and if for the fifth then for the sixth, and so on, as far as we please.

Hence we say that the **Binomial Formula** holds for any positive, integral exponent n. Stated as a formula this is:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

WRITTEN EXERCISES

1. Write out the fifth term in the expansion of $(a+b)^k$.

2. Multiply

$$\frac{k(k-1)(k-2)(k-3)}{2\cdot 3\cdot 4}a^{k-4}b^4 \text{ by } a \text{ and } \frac{k(k-1)(k-2)}{2\cdot 3}a^{k-3}b^3$$

by b and add the products.

222. The General Term. We can easily make a rule for writing the *rth term after the first*. For instance, in the 4th term after the first, the exponent of b is 4, the exponent of a is $n-4$, the last factor in the denominator is 4, and the *number* of factors in the numerator is 4.

Hence the **rth term after the first**

$$= \frac{n(n-1)(n-2)\cdots \text{to } (r \text{ factors})}{2\cdot 3\cdots r}a^{n-r}b^r.$$

E.g. the 5th term after the first in the expansion of $(a+b)^9$ is

$$\frac{9\cdot \overset{\cancel{2}}{\cancel{8}}\cdot 7\cdot \overset{2}{\cancel{6}}\cdot \cancel{5}}{\cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}}a^{9-5}b^5 = 126\,a^4b^5.$$

CHAPTER XXI

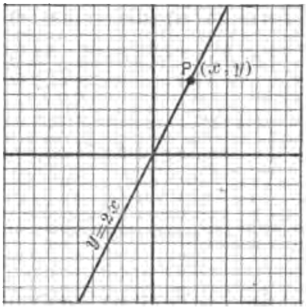
VARIABLES AND FUNCTIONS*

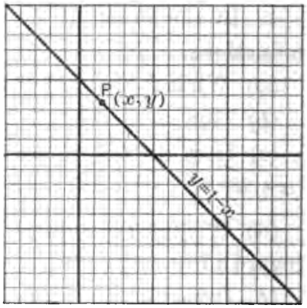
223. Definitions. In the graph of the equation $y = 2x$ we may think of the point P represented by (x, y) as moving along the line. As the point moves the values of x and y both change, subject to the fixed relation that y is always twice as great as x.

Variable. An algebraic expression whose value changes in any one problem or discussion is called **a variable.**

E.g. As P moves along the line $y=2x$, both x and y are variables.

Function. When two variables are connected by a fixed relation so that the value of one depends on the value of the other, then *one variable is said to be a function of the other.*





E.g. In the equation $y = 2x$, if any value is given to x, this determines the corresponding value of y. Hence y is a function of x.

Increasing Function. In the equation $y = 2x$, y is said to be an *increasing function* of x, because *y increases as x increases.*

---

* Articles 223–226 may be omitted without destroying the continuity.

Blaise Pascal was born at Clermont, France, in 1623, and died at Paris in 1662. He was a celebrated philosopher and mathematician.

At the age of twelve years, Pascal was found drawing charcoal figures on the pavèment and proving theorems in geometry. At the age of sixteen he wrote a treatise on algebraic geometry which displayed great power. At nineteen he invented a machine for performing arithmetical operations,—the forerunner of our modern calculating machines.

In 1653 Pascal devised a scheme for writing down the coefficients of the binominal expansion, which he called the "Arithmetical Triangle," and which has ever since been known as the "Pascal Triangle."

Pascal wrote extensively on both philosophy and mathematics.

Decreasing Function. In the equation $y = 1 - x$, y is said to be a *decreasing function* of x, because *y decreases as x increases.*

E.g. If x increases from 0 to 1, y decreases from 1 to 0.

The graphs of $y = 2x$ and $y = 1 - x$ show the relations of the variables.

E.g. Think of the values of x as increasing from 0 toward the right. Then in $y = 2x$ the values of y *ascend*, while in $y = 1 - x$ the values of y *descend.*

ORAL EXERCISES

1. A man walks at a uniform rate of 3 miles an hour. Using d for the distance walked and t for the number of hours, express d as a function of t.

Is d an increasing or decreasing function of t?

2. A train moves at a uniform speed of 30 miles an hour. Express the distance it moves as a function of the time.

3. A pipe pours out 24 gallons of water a minute. Express the amount poured out as a function of the time.

224. Definition. If k is a constant, and if $y = kx$, then y is said to **vary as x**. This is written $y \propto x$, and is read *y varies as x.*

WRITTEN EXERCISES AND PROBLEMS

1. If $y \propto x$ and $y = 3$ when $x = 5$, find y when $x = 9$.

Solution. Let $y = kx$. If $y = 3$ when $x = 5$, then $3 = k \cdot 5$ and $k = \frac{3}{5}$. Hence, when $x = 9$, $y = kx = \frac{3}{5} \cdot 9 = \frac{27}{5} = 5\frac{2}{5}$.

2. If $y \propto x$ and $y = 7$ when $x = 3$, find y when $x = 7$.

3. If $y \propto x$ and $y = a$ when $x = b$, find y when $x = c$.

4. The circumference of a circle varies as the diameter, that is, $c = \pi \cdot d$. Find the circumference of a circle whose diameter is 10 feet if a circle whose diameter is 1 foot has a circumference of approximately 3.1416 feet.

5. The interest on a fixed sum of money varies as the length of time it is invested. If a certain sum draws $145 in three months, how much will it draw in 11 months?

6. The interest on a sum of money invested for a fixed time at a fixed rate varies as the amount invested. If the interest on $800 is $125, what will be the interest on $540 at the same rate and for the same time?

7. If a steamer uses 340 tons of coal in going 425 miles, how much will it use in going 3240 miles?

8. If an automobile uses $8\frac{1}{2}$ gallons of gasoline in going 96 miles, how many gallons will it use in going 370 miles? How far will it go in using 32 gallons?

9. The weight of an object below the earth's surface varies directly as the distance from the earth's center. If an object weighs 100 pounds at the earth's surface (4000 miles from the center), what will it weigh 2000 miles from the center? 500 miles from the center? How far from the center must it be to weigh just one pound?

225. Inverse Variation. If the number of men employed on a certain piece of work is doubled, the time required to finish the work is divided by 2. If the number of men is trebled, the time is divided by 3, and so on.

If n is the number of men and t the number of days required to finish a piece of work, then the relation described above may be expressed by $t = \frac{k}{n}$, where k is some fixed number depending on the piece of work to be done.

When $t = \frac{k}{n}$, we say that t varies inversely as n, and this is also written $t \propto \frac{1}{n}$.

EXERCISES

Give the results orally in Examples 1 to 4.

1. If $t = \frac{k}{n}$, and if $n = 4$ when $t = 2$, find k.

2. If $t = \frac{k}{n}$, and if $n = 7$ when $t = 3$, find k.

3. If $t = \frac{k}{n}$, and if $n = 15$ when $t = \frac{1}{5}$, find k.

4. If $t = \frac{k}{n}$, and if $n = \frac{1}{3}$ when $t = 21$, find k.

5. If a piece of work can be finished in 84 days by 16 men, how long will it take 40 men to do it?

6. If 126 men can do a piece of work in 80 days, how many men will be required to do this work in 120 days?

226. Variation as the Square of a Variable. We have seen (page 261) that $a = \pi r^2$ where r is the radius of a circle and a its area. We say that a varies directly as r^2.

It follows at once that the weight of a circular disc cut from a sheet of metal of even thickness varies as the square of the diameter.

WRITTEN EXERCISES

1. If $w = kr^2$, and if $w = 10$ when $r = 1$, find w when $r = 1\frac{1}{2}$.

2. If $y = kx^2$, and if $y = 4$ when $x = 2$, find k. Also find y when $x = 4$.

3. If $y = kx^2$, and if $y = 1$ when $x = 3$, find y when $x = 15$.

4. A circular piece of metal one foot in diameter weighs 8 pounds. How much will a circular piece weigh if it is cut from the same sheet of metal and is $2\frac{1}{2}$ feet in diameter?

5. The distance passed over by a falling body varies directly as the square of the time.

If a falling body goes 16 feet in one second, how far will it go in 7 seconds?

CHAPTER XXII

REVIEW EXERCISES

REVIEW EXERCISES FOR CHAPTER I

1. Write a formula for the area of a rectangle.

2. Write a formula for the volume of a rectangular solid.

3. If $a = 2$, $b = 3$, $c = 4$, find the value of

$$b + c - a;\ 2b - c + a;\ \frac{4b + c}{a};\ \frac{3c - 2b}{a}.$$

4. Perform the indicated operations:

$20k + 10k - 16k$; $15n + 8n - 3n$;

$50ax - 20ax - 7ax$; $63ac + 7ac - 50ac$.

5. Multiply $a + b + c$ by x; $m - n - 2$ by a; $x - y - z$ by 4.

6. Divide $42 + 12$ by 3 without first adding.

7. Divide $12x - 8y + 16z$ by 4; $6rx + 12sx$ by $6x$.

8. Multiply $3 \cdot 4 \cdot 5$ by 2 in four different ways.

9. Multiply $6ax$ by 4; $10abc$ by d; $16xyz$ by 3; $8mn$ by 7.

10. Divide $4 \cdot 8 \cdot 12$ by 2 in three different ways.

11. Divide $10abc$ by $5c$; $18mn$ by 9; $12xy$ by 3. Divide $10(2x + 4y + 6z)$ by 2 in two different ways.

12. Find the value of $3 \cdot 2 + 4 \cdot 5 - 2 \cdot 7$. What operations are performed first?

13. Find the value of $3(2 + 8 - 4) - 2(6 - 1)$.

14. If $a = 2$, $b = 1$, $c = 3$, $d = 4$, find the value of each of the following:

$c \cdot d \div a - 2b$; $6abc \div 2d$; $4ad - 3(c - 2b)$; $(10ac - 2b) \div d$.

REVIEW EXERCISES FOR CHAPTER II

1. Solve orally the equations:

$2x + 3x = 20;$ $\quad 2(3x - x) = 8;$

$16y - 2(3y + y) = 24;$ $\quad \frac{1}{2}n + 3n - n = 15.$

Solve the following equations:

2. $4n + 5n + 6 = 3n + 18.$

3. $2x + 4x - 2x = 6x - 3x + 15.$

4. $7(x + 2) + 2(4x - 1) + 6 = 6(x + 4) + 3(2x - 1).$

5. Six times a certain number plus 12 equals 42. What is the number?

6. For how many years must $5000 be invested at 6% to yield $1800 interest?

7. The greater of two numbers is 8 times the less, and their sum is 90. Find the numbers.

8. Find a number such that when 5 times the number is subtracted from 12 times the number the remainder is 42.

9. Find three consecutive integers such that 4 times the first is 7 greater than 3 times the third.

10. A rectangle is twice as long as it is wide. The perimeter exceeds the length by 40. Find the dimensions.

11. The area of Louisiana is (nearly) 4 times that of Maryland, and the sum of their areas is 60,930 square miles. Find the (approximate) area of each state.

12. The horse power of a certain steam yacht is 12 times that of a motor boat. The sum of their horse powers is 195. Find the horse power of each.

13. At a football game there were 2000 persons. The number of women was 3 times the number of children, and the number of men was 6 times the number of children. How many men, women, and children were there?

REVIEW EXERCISES FOR CHAPTER III

Perform the following indicated operations:

1. $-8-(-9)-(+7)+8$.
2. $16+(-18)-(+2)+4$.
3. $12x+(-7x)-(-3x)+2x$.
4. $34n+(-30n)-(-7n)$.

5. $8\cdot(-3)\cdot(-1)$.
6. $[48\div(-6)]\div(-1)$.
7. $3\cdot(-2)(-3)(-1)$.
8. $(-12)\div[(-2)\div(-2)]$.
9. $(-42)\cdot(-2)\div(-7)$.
10. $[(7ab)\div(-a)]\cdot(-3)$.

11. What is meant by the average of several numbers?

12. Find the average of 20, 16, 8, 4, 0, -8, -12.

13. Find the average of -8, -32, 14, 26, -40.

Solve the following:

14. $x+6=4$.
15. $3x+12=6$.
16. $8+x=4$.
17. $3x-8=-16$.
18. $2(x+6)=3(x+5)$.
19. $2x+4=3x+8$.
20. $5x+7-2x=2x+9$.
21. $3n+2(n+4)=4n+14$.
22. $8+6x+3(2+x)=5x+26$.
23. $4(x+3)+2(3x+1)=5(x+2)+2(x+3)+19$.
24. $7(x+1)+3(2x+3)=4(x+5)+7(x+2)-8$.
25. $7(2x-3)+5(4x-1)=3(x+1)+2$.

26. If $-n$ represents a negative integer, how do you represent the next integer to the right on the number scale? the next to the left?

27. If $-2n$ represents a negative even integer, how do you represent the next even integer to the right? the next to the left?

28. Do negative numbers apply to all things to which positive numbers apply? Can there be a negative number of books on a shelf?

REVIEW EXERCISES FOR CHAPTER IV

1. Add $3x + 4y - 3z$, $5x - 2y - z$, and $3y - 5x + 7z$.
2. From $15a + 4b - 13bc$ subtract $3a - 8b + 2bc$.
3. Subtract $7x - 5y - 7a$ from $6x + 5y + 3a$.
4. From $5x - 4y - 9z$ subtract $3x - 8y + 2z$.
5. Add $5a + 3b - 2c$ and $11a - 7b + 8c$.
6. Add $11axy + 13x - 14y$, $2y - 4x$, and $3y + x - 8axy$.
7. Simplify $(5x - 3b) + (2x + b) - (4x - 2b - x + 5b)$.
8. Add $19b + 3c$, $2b - 7c$, $2c - 14b$, and $c + 8b$.
9. Simplify $-(a - 3b - c) - (2c - a - 5b) + (a - c + b)$.
10. Subtract $2x + 4y + z$ from $13x - 3y - 5z + 8$.
11. Solve $5(x - 7) + 3(14 - x) + 60 = 1 - 10x$.
12. Solve $13(1 - x) - 6(2x - 5) = 80 + 12x$.
13. Add $7x - 3y - 4$, $5x + 2y + 5$, and $3y - 8x - 6$.
14. Add $13a + 4b - 9c$, $2c - 8b - 16a$, and $8a - 5b - 8c$.
15. Simplify $8x - [2x + 3(x - 1) - (2x - 3)]$.
16. From $17b - 4a - 2c - 19$ subtract $8c - 5a - 8b + 4$.
17. Simplify $3 - (3 - 2 + 6 + 8 - 3) + 8 - (9 - 3 + 8)$.
18. Solve $3(4 - x) - 2(5 - 6x) = 8x + 4$.
19. Simplify $12 + (2a - 3c - 4b) - (3b - c - a - 8)$.
20. Simplify $5x - (3x - 2 + 2y + x) + 13y - (6 - 3x)$.
21. Add $y - 20$, $4y + 6$, $2y + 4x - 13$, and $2x - 8y - 40$.
22. Subtract $16 - x + 2z - 4y$ from $3x - 5z - 8y$.
23. Solve $19 + (2x - 7) - (31 - 4x - 8 - 2x) = 5x + 7$.
24. Solve $16 + 5x - (8x + 9 - 4x + 17) = 8x - 3$.
25. Solve $6x - 3 - (4x + 8 - 9x) - (5x - 2) = x + 11$.

26. The sum of two numbers is 16. Seven times one is 8 less than 5 times the other. What are the numbers?

27. From a certain number a there is subtracted 3 times the remainder when 8 is subtracted from $2a$. Express the result in terms of a.

REVIEW EXERCISES FOR CHAPTER V

Multiply as indicated:

1. $(x+1)(x-2)$.
2. $(a+b)(2a+b)$.
3. $(2x-y+1)(x-y)$.
4. $(8x-3y)(2x+5y)$.
5. $(a-1+b-c-d)(4a+5b+3c-2d)$.
6. $(4ax-3ay+5az-8)(x+y-z+2)$.
7. $(3a-2b+4c)(2a+3b-c)$.

Divide as indicated:

8. $a^3-12a^2+27a+40$ by $a-5$.
9. $x^5-5x^4y+11x^3y^2-14x^2y^3+9xy^4-2y^5$ by $x^2-3xy+2y^2$.
10. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.
11. $a^3+5a^2-2a-24$ by $a^2+7a+12$.
12. $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$ by $a^2-2ab+b^2$.
13. $x^5-5x^3y^2-5x^2y^3+y^5$ by $x^2-3xy+y^2$.
14. Add $12a^2b^2c+8ax$, $6ax-8a^2b^2c$, and $2ax+3a^2b^2c$.
15. Add $5xy^2+3x^2y+4xy$, $2x^2y-6xy^2-3xy$, and $4xy$.
16. Add $6ab-3c-2a$, $2c-4ab-5a$, $5c-a+ab$.
17. From $35ab-8x-9z+13$ subtract $16ab-4z+5x+8$.
18. Subtract $5a-8x-6y$ from $13x+14y-15z-4a$.
19. From $9y-4x-6z-3b$ subtract $8-9y-3x-2z$
20. Solve $(n-4)(6-3n)-(6-n)^2-10=-4n(n-4)$.
21. $(n+2)^2+(n-1)^2+(n+1)^2=3n(n+2)+60n+130$.
22. $2x+4-6(5x-8-7x)+2-4x=6(2-3x)-42$.

23. A man bought a tract of coal land and sold it a month later for \$93,840. If his gain was at the rate of 24 % per annum, what did he pay for the land?

24. The melting point of copper is 250 degrees (Centigrade) lower than 4 times that of lead. Ten times the number of degrees at which lead melts minus twice the number at which copper melts equals 1152. Find the melting point of each.

REVIEW EXERCISES FOR CHAPTER VI

Use the formulas for $(a \pm b)^2$ for the following:

1. $[(a+b)+(c-d)]^2$. 4. $[7x-(4r-s)]^2$.
2. $[(a+3)-(b+c)]^2$. 5. $[(m^2-3)-2(m^3+n)]^2$.
3. $[(3a-2b)+5]^2$. 6. $[3(2+y)-2(3+x)]^2$.

Use the formula for $(a+b)(a-b)$, for the following:

7. $[a+b+(c-d)][a+b-(c-d)]$.
8. $[x+y+(u+v)][x+y-(u+v)]$.
9. $[4x-(a-2b)][4x+(a-2b)]$.
10. $[a+2b-(x-y^2)][a+2b+(x-y^2)]$.
11. $(11b^3x-3bx^3)(11b^3x+3bx^3)$.

Find the following as indicated:

12. $(2x-3y+z)^2$. 14. $(x^3+y^3) \div (x+y)$.
13. $(2x-3y)^3$. 15. $(8a^3-125b^3) \div (2a-5b)$.
16. $16c-(41-7c)+(15-8c)$.
17. $-(5a-3c)-(2c-8a)+3a$.
18. $-(-12x-7y-15x)-(-9y+8x+3y)$.
19. $(19x+4y-32x-17x)-12x-(49y+18x-70x)$.

Solve the following equations:

20. $7(m+6)+10m=42-8(2m+2)+181$.
21. $20-3(x-4)+2x=2x+17$.
22. $6(x-3)-2(2-x)=2(x+1)-6$.

23. If two numbers differ by d and if the greater of the numbers is x, how do you represent the other?

24. A father is 3 times as old now as his son was 7 years ago. If the son's age now is represented by x, how is the father's age represented?

25. A picture inside the frame is w inches wide and $w+6$ inches long. The frame is 4 inches wide. Express the area of the frame in terms of w.

REVIEW EXERCISES FOR CHAPTER VII

Factor:

1. x^2+5x+4.
2. x^2-5x+4.
3. a^4-7a^2+12.
4. a^6-a^3-12.
5. $b^2+3b-18$.
6. b^4-b^2-56.
7. a^3+8.
8. $27a^3+b^3$.
9. b^3-27.
10. $8a^3-b^3$.
11. $1+64x^3$.
12. a^6-b^6.
13. $1-64x^3$.
14. w^6+27a^6.
15. w^6-8a^3.
16. $27a^3-8b^3$.
17. x^6+y^6.
18. $125a^3+b^3$.
19. $1+125x^6$.
20. $27x^6-1$.
21. $1-8x^3y^3$.
22. $1+8x^3y^3$.
23. $8x^3+27y^3$.
24. $8x^3-27y^3$.
25. c^4-31c^2+220.
26. $ac+d^5a-b^4c-b^4d^5$.
27. $26+39n-22m-33mn$
28. $12x^2+11x-56$.
29. $a^2+4ab+4b^2-(a^2-4ab+4b^2)$.
30. $(3x-1)^2-(x^2+4y^2-4xy)$.
31. $(x+3y)^2+(x-2y)^2+2(x+3y)(x-2y)$.
32. $16(a+b)^2-8(a-b)(a+b)+(a-b)^2$.
33. $256x^2-(49x^2+4y^4-28xy^2)$.

Factor the following:

34. $a^2+4ab+4b^2-(x^2-2xy+y^2)$.
35. $(3x-2)^2-(4x^2+9y^2-12xy)$.
36. $x^2+4xy+4y^2-(a^2+2ab+b^2)$.
37. $16x^4y^2-(4x^2+9y^2+12xy.)$
38. $(a+b)^2-(4a^2+9b^2-12ab)$.
39. $a^4+a^2b^2+b^4$.
40. $16x^4+20x^2y^2+9y^4$.

41. The Nile is 100 miles more than twice as long as the Danube. Ten times the length of the Danube minus 4 times the length of the Nile equals 3400 miles. How long is each river?

42. Lead weighs 259 pounds more per cubic foot than cast iron, and 166 pounds more than bronze; while a cubic foot of bronze weighs 807 pounds less than 3 cubic feet of iron. Find the weight per cubic foot of each metal.

REVIEW EXERCISES FOR CHAPTER VIII

Solve the equations:

1. $(x-1)(x+1)(x-3)=0$.

2. $x(x^2-4)(x^2-9)=0$.

3. $x^2+7x+12=0$.

4. $x^3+3x^2-4x-12=0$.

5. $x^3-3x^2-4x+12=0$.

6. $(x-1)(2x-2)+(x-5)^2=(3-x)(24-3x)-7$.

7. $(17x+3)(x-1)+8=(2-x)(6-17x)+19$.

Perform the operations indicated:

8. $(a-2)(6a-4)+2(a-1)^2=(6-a)(30-8a)+4$.

9. $5-(a+b-c-d+8)+(3+a+c-d)-5$.

10. Add $6a+9$, $8a-13$, $46a-8$, and $6-54a$.

11. From $3-4a-5c+8x^2$ subtract $2x^2-2a-4c+8$.

12. $(4ab-6ac-5ad)(b-c+d)$.

13. From $6(a+2)+3(c+4)-2(b-d)$
subtract $2(a+2)-2(c+4)+3(b-d)$.

14. The sum of two numbers is 13 and the sum of their squares is 97. Find the numbers.

15. The sum of two numbers is 38 and their product is 240. Find the numbers.

16. The sum of two numbers is 23 and their product is 120. Find the numbers.

17. The difference between two numbers is 14 and their product is 176. Find the numbers.

18. A rectangular field is 10 rods longer than it is wide, and its area is 1200 square rods. Find its dimensions.

19. An open box is made from a square piece of tin by cutting out a 6-inch square from each corner and turning up the sides. How large is the original square if the box contains 150 cubic inches?

REVIEW EXERCISES FOR CHAPTER IX

Find the H. C. F. of the following:

1. $x^2-y^2,\ x^3-y^3,\ x^2-2xy+y^2.$
2. $x^2+2xy+y^2,\ x^3+y^3,\ x^2+xy.$
3. $a^2+7a+12,\ a^2-4,\ a^2-9.$
4. $125+m^3,\ 25-m^2,\ m^2+10m+25.$

Find the L. C. M. of the following:

5. $x^2-11x+30,\ x^2-36,\ x^2-25.$
6. $a^2-a-6,\ a^2-7a+12,\ a^2-2a-8.$
7. $a-3,\ a^2+3a+9,\ a^3-27.$
8. $x^2-3x+2,\ x^2-5x+6,\ x^2-4x+3.$

Factor the following:

9. $(2-x)^2-2(2-x)(x-1)+(x-1)^2.$
10. $(2+y)^2+2(2+y)(1+x)+(1+x)^2.$
11. $(3a-2b)^2-10(3a-2b)+25.$
12. $(6a-b)^2+(2a+1)^2-2(6a-b)(2a+1).$
13. $25(a+b)^2+50(a+b)(a-b)+25(a-b)^2.$
14. $x^2+12x(a+b+c)+36(a+b+c)^2.$
15. $49(m-3)^4+36(m+1)^6-84(m-3)^2(m+1)^3.$
16. $16(x-y)^2-16(x-y)(x+y)+4(x+y)^2.$
17. $-30(a+b)(a-b)^2+25(a-b)^4+9(a+b)^2.$

18. Express the average of the numbers 3, 8, -9, 12; also of the numbers $3a$, b, $2c$, $-5b$.

19. Express the sum of the squares of four consecutive even integers of which $2n$ is the smallest.

20. Express the sum of the squares of four consecutive odd integers of which $2n+1$ is the greatest.

21. A wall l feet long and h feet high has three windows each k feet wide and m feet high. By how much does the area of the wall exceed that of the windows?

REVIEW EXERCISES FOR CHAPTER X

Perform the following indicated operations:

1. $\frac{x-1}{x-2}+\frac{x^2-3}{4-x^2}+\frac{x}{2+x}.$

2. $\frac{1}{(a-b)(b-c)}+\frac{1}{(c-b)(c-a)}-\frac{1}{(a-c)(b-a)}.$

3. $\left(a-b+\frac{4ab}{a-b}\right)\div\left(\frac{(a+b)(b+2a)}{2a^2+3ab+b^2}\right).$

4. $\dfrac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}}.$

5. $\dfrac{\frac{x-y}{x}-\frac{x+y}{y}}{\frac{x-y}{y}+\frac{x+y}{x}}.$

6. $\left\{\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right\}\div\left\{\frac{a+x}{a-x}-\frac{a-x}{a+x}\right\}.$

7. $\left(2a-\frac{a^2-b^2}{a}\right)\left(3b+\frac{a^2+b^2}{b}\right)\div\left(\frac{a^2}{b^2}+5+\frac{4b^2}{a^2}\right).$

8. $\frac{x-y}{x-2z}-\frac{x+y}{x+2z}+\frac{y-4z^2}{4z^2-x^2}.$

9. $\left(4-\frac{a^2+2ab+b^2}{ab}\right)\left(\frac{4ab}{a^2-2ab+b^2}+1\right).$

10. $\frac{a^3+27}{a^3-8}\div\frac{a^2b-3ab+9b}{a^2+2a+4}.$

11. $\left(y+\frac{xy}{y-x}\right)\left(y-\frac{xy}{x+y}\right)\left(\frac{y^2-x^2}{y^2+x^2}\right).$

12. $\left(\frac{x-2}{x-3}-\frac{x+3}{x+4}\right)\div\left(\frac{1}{x+1}+\frac{7}{x-3}\right).$

13. The world's gold production in 1908 was 29 million dollars less than 3 times that of 1893, and the production in 1900 was 59 million less than twice that of 1893. The production of 1900 and 1908 together amounted to 697 million. How much was produced each year?

REVIEW EXERCISES FOR CHAPTER XI

Solve:

1. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

2. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

3. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

4. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

5. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

6. $\frac{x}{x-1}-\frac{x}{x+1}=\frac{4}{(x-1)(x+1)}.$

7. $-\frac{x-1}{x+2}+\frac{x+1}{x-3}=\frac{27}{(x+2)(x-3)}.$

8. $\frac{2x-1}{x+2}-\frac{4x-1}{2x-3}=\frac{-10}{(x+2)(2x-3)}.$

9. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

10. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

11. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

12. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

Perform the following indicated operations:

13. $16ax+4-(8-8ax-a)-(12ax-13-ax).$

14. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

15. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.

16. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.

17. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.

18. How do you represent a fraction whose numerator is 3 less than twice that of $\frac{m}{n}$ and whose denominator is equal to 3 times the sum obtained by adding 2 to the denominator of this fraction?

19. There are two numbers such that if one half their product is divided by twice their sum the result is 12 times their difference. Write an equation representing this relation between the numbers.

REVIEW EXERCISES FOR CHAPTER XII

1. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a}=\frac{c+d}{c}$.

2. Which is the greater ratio, $\frac{x+3y}{x+4y}$ or $\frac{x+5y}{x+6y}$, x and y both being positive?

3. Find a fourth proportional to 25, 75, and 100.

Solve:

4. $(9x-3)(4-x)+(x-3)^2=-8(x+2)^2+94$.

5. $(x+1)^2+(x+2)^2+(x+3)^2=(3x-1)(x+12)-43$.

6. $(2x+5)(x-7)-(x-1)^2=(x+1)(x+2)-28$.

7. $3(5-x)^2-(2x-1)(x-1)=(x-7)(x+10)+17x+50$.

8. $(32+x)(4x-1)+(5-x)^2+(x-1)^2=6(x+1)^2+194$.

9. $(2x-7)(5-x)-(2-5x)(1-x)=-x(7x-34)-17$.

10. $\frac{x+8}{2}-\frac{x-9}{12}+\frac{x-17}{6}=\frac{4x-7}{2}+\frac{2x+6}{3}+\frac{5-31x}{12}$.

11. $\frac{3x-1}{6}-\frac{3x+3}{3}+\frac{x-1}{2}=\frac{x+5}{6}+4x-\frac{20}{3}$.

12. $\frac{x-b}{a}+\frac{x+a}{b}=2$.

13. $\frac{3x-16}{2}+\frac{21}{x-8}=\frac{6x-11}{4}$.

REVIEW EXERCISES FOR CHAPTER XI

Solve:

1. $\frac{x}{4}+\frac{x}{8}+\frac{x}{16}-\frac{x}{2}+\frac{x}{32}=-2.$

2. $\frac{y-3}{4}-\frac{y+9}{10}=\frac{y-11}{4}-2.$

3. $\frac{y-4}{3}+\frac{y+2}{3}+\frac{y+8}{3}=2y-20.$

4. $\frac{y}{2}+\frac{y+20}{4}+\frac{y+5}{5}=25.$

5. $\frac{y}{3}-\frac{y+20}{5}+\frac{y-5}{5}+\frac{y-10}{2}=15.$

6. $\frac{x}{x-1}-\frac{x}{x+1}=\frac{4}{(x-1)(x+1)}.$

7. $-\frac{x-1}{x+2}+\frac{x+1}{x-3}=\frac{27}{(x+2)(x-3)}.$

8. $\frac{2x-1}{x+2}-\frac{4x-1}{2x-3}=\frac{-10}{(x+2)(2x-3)}.$

9. $\frac{a}{3}+\frac{a+7}{4}-\frac{a-3}{3}=\frac{a+227}{5}-1.$

10. $\frac{a-1}{2}+\frac{a+1}{2}+\frac{a-3}{12}=2+a.$

11. $\frac{a+1}{4}+\frac{a-3}{4}+\frac{a-7}{4}=2a-26.$

12. $\frac{n+1}{3}+\frac{n+3}{4}+\frac{n-1}{4}=\frac{n+13}{3}+\frac{n-2}{3}.$

Perform the following indicated operations:

13. $16ax+4-(8-8ax-a)-(12ax-13-ax).$

14. $a^2b-(3b-8a^2-7)+3ab^2-(4ab^2+8-2a^2).$

15. Add $15ax^2+3bc^2$, $2bc^2-7ax^2$, and $5+2ax^2-5bc^2$.

16. Add $16-7ab-2a^2+5ab$, $4a^2-2ab$, and $5ab-8$.

17. Add $51x^2y-35+12a^2$, $41-17a^2-57x^2y$, and $3x^2y$.

18. How do you represent a fraction whose numerator is 3 less than twice that of $\frac{m}{n}$ and whose denominator is equal to 3 times the sum obtained by adding 2 to the denominator of this fraction?

19. There are two numbers such that if one half their product is divided by twice their sum the result is 12 times their difference. Write an equation representing this relation between the numbers.

REVIEW EXERCISES FOR CHAPTER XII

1. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{a+b}{a}=\frac{c+d}{c}$.

2. Which is the greater ratio, $\frac{x+3y}{x+4y}$ or $\frac{x+5y}{x+6y}$, x and y both being positive?

3. Find a fourth proportional to 25, 75, and 100.

Solve:

4. $(9x-3)(4-x)+(x-3)^2=-8(x+2)^2+94$.

5. $(x+1)^2+(x+2)^2+(x+3)^2=(3x-1)(x+12)-43$.

6. $(2x+5)(x-7)-(x-1)^2=(x+1)(x+2)-28$.

7. $3(5-x)^2-(2x-1)(x-1)=(x-7)(x+10)+17x+50$.

8. $(32+x)(4x-1)+(5-x)^2+(x-1)^2=6(x+1)^2+194$.

9. $(2x-7)(5-x)-(2-5x)(1-x)=-x(7x-34)-17$.

10. $\frac{x+8}{2}-\frac{x-9}{12}+\frac{x-17}{6}=\frac{4x-7}{2}+\frac{2x+6}{3}+\frac{5-31x}{12}$.

11. $\frac{3x-1}{6}-\frac{3x+3}{3}+\frac{x-1}{2}=\frac{x+5}{6}+4x-\frac{20}{3}$.

12. $\frac{x-b}{a}+\frac{x+a}{b}=2$.

13. $\frac{3x-16}{2}+\frac{21}{x-8}=\frac{6x-11}{4}$.

REVIEW EXERCISES FOR CHAPTER XIV

Solve:

1. $\begin{cases} 6y - x = 7 + 4y, \\ 5x + 8y = 1. \end{cases}$

2. $\begin{cases} x - y = 37, \\ 2x + 3y = 314 + 13y. \end{cases}$

3. $\begin{cases} 2x - 3y = y + 6, \\ x - 2y = 4y + 3. \end{cases}$

4. $\begin{cases} 5x + 3y = 0, \\ 2x + y = 1. \end{cases}$

5. $\begin{cases} 2x + 3y = 6x - 1, \\ 3x - 2y = 3. \end{cases}$

6. $\begin{cases} 5x - 3y = 0, \\ 2x + 2 - 6y = 2 - x. \end{cases}$

7. $\begin{cases} \dfrac{5x-3}{4} - \dfrac{2y-5}{2} = \dfrac{1}{2}, \\ \dfrac{3x+5}{2} + \dfrac{y-10}{5} = 6. \end{cases}$

8. $\begin{cases} \dfrac{2x-9}{3} - \dfrac{4y-2}{5} = -1, \\ \dfrac{7x+4}{2} + \dfrac{y+6}{4} = 3. \end{cases}$

9. $\begin{cases} \dfrac{a+7}{3} + \dfrac{4b-1}{6} = \dfrac{1}{2}, \\ \dfrac{4a+7}{3} + \dfrac{7b+3}{5} = -5. \end{cases}$

10. $\begin{cases} x + 2y + 2z = 3, \\ 3x - 4y + z = 19, \\ -2x + 6y + 3z = 0. \end{cases}$

11. $\begin{cases} 2x + 5y + 7z = 7, \\ 3x - 9y - 2z = 23, \\ -x + 3y + 3z = -10. \end{cases}$

12. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} + \dfrac{z}{6} = 6, \\ \dfrac{x}{2} + \dfrac{y}{8} - \dfrac{z}{12} = 3, \\ \dfrac{x}{6} - \dfrac{y}{2} + \dfrac{z}{3} = 1. \end{cases}$

13. $\begin{cases} x + y - z = 35, \\ \dfrac{x}{3} + \dfrac{y}{5} + z = 15, \\ \dfrac{x}{5} + \dfrac{y}{5} - z = 3. \end{cases}$

14. If r represents the rate in miles per hour at which a train is moving, how far will it go in t hours? Another train runs 10 miles per hour faster. Express in symbols the sum of the distances which these two trains travel in t hours.

15. If r_1 is the rate of a current and r_2 the rate of a steamer in still water, express the distance which the steamer can go in t hours: (*a*) with the current; (*b*) against the current.

REVIEW EXERCISES FOR CHAPTER XV

1. Describe the method of locating a point geographically on the earth's surface.

2. Describe the method for locating a point graphically on squared paper.

3. What is meant by a *linear* equation in two unknowns? Write such an equation.

4. How many points do you need to locate on the graph of a linear equation before you can draw the whole graph?

5. Construct graphs of the equations $3y - 2x = 12$ and $5x + 2y = 8$.

6. What is the relation between the graphs of the equations $2x + 3y = -6$ and $2x - 4y = -12$?

7. What is meant by *independent* equations? by *simultaneous* equations?

8. Write a pair of equations which are not independent. Construct their graphs.

9. Write a pair of equations which are not simultaneous Construct their graphs.

Solve the following equations:

10. $\dfrac{x}{x-2} + \dfrac{4x}{x+3} = 5.$

11. $\dfrac{x+a}{a+b} + \dfrac{x+b}{a-b} = 1.$

12. $\begin{cases} x + 2y = 4, \\ 2x + y = -1. \end{cases}$

13. $\begin{cases} 5x + 9y = 19, \\ 3x - y = 5. \end{cases}$

14. $\begin{cases} 3x - 7y = -11, \\ 2x + y = 4. \end{cases}$

15. $\begin{cases} 5x - 3y = 4 - 2x + 7y, \\ 5y + x = 7. \end{cases}$

16. $\begin{cases} x - 3y + z = 10, \\ 2x + y - z = 1, \\ 3x - 2y + 5z = 31. \end{cases}$

17. $\begin{cases} x + y + z = 1, \\ 3x + 4y - z = 1, \\ -2x - y + 3z = 5. \end{cases}$

REVIEW EXERCISES FOR CHAPTER XVI

Find the square roots of:

1. $x^6 + 2x^4 + 2x^3 + x^2 + 2x + 1$.

2. $x^8 + 2x^6 + 3x^4 + 2x^2 + 1$.

3. $x^6 + 4x^5 + 10x^4 + 16x^3 + 17x^2 + 12x + 4$.

4. $4a^4 - 12a^3 - 7a^2 + 24a + 16$.

5. $a^2 + \frac{2a}{b} + \frac{2a}{c} + \frac{2}{bc} + \frac{1}{b^2} + \frac{1}{c^2}$.

6. $\frac{1}{4}a^4 - \frac{1}{3}a^2b^3 + \frac{1}{9}b^6 + \frac{1}{4}a^2c^4 + \frac{1}{16}c^8 - \frac{1}{6}b^3c^4$.

7. $\frac{4a^4}{b^4} - \frac{4a^3}{b^3} - \frac{13a^2}{3b^2} + \frac{8a}{3b} + \frac{16}{9}$.

8. $\frac{x^2}{y^2} + \frac{2x}{z} + \frac{y^2}{z^2} - \frac{2y}{x} + \frac{z^2}{x^2} - \frac{2z}{y}$.

Approximate the square roots of each of the following to two places of decimals:

9. 7.9482. 11. 390.07. 13. .0048.

10. 4578.9. 12 9.176. 14. .04791.

Simplify the following:

15. $\sqrt{18}$; $\sqrt{20}$; $\sqrt{a^2b^3}$; $\sqrt{a^2b^2x}$. 16. $\sqrt{\frac{2}{3}}$; $\sqrt{\frac{2}{5}}$; $\sqrt{\frac{3}{a}}$; $\sqrt{\frac{1}{6}}$.

17. $\sqrt{45} + \sqrt{\frac{1}{5}}$; $\sqrt{48} + \sqrt{75} - \sqrt{3}$.

18. Divide $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$ by $x + y$.

19 Divide $x^6 - y^6$ by $x^2 + xy + y^2$.

20. Divide $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.

21. The older of two sisters is now 8 years less than twice as old as the other. If x represents the age of the younger sister, represent in symbols twice the sum of their ages 7 years ago.

22. A rear wheel of a wagon has a circumference 4 feet greater than that of a front wheel. If the circumference of the rear wheel is x feet, represent in symbols the number of revolutions each wheel must make to go one mile.

REVIEW EXERCISES FOR CHAPTER XVII

Simplify the following:

1. $\sqrt{32}+\sqrt{72}-\sqrt{50}$.
2. $2a\sqrt{a^3b}-b\sqrt{b^3a}+\sqrt{ab}$.
3. $\sqrt{a^3+2a^2b+ab^2}+\sqrt{a^3-2a^2b+ab^2}-2\sqrt{a^3}$.
4. $\sqrt{a^3-a^2b}-\sqrt{b^2a-b^3}+\sqrt{a^3b^2-a^2b^3}$.
5. $\sqrt{(x+y)(x^2-y^2)}-\sqrt{(x+y)^2(x-y)}-\sqrt{x^3-x^2y}$.

Solve the following equations:

6. $\sqrt{x^2-8}+x=8$.
7. $\sqrt{5a-24}+4=\sqrt{5a}$.
8. $\sqrt{2a-1}=7-\sqrt{2a+6}$.
9. $\sqrt{x+2}=\sqrt{x-6}+2\sqrt{x-5}$.
10. $\sqrt{2x^2-2}=x+1$.
11. $\sqrt{2x+7}=\sqrt{x}+2$.
12. $\dfrac{\sqrt{a}}{\sqrt{x-a}}-\dfrac{\sqrt{x+2a}}{2\sqrt{a}}=0$.
13. $\dfrac{\sqrt{5x+1}}{\sqrt{2x-5}}=\sqrt{\dfrac{9x+1}{3x-5}}$.
14. $\sqrt{x+2}=3-\sqrt{3-x}$.
15. $\dfrac{9x+8}{\sqrt{8x+1}}=\sqrt{10x+19}$.

* Rationalize the denominators of the following:

16. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$.
17. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.
18. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.
19. $\dfrac{\sqrt{x-a}+b}{\sqrt{x+a}-c}$.
20. $\dfrac{3}{\sqrt{7}-\sqrt{4}}$.
21. $\dfrac{\sqrt{7}+\sqrt{2}}{\sqrt{7}-\sqrt{2}}$.
22. $\dfrac{a+b}{\sqrt{a}+\sqrt{b}}$.
23. $\dfrac{2+\sqrt{3}}{4-\sqrt{3}}$.
24. $\dfrac{\sqrt{8}+7}{\sqrt{8}-7}$.
25. $\dfrac{\frac{1}{3}\sqrt{2}-1}{\frac{1}{3}\sqrt{2}+1}$.

26. A and B working together can do a piece of work in 12 days. B and C working together can do it in 13 days, and A and C working together can do it in 10 days. How long will it require each to do it when working alone?

REVIEW EXERCISES FOR CHAPTER XX

Expand by the binomial formula:

1. $(3x-2y)^3$.
2. $(2x+y)^5$.
3. $\left(x+\frac{1}{x}\right)^6$.
4. $(2x+3y)^4$.
5. $(a+b)^7$.
6. $(x-y)^8$.
7. Find the 6th term of $(x+y)^{14}$.
8. Find the 8th term of $(2a-b)^{10}$.
9. Find the 7th term of $(a-3b)^8$.
10. Find the 13th term of $(2a-c)^{15}$.
11. What is the value of x^2-3x if $x=1+\sqrt{5}$? If $x=1-\sqrt{5}$?
12. What is the value of $3x^2-5x+6$ if $x=\frac{2-\sqrt{3}}{2}$? If $x=\frac{2+\sqrt{3}}{2}$?
13. What is the value of $5x^2+7x$ if $x=\frac{a+\sqrt{b}}{2}$? If $x=\frac{a-\sqrt{b}}{2}$?

Find the square root of each of the following:

14. $25a^2+c^2+9b^2-10ac+30ab-6bc$.
15. $x^2+y^2+4z^2+4v^2-2xy+4xz-4xv-4yz+4yv-8zv$.
16. $x^6-2x^5-x^4+3x^2+2x+1$.
17. $x^4-6x^3+13x^2-12x+4$.
18. Divide $6x^4+x^3+12x^2+8$ by $2x^2-x+2$.
19. Divide $3x^6+4x^5-x^4+6x^3-12x^2+8x-12$ by $3x^2-2x+3$.
20. Divide $3a^7-5a^5+8a^3+2a^2-18a+12$ by a^3-a+2.
21. Divide $6a^4+10a^3-9a^2+11a-6$ by $2a^2+4a-3$.
22. A picture inside the frame is w inches wide and l inches long. The frame is a inches wide. Express the area of the frame in terms of a, w, and l.

REVIEW EXERCISES FOR CHAPTER XXI

1. If $y \propto x$ and if $y = 9$ when $x = 17$, find y when $x = 12$.

2. If $y = \frac{k}{x}$ and if $y = 4$ when $x = \frac{1}{3}$, find y when $x = \frac{3}{5}$.

3. If $y \propto x^2$ and if $y = 7$ when $x = 1$, find y when $x = 4$.

4. If $y \propto x^2$ and if $y = 3$ when $x = 3$, find y when $x = 7$.

5. If $y \propto x^2$ and if $y = 6$ when $x = 4$, find y when $x = 10$.

6. If a certain sum of money yields $350 interest in 8 months, how much will the same sum yield in $17\frac{1}{2}$ months?

7. If $2500 yields $680 interest during a certain time, how much money will be required to yield $390 during the same time?

8. If 60 men can do a certain piece of work in 45 days, how long will it take 35 men to do it?

9. Over what distance will a falling body pass in 5 seconds, starting from rest?

10. Over what distance will a body fall in 10 seconds, if it starts with a velocity of 15 feet per second?

11. If a circular disk weighs 16 lbs., how much will a disk weigh if its diameter is 3 times that of the first disk and if it is of the same material and thickness?

12. $$\frac{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}}{\frac{(a-b-c)(a+b+c)}{ab}}.$$

14. $$\frac{\frac{a}{b^2}}{1+\frac{a^2}{b^2}}+\frac{\frac{a}{(b-a)^2}}{\frac{b+a}{b-a}}.$$

13. $$\frac{\left(a-\frac{3}{a}\right)^2+16}{\left(a+\frac{3}{a}\right)^2+4}.$$

15. $$\frac{2+\frac{1}{x-2}+x}{x+\frac{1}{x+2}-2}.$$

MISCELLANEOUS REVIEW EXERCISES

Solve the following:

1. $(x-2)^2-(x-1)(x+2)=6-5x.$

2. $(x-2)(x+2)+(3x-1)(2-x)=(x-2)(5-2x).$

3. $(x-3)^2+(2x+5)^2=(5x-3)(x+5)-7.$

4. $(2-x)^2-(2x-1)^2=(-3x+1)(4+x)-4.$

5. $\begin{cases} 2x-3y-1=0, \\ 5x+2y=12. \end{cases}$

6. $\begin{cases} x+y=a, \\ x-y=b. \end{cases}$

7. $\begin{cases} ax+by=c, \\ ax-by=d. \end{cases}$

8. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=c, \\ \dfrac{x}{a}-\dfrac{y}{b}=d. \end{cases}$

9. $\begin{cases} 3ax-by=2, \\ 2x+3by=6. \end{cases}$

10. $\begin{cases} x-y-3z=-6, \\ 2x+y-z=11, \\ -x+3y+z=16. \end{cases}$

11. $\begin{cases} 2x-3by=c, \\ 2ax-5y=d. \end{cases}$

12. $\begin{cases} 2x-y+3z=20, \\ x+4y-z=-2, \\ 5x+y-6z=6. \end{cases}$

13. $(x-1)^2-(x-8)(2x-1)=-x^2+98.$

14. $(7+x)(x-4)+(1-x)^2=-23+2x^2.$

15. $(12-4x)(2-x)-4(1+x)^2=5x+119.$

16. $(x-17)(59-2x)-(1-x)^2=(6-3x)(x-2)+384.$

17. $(3x-2)+(x-1)^2+(x-2)^2=2(x-1)(x-2)+5.$

18. $(6-3x)(2+x)+16(x-1)^2=13(x+4)^2+364.$

19. $7x+(8x+4)\div 2=4x+9.$

20. $6x+4(4x+2)=85-3(2x+7).$

21. $8+7(6+6n)+2n=2(4n+5)+18n+49.$

22. $5(9x+3)+6x=24x-4(3x+2)+36.$

23. Find the average of the following temperatures: 7 A.M., $-4°$; 8 A.M., $-2°$; 9 A.M., $-1°$; 10 A.M., $+1°$; 11 A.M., $+5°$; 12 M., $+7°$.

Solve the following:

24. $\begin{cases} 3x - 7by = 2, \\ 2ax + 2y = 4. \end{cases}$

25. $\begin{cases} 6x + 3y = 1, \\ 5ax - 2by = c. \end{cases}$

26. $\dfrac{7(12x+8)}{4} + 13 + 5x - 6 = 47$

27. $\dfrac{x+2}{a} = \dfrac{x-1}{b} + \dfrac{3x+2}{ab}.$

28. $\begin{cases} \dfrac{7x-3}{2} - \dfrac{ay+1}{3} = 1, \\ \dfrac{bx+5}{4} + \dfrac{y-3}{3} = 6. \end{cases}$

29. $\begin{cases} \dfrac{2by}{3a} - 3x = 2, \\ 5y + \dfrac{2ax}{b} = 5. \end{cases}$

30. $\begin{cases} 2x - y - z = 8, \\ 3x + 2y + z = 24, \\ -x - 3y + 5z = 16. \end{cases}$

31. $\begin{cases} 8x - 3y - z = 0, \\ 2x + 2y + 3z = 10, \\ -x + y + 6z = 8. \end{cases}$

32. $\dfrac{a-b}{2x-1} = \dfrac{4c}{(2x+1)(2x-1)} - \dfrac{a-b}{2x+1}.$

33. $(1-3x)^2 + (2x+1)^2 = 5x^2 + (2x+6)(4x+27).$

34. $(1+2x)(2-3x) + (x-4)(x+4) = (x+14)(18-5x) - 1.$

35. A picture inside the frame is 12 inches long and 8 inches wide. If the frame is a inches wide, express its area in terms of a.

36. If h is the length of the hypotenuse of a right triangle and a the length of one side, express the length of the third side in terms of h and a.

37. A rectangular piece of tin is w inches wide and l inches long. If a square a inches on a side is cut out of each corner, express in terms of w, l, and a the volume of a box formed by turning up the sides.

38. A farmer plows a strip a rods wide around a rectangular field w rods wide and l rods long. Express in terms of w, l, and a the area plowed.

Solve the following:

39. $2(x+5)(x-5)=(x-5)(x+1)+(x-2)^2-1.$

40. $\begin{cases} \dfrac{x-1}{2}+\dfrac{y+2}{3}=3\frac{1}{6}, \\[2mm] \dfrac{x+1}{3}-\dfrac{y-1}{4}=\frac{7}{6}. \end{cases}$

41. $\begin{cases} \dfrac{x-a}{2}+\dfrac{y-b}{3}=1, \\[2mm] \dfrac{x+a}{3}+\dfrac{y+b}{2}=1. \end{cases}$

42. $\begin{cases} 2\,ax-3\,by=c, \\ 2\,ax+3\,by=d. \end{cases}$

43. $\begin{cases} x+y+z=6, \\ 2\,x-y+z=3, \\ 3\,x+2\,y-2\,z=1. \end{cases}$

44. $\begin{cases} x-y+z=4, \\ x-2\,y+4\,z=6, \\ 2\,x+y-3\,z=10. \end{cases}$

45. $\begin{cases} x+y+z=0, \\ 2\,x-4\,y+z=-3, \\ 3\,x+2\,y+4\,z=3. \end{cases}$

46. $\begin{cases} x+ay=c, \\ bx+y=d. \end{cases}$

47. $\begin{cases} ax+3\,y=2\,c, \\ bx-2\,y=3\,d. \end{cases}$

48. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\[2mm] \dfrac{1}{y}+\dfrac{1}{z}=b, \\[2mm] \dfrac{1}{z}+\dfrac{1}{x}=c. \end{cases}$

49. $\begin{cases} ax+by=1, \\ cx+dy=4. \end{cases}$

50. $\begin{cases} ax-by=c, \\ cx+dy=e. \end{cases}$

51. $\begin{cases} x+y+z=a, \\ 2\,x-2\,y+2\,z=b, \\ 3\,x-y-z=c. \end{cases}$

52. $\begin{cases} ax-y+bz=a, \\ x+ay-z=1, \\ bx-y+az=b. \end{cases}$

53. If w and l are the length and width of a rectangle, express in symbols the length of its diagonal.

54. The lengths of the two sides of a right triangle are 16 and 24 respectively. Express the length of the hypotenuse in the simplest form without approximating a square root.

55. A father is now twice the age of his son. If x represents the son's age now, express twice the sum of their ages 5 years ago.

56. One number is 3 less than 4 times another. Express one third the sum of the numbers if x represents one of them.

57. If h is the digit in hundreds' place, t the digit in tens' place, and u the digit in units' place of a certain number, express the number obtained by inverting the order of the digits of the given number.

58. h, t, and u are the digits in hundreds', tens', and units' places of a number. Express the number obtained by increasing each digit by 2.

59. $\frac{12(5+4x)}{6} - \frac{5(6+4x)}{2} + 50 = x + 18.$

60. $15 + \frac{21(3+x)}{7} + \frac{2(6+18x)}{3} = \frac{3(9x+12)}{3} + 28.$

61. $\frac{11(5x+25)}{5} + \frac{3(6x-2)}{2} = \frac{7(4x+8)}{4} + \frac{12x+36}{3} + 35.$

62. $\frac{2(x+1)}{x-1} - 3 = -\frac{x-1}{x+1}.$

63. $\frac{x+a}{x-a} = 5 - \frac{4(x-a)}{x+a}.$

64. $\begin{cases} \frac{2a+5}{3} + \frac{3b-10}{4} = 0, \\ \frac{3a+6}{8} - \frac{5-4b}{7} = 2. \end{cases}$

PROBLEMS ON MOTION

1. A sparrow flies 135 feet per second and a hawk 149 feet per second. The hawk in pursuing the sparrow passes a certain point 7 seconds after the sparrow. In how many seconds from this time does the hawk overtake the sparrow?

2. A courier starts from a certain point, traveling r_1 miles per hour, and a hours later a second courier starts, going at the rate of r_2 miles per hour. In how long a time will the second overtake the first, supposing r_2 to be greater than r_1?

If the second courier requires t hours to overtake the first, the latter had been on the way $t + a$ hours. Thus the distance covered by the second courier is r_2t and by the first $r_1(t+a)$. As these numbers are equal we have

$$r_2t = r_1(t+a).$$

This formula summarizes the solution of all problems like 3 and 4 on the next page.

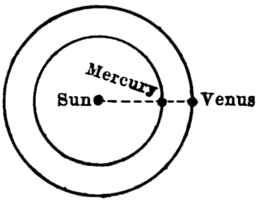
3. In an automobile race A drives his machine at an average rate of 53 miles per hour, while B, who starts $\frac{1}{4}$ hour later, averages 57 miles per hour. How long does it require B to overtake A? Use the formula on page 343.

4. A freight steamer leaves New York for Liverpool, averaging $10\frac{1}{2}$ knots per hour, and is followed 4 days later by an ocean greyhound, averaging $25\frac{1}{2}$ knots per hour. In how long a time will the latter overtake the former?

5. One athlete makes a lap on an oval track in 26 seconds, another in 28 seconds. If they start together in the same direction, how soon will the first gain one lap? Two laps?

Let one lap be the unit of distance. Since the first covers one lap in 26 seconds, his rate per second is $\frac{1}{26}$. Likewise the rate of the other is $\frac{1}{28}$. If t is the required number of seconds, the distance covered by the first is $\frac{1}{26}t$ and by the second $\frac{1}{28}t$. If the first goes one lap farther than the second, the equation is $\frac{1}{26}t = \frac{1}{28}t + 1$.

6. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?



7. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, as in the figure, how long before they will again be in this position?

Note that the problem may be solved just as if the two planets were moving in the same orbit at different rates.

8. Saturn goes around the sun in 29 years and Jupiter in 12 years. Find the time between two conjunctions.

9. Uranus makes the circuit of its orbit in 84 years and Neptune in 164 years. If they start in conjunction, how long before they will be in conjunction again? *Ans.* $172\frac{1}{5}$ years.

10. The hour hand of a watch makes one revolution in 12 hours and the minute hand in 1 hour. How long is it from the time when the hands are together until they are again together?

11. One object makes a complete circuit in a units of time and another in b units (of the same kind). In how many units of time will one overtake the other, supposing b to be greater than a?

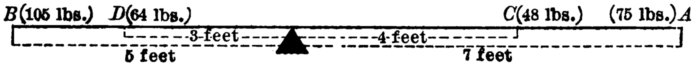
The solution of this problem summarizes the solution of all problems like those from 5 to 10.

12. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

PROBLEMS INVOLVING THE LEVER

1. A teeter board is in balance when two boys, A and B, weighing 105 and 75 pounds respectively, are seated at distances 5 and 7 feet from the fulcrum, because $7 \cdot 75 = 5 \cdot 105$. If now two boys weighing 48 and 64 pounds are seated on the same board with the other boys, the teeter will again be in balance if their distances are 4 and 3 feet, because

$$7 \cdot 75 + 4 \cdot 48 = 5 \cdot 105 + 3 \cdot 64$$



The weight of the boy multiplied by his distance from the fulcrum is called his **leverage**. The sum of the leverages on the two sides must be the same. Hence, if the teeter balances when two boys, weighing respectively w_1 and w_2 pounds, are at distances d_1 and d_2 on one side, and two boys, weighing w_3 and w_4 pounds, are at distances d_3, d_4 on the other side, then

$$\boldsymbol{w_1d_1 + w_2d_2 = w_3d_3 + w_4d_4}.$$

2. If two boys weighing 75 and 90 pounds sit at distances of 3 and 5 feet respectively on one side and one weighing 82 pounds sits at 3 feet on the other side, where should a boy weighing 100 pounds sit to make the board balance?

3. A beam carries a weight of 240 pounds 7½ feet from the fulcrum and a weight of 265 pounds at the opposite end which is 10 feet from the fulcrum. On which side and how far from the fulcrum should a weight of 170 pounds be placed so as to make the beam balance?

4. Two boys, A and B, having a 50-lb. weight and a teeter board, proceed to determine their respective weights as follows: They find that they balance when B is 9 feet and A is 7 feet from the fulcrum. If B places the 50-lb. weight on the board beside him, they balance when B is 3 and A is 4 feet from the fulcrum. How heavy is each boy?

5. C is 6½ feet from the point of support and balances D, who is at an unknown distance from this point. C places a 33-lb. weight beside himself on the board and, when 4⅔ feet from the fulcrum, balances D, who remains at the same point as before. D's weight is 84 pounds. What is C's weight, and how far is D from the fulcrum?

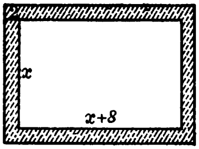
6. E weighs 95 pounds and F 110 pounds. They balance at certain unknown distances from the fulcrum. E then takes a 30-pound weight on the board, which compels F to move 3 feet farther from the fulcrum. How far from the fulcrum was each of the boys at first?

7. A who weighs 60 pounds sits 3 feet from the fulcrum. B who weighs 90 pounds sits 4 feet from the fulcrum on the same side as A. How far from the fulcrum on the opposite side must C, who weighs 108 pounds, sit in order to make the teeter board balance?

PROBLEMS INVOLVING GEOMETRY

1. A picture is 4 inches longer than it is wide. Another picture, which is 12 inches longer and 6 inches narrower, contains the same number of square inches. Find the dimensions of the pictures.

2. A picture, not including the frame, is 8 inches longer than it is wide. The area of the frame, which is 2 inches wide, is 176 square inches. Find the dimensions of the picture.



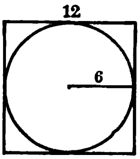
3. A picture, including the frame, is 10 inches longer than it is wide. The area of the frame, which is 3 inches wide, is 192 square inches. What are the dimensions of the picture?

4. The base of a triangle is 11 inches greater than its altitude. If the altitude and the base are both decreased 7 inches, the area is decreased 119 square inches. Find the base and the altitude of the triangle.

5. The base of a triangle is 3 inches less than its altitude. If the altitude and the base are both increased by 5 inches, the area is increased by 155 square inches. Find the base and the altitude of the triangle.

6. A square is inscribed in a circle and another circumscribed about it. The area of the strip inclosed by the two squares is 25 square inches. Find the radius of the circle.

7. Find the sum of the areas of a circle of radius 6 and the square circumscribed about the circle.



The area of the circle is $6^2\pi = 36\pi$, and the area of the square $4 \cdot 6^2 = 4 \cdot 36$; *i.e.* the square contains 4 squares whose sides are 6. The sum of the areas is

$$4 \cdot 36 + 36\pi = (4+\pi)36 = (4 + 3\tfrac{1}{7})\,36.$$

8. Find an expression for the sum of the areas of a circle of radius r and the circumscribed square. (Solve Example 7 by substituting in the formula here obtained.)

9. If the sum of the areas of a circle and the circumscribed square is 64, find the radius of the circle.

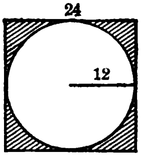
By the formula obtained under Example 8,

$$64 = (4 + \pi)\, r^2 = \tfrac{50}{7}\, r^2.$$

Hence,
$$r = \sqrt{8.96} = 2.99.$$

10. If the sum of the areas of a circle and the circumscribed square is 640 square feet, find the radius of the circle.

11. The sum of the areas of a circle and the circumscribed square is a. Find an expression representing the radius of the circle. (Replace π by $3\frac{1}{7}$ before simplifying.)



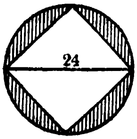
12. If the radius of a circle is 12, find the difference between the areas of the circle and the circumscribed square.

13. If the radius of a circle is r, find the difference between the areas of the circle and the circumscribed square. (Solve Example 12 by the use of the formula obtained here.)

14. If the radius of a circle is 16, find the area of the inscribed square. (This is the same problem as finding the area of a square whose diagonal is 32.)

15. If the radius of a circle is r, find an expression representing the area of the inscribed square.

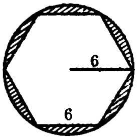
16. If the radius of a circle is 12, find the difference between the area of the circle and the area of the inscribed square.



17. If the radius of a circle is r, find an expression representing the difference between the areas of the circle and of the inscribed square.

18. The radius of a circle is 10. Find the area of an inscribed hexagon.

19. The radius of a circle is 6. Find the difference between the areas of the circle and the inscribed hexagon.



20. Find an expression representing the difference between the areas of a circle with radius r and the inscribed regular hexagon.

MISCELLANEOUS PROBLEMS

1. Divide the number 645 into two parts, such that 13 times the first part is 20 more than 6 times the other.

2. Divide the number a into two parts, such that b times the first part is c more than d times the second part.

3. The sum of three numbers is 98. The second is 7 greater than the first, and the third is 9 greater than the second. What are the numbers?

4. The sum of three numbers is s. The second is a greater than the first, and the third is b greater than the second. What are the numbers?

5. One boy runs around a circular track in 26 seconds, and another in 30 seconds. In how many seconds will they again be together, if they start at the same time and place and run in the same direction?

6. A bird flying with the wind goes 65 miles per hour, and flying against a wind twice as strong it goes 20 miles per hour. What is the rate of the wind in each case?

7. A steamer going with the tide makes 19 miles per hour, and going against a current $\frac{1}{2}$ as strong it makes 13 miles per hour. What is the speed of the steamer in still water?

8. Find the time between 4 and 5 o'clock when the hands of the clock are 30 minute spaces apart.

9. A man takes out a life insurance policy for which he pays in a single payment. Thirteen years later he dies and the company pays \$ 12,600 to his estate. It was found that his investment yielded 2 % simple interest. How much did he pay for the policy ?

10. After deducting a commission of 3 % for selling bonds, a broker forwarded \$ 824.50. What was the selling price of the bonds ?

11. A broker sold stocks for \$ 1728 and remitted \$ 1693.44 to his employer. What was the rate of his commission ?

12. The difference between the areas of a circle and its circumscribed square is 12 square inches. Find the radius of the circle. (See problem 11, page 348.)

13. The difference between the areas of a circle and its inscribed square is 12 square inches. Find the radius of the circle.

14. The difference between the areas of a circle and the regular inscribed hexagon is 12 square inches. Find the radius of the circle.

15. The altitude of an equilateral triangle is 6. Find its side and also its area. Find the side and area if the altitude is h.

16. The radius of a circle is 3 feet. Find the area of the regular circumscribed hexagon. Find the area if the radius is r feet.

17. The radius of a circle is r. Find the difference between the areas of the circle and the regular circumscribed hexagon.

18. The difference between the areas of a circle and the regular circumscribed hexagon is 9 square inches. Find the radius of the circle.

19. A circle is inscribed in a square and another is circumscribed about it. The area of the ring formed by the two circles is 25 square inches. How long is the side of the square?

20. In a building there are at work 18 carpenters, 7 plumbers, 13 plasterers, and 6 hod carriers. Each plasterer gets $1.90 per day more than the hod carriers, the carpenters get 35 cents per day more than the plasterers, and the plumbers 50 cents per day more than the carpenters. If one day's wages of all the men amount to $183.45, how much does each get per day?

21. A train running 46 miles per hour leaves Chicago for New York at 7 A.M. Another train on the same road running 56 miles per hour leaves at 9.30 A.M. Find when the trains will be 15 miles apart. (Two answers.)

22. There is a number consisting of three digits, those in tens' and units' places being the same. The digit in hundreds' place is 4 times that in units' place. If the order of the digits is reversed, the number is decreased by 594. What is the number?

23. A hound pursuing a deer gains 400 yards in 25 minutes. If the deer runs 1300 yards a minute, how fast does the hound run? If the hound gains v_1 yards in t minutes, and the deer runs v_2 yards per minute, find the speed of the hound.

24. The altitude of Popocatepetl is 1716 feet less than that of Mt. Logan, and the altitude of Mt. St. Elias is 316 feet greater than that of Popocatepetl. Find the altitude of each mountain, the sum of their altitudes being 55,384 feet.

25. It is 4 times as far from New York City to Cincinnati as from New York to Baltimore. Twice the distance from New York to Cincinnati minus 5 times that from New York to Baltimore equals 567 miles. How far is it from New York to each of the other cities?

26. A disabled steamer 240 knots from port is making only 4 knots an hour. By wireless telegraphy she signals a tug, which comes out to meet her at 17 knots an hour. In how long a time will they meet? If the steamer is s knots from port and is making v_1 knots per hour, and if the tug makes v_2 knots per hour, find how long before they will meet.

27. A motor boat, going 11 miles per hour, starts $7\frac{2}{3}$ miles behind a sailboat going $6\frac{1}{2}$ miles per hour. How far apart will they be in $1\frac{1}{3}$ hours? If the motor boat starts s miles behind the sailboat and runs v_1 miles per hour, while the sailboat runs v_2 miles per hour, how far apart will they be in t hours?

28. An ocean liner making 21 knots an hour leaves port when a freight boat making 8 knots an hour is already 1240 knots out. In how long a time will the two boats be 280 knots apart? Is there more than one such position? If the liner makes v_1 knots per hour and the freight boat, which is s_1 knots out, makes v_2 knots per hour, how long before they will be s_2 knots apart?

29. A passenger train running 45 miles per hour leaves one terminal of a railroad at the same time that a freight running 18 miles per hour leaves the other. If the distance is 500 miles, in how many hours will they meet? If they meet in 8 hours, how long is the road? If the rates of the trains are v_1 and v_2 and the road is s miles long, find the time.

30. The melting temperature of glass is 276 degrees (Centigrade) higher than twice that of zinc. One half the number of degrees at which glass melts plus 7 times the number at which zinc melts equals 3434. Find the melting point of each.

31. The melting temperature of nickel is 496 degrees (Centigrade) higher than that of silver. Three times the number of degrees at which nickel melts plus 2 times the number at which silver melts equals 6258. Find the melting point of each.

INTERMEDIATE ALGEBRA

INTERMEDIATE ALGEBRA

CHAPTER I

DEFINITIONS AND FUNDAMENTAL LAWS

1. Algebra and Arithmetic. Algebra differs from Arithmetic in three important respects:

1. *Letters are used systematically to represent numbers.*
2. *The equation is used extensively in the solution of problems.*
3. *The number system is extended to include negative and imaginary numbers.*

2. The operations of *addition, subtraction, multiplication, division,* and *extraction of roots* are called *algebraic operations.*

3. Addition. Numbers which are to be added are called *addends,* and the result obtained by adding numbers is called their *sum.*

4. Subtraction. The process of finding one of two numbers when the other number and their sum are given is called *subtraction.* The given number is the *subtrahend;* the given sum, the *minuend;* and the number to be found is the *remainder.*

5. Multiplication. The result obtained by multiplying one number by another is called their *product.* The numbers multiplied together are the *factors* of the product.

6. Division. The process of finding one of two numbers when the other number and their product are given is called *division.* The given number is the *divisor;* the given product, the *dividend;* and the number to be found is the *quotient.*

7. Symbols of Operation. The symbols $+$, $-$, $\times$, $\div$, $\sqrt{}$, are used with the same meaning as in arithmetic.

8. Coefficient. In the product of two factors, either factor is called the coefficient of the other.

Thus, in $6a$, 6 is the coefficient of a, and a is the coefficient of 6. In $2ax$, $2a$ is the coefficient of x, $2x$ is the coefficient of a, and 2 is the coefficient of ax.

9. Exponent. A number written a little above and to the right of another number is called an *exponent*.

If an exponent is a positive integer, it indicates how many times the number under it is used as a factor.

Thus, $a^3 = a \cdot a \cdot a$, and $x^4 = x \cdot x \cdot x \cdot x$.

10. Powers. The product obtained by multiplying a number by itself one or more times is called a *power* of the number.

11. Base. A number multiplied by itself to form a power is called the *base* of the power.

Thus, in x^4, x is the base, x^4 is the power of the base, and 4 is the exponent of the power.

12. Roots. One of the two equal factors of a number is a *square root* of the number; one of its three equal factors is a *cube root* of the number; and in general, one of the n equal factors of a number is an *nth root of the number*.

Thus, 2 is a square root of 4, and 3 is a cube root of 27.

13. The Radical Sign. The symbol $\sqrt{}$ is called the *radical sign*. When placed over a number, it indicates that a root of that number is to be taken. A small numeral placed in the radical sign is called the index of the root, and shows what root is to be taken.

Thus, $\sqrt[3]{8}$ means a cube root of 8, and $\sqrt[4]{16}$ means a fourth root of 16. That is, $\sqrt[3]{8} = 2$ and $\sqrt[4]{16} = 2$.

The square root is indicated by the radical sign without an index.

Thus, $\sqrt{4}$ means a square root of 4.

14. Rational and Irrational Roots. A root which can be expressed in the form of an integer, or as the quotient of two integers, is said to be *rational,* while one which cannot be so expressed is *irrational.*

Thus, $\sqrt[3]{8} = 2$, $\sqrt{a^2 + 2\,ab + b^2} = a + b$, and $\sqrt{\frac{9}{4}} = \frac{3}{2}$ are rational roots, while $\sqrt[3]{4}$ and $\sqrt{a^2 + ab + b^2}$ are irrational roots.

15. Rational and Irrational Expressions. An algebraic expression which involves a letter in an irrational root is said to be *irrational with respect to that letter;* otherwise the expression is *rational with respect to the letter.*

Thus, $a + b\sqrt{c}$ is rational with respect to a and b, and irrational with respect to c.

16. Signed Numbers. There are many situations in which we start at a certain point and measure in opposite directions. Thus, latitude is measured *north* and *south* from the equator; longitude is measured *east* and *west* from the meridian of Greenwich; temperature is measured *above* and *below* zero, etc.

In algebra we have the signed numbers which are directly applicable to such cases.

The signed numbers of algebra consist of the ordinary numbers of arithmetic, together with numbers *opposite* to these, which are called *negative* numbers. The numbers of arithmetic are called *positive* numbers.

A negative number is indicated by placing the sign $-$ before it. A number with no sign before it is positive, but sometimes the sign $+$ is used to point out especially that a number is positive.

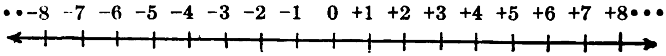
Thus, temperature is often designated as $+$ or $-$, according as it is above or below zero.

17. Absolute Value of Signed Numbers. The absolute value of a signed number is the value of the number when the sign is omitted.

Thus, the absolute value of $+3$, and also of -3, is 3.

18. Graphic Representation of Signed Numbers. One of the most extensive uses of signed numbers is for marking points on a straight line.

On an unlimited straight line mark one point zero. On both sides of it lay off equal divisions as shown in the figure. The points to the right of the zero point are marked + and those to the left are marked −.



19. Addition of Signed Numbers. In addition, positive and negative numbers tend to cancel each other.

Thus, $+3+(-3)=0$, since -3 cancels all of $+3$, leaving zero.
Similarly, $+6+(-4)=+2$, since -4 cancels 4 out of 6, leaving $+2$.
Again, $+6+(-10)=-4$, since $+6$ cancels -6, leaving -4.

Signed numbers are added according to the following **rule**:

To add two numbers with like signs, find the sum of their absolute values, and prefix to this their common sign.

To add two numbers with opposite signs, find the difference of their absolute values, and prefix to this the sign of that one whose absolute value is the greater.

In case the signs of two numbers are opposite, and their absolute values are equal, their sum is zero.

20. Subtraction of Signed Numbers. The method of subtracting signed numbers is obtained directly from the definition of subtraction.

Thus, $8-(3)=5$, because $5+3=8$.
Similarly, $8-(-3)=11$, because $11+(-3)=8$,
$-8-(+3)=-11$, because $-11+3=-8$,
and $-8-(-3)=-5$, because $-5+(-3)=-8$.

These examples lead to the following **rule**:

To subtract one signed number from another signed number, change the sign of the subtrahend and add it to the minuend.

21. Multiplication of Signed Numbers. The rule for multiplying signed numbers may be found by considering the case in which the multiplier is an integer.

Thus, $3 \times 4 = 4 + 4 + 4 = 12,$

$3 \times (-4) = (-4) + (-4) + (-4) = -4 - 4 - 4 = -12,$

$-3 \times 4 = -(+4) - (+4) - (+4) = -4 - 4 - 4 = -12,$

and $-3 \times (-4) = -(-4) - (-4) - (-4) = +4 + 4 + 4 = +12.$

These examples lead to the following **rule**:

If two numbers have the same sign, their product is positive; if they have opposite signs, their product is negative.

The absolute value of the product is the product of the absolute values of the factors.

22. Division of Signed Numbers. The method of dividing signed numbers is obtained from the definition of division.

Thus, $8 \div 4 = 2$, because $2 \times 4 = 8$,

$8 \div (-4) = -2$, because $(-2) \times (-4) = 8$,

$-8 \div (-4) = 2$, because $2 \times (-4) = -8$,

and $-8 \div 4 = -2$, because $(-2) \times 4 = -8$.

Such examples lead to the following **rule**:

The quotient of two signed numbers is positive if the dividend and divisor have like signs; it is negative if they have opposite signs.

ORAL EXERCISES

Give the result in each of the following:

1. $8 + (-9)$
2. $9 + (-3)$
3. $7 - (-3)$
4. $4 - (-8)$
5. $3 - 8$
6. $-3 - (-4)$
7. $-9 - (-3)$
8. $-8 - 6$
9. $3 \times (-7)$
10. -6×4
11. $-3 \times (-9)$
12. $-4 \times (-12)$
13. $12 \div (-6)$
14. $16 \div (-4)$
15. $-8 \div 2$
16. $-18 \div 6$
17. $-21 \div (-7)$
18. $(-40) \div (-8)$
19. $(-36) \div 9$
20. $32 \div (-16)$
21. $-48 \div (-8)$

23. Fundamental Laws. While the operations of algebra are the same as those of arithmetic, they are sometimes carried out in a different manner, the difference being due largely to the representation of numbers by means of letters.

This leads to the study of certain *fundamental laws*, according to which these operations are carried out.

24. The Commutative Law of Addends. *Numbers to be added may be arranged in any order.*

Thus, $a + b + c = a + c + b = b + a + c$, etc.

25. The Associative Law of Addends. *Numbers to be added may be grouped in any manner.*

Thus, $a + b + c = a + (b + c)$.

The expression $a + b + c$ means that a is to be added to b, and then this sum is added to c, while $a + (b + c)$ means that b and c are added first and then a is added to the sum.

26. The Commutative Law of Factors. *Factors may be arranged in any order.*

Thus, $abc = acb = bac = bca$, etc.

27. The Associative Law of Factors. *Factors may be grouped in any manner.*

Thus, $abc = a(bc)$.

The expression abc means that b is multiplied by a, and c by this product; while $a(bc)$ means that c is first multiplied by b and this product is multiplied by a.

28. The Distributive Law of Multiplication with Respect to Addition. *The sum of two numbers may be multiplied by a third number by multiplying each number separately and adding the products.*

That is, $a(b + c) = ab + ac$.

If the equation $a(b + c) = ab + ac$ is read from right to left, it may be stated in words as follows:

To add numbers having a common factor, add the coefficients of the common factor and multiply the sum by the common factor.

29. If $a = \frac{1}{m}$, we have: $\frac{1}{m}(b + c) = \frac{1}{m} \cdot b + \frac{1}{m} \cdot c = \frac{b}{m} + \frac{c}{m}$, which is the distributive law of division with respect to addition.

This may be stated in words as follows:

To divide the sum of two numbers, divide each one separately and add the quotients.

If c is a negative number, $a(b + c) = ab + ac$ gives rules for multiplying or dividing the *difference of two numbers.*

30. The commutative and associative laws of factors give rise to the following **rule**:

To multiply or divide a product, multiply or divide any one of the factors, leaving the other factors unchanged.

Thus, $4 \cdot (2 \cdot 3) = (4 \cdot 2) \cdot 3 = 8 \cdot 3 = 24$; or $4 \cdot (2 \cdot 3) = (4 \cdot 3) \cdot 2 = 12 \cdot 2 = 24$. Also $\frac{2 \cdot 4 \cdot 6}{2} = 1 \cdot 4 \cdot 6 = 2 . 2 \cdot 6 = 2 \cdot 4 \cdot 3 = 24$.

31. Division by Zero. Using the signed numbers of algebra, we have:

1. Any two numbers have one and only one sum.
2. Any two numbers have one and only one difference.
3. Any two numbers have one and only one product.
4. Any number divided by any number gives one and only one quotient *provided the divisor is not zero.*

In case the divisor is zero, we need to consider two cases:

(*a*) If the dividend a is different from zero, then there is no number k such that

$$\frac{a}{0} = k,$$

for by the definition of division, $a = 0 \cdot k = 0$, which contradicts the provision that a is different from zero.

(*b*) If the dividend a is zero, then $\frac{a}{0}$ equals any number whatever. That is

$$\frac{0}{0} = k,$$

for any value of k, since $0 = 0 \cdot k$ for all values of k.

Hence *dividing a number different from zero by zero is an impossible operation. Dividing zero by zero leads to an indeterminate result. In no case, therefore, is division by zero permitted.*

32. Symbols of Aggregation. Parentheses are used to indicate that some operation is to be extended over the whole expression inclosed by them. Thus, $2(x+y)$ means that the sum of x and y is to be multiplied by 2, while $2x+y$ means that x alone is to be multiplied by 2.

Instead of parentheses, brackets [ ], braces { }, or the vinculum $\overline{\quad\quad}$, may be used with the same meaning. All such symbols are called symbols of aggregation.

Thus, $2(x+y)$, $2[x+y]$, $2\{x+y\}$, $2\,\overline{x+y}$ all mean the same thing.

In the sign $\sqrt{\quad}$, the horizontal bar is a sign of aggregation, indicating that the root of the whole expression under the bar is to be taken. Thus, $\sqrt{a+b}$ means the square root of the sum of a and b.

The dividing line of a fraction is a symbol of aggregation, indicating that the whole of the numerator is to be divided by the whole of the denominator.

33. Order of Operations. In a succession of indicated operations the final result depends in some cases upon the order in which the operations are performed, while in some cases it does not.

Thus, $36 \div 6 \div 3 = 6 \div 3 = 2$, if 36 is divided by 6 first and this result by 3. On the other hand, $36 \div 6 \div 3 = 36 \div 2 = 18$, if 6 is divided by 3 first, and then 36 divided by the quotient.

To prevent mistakes, and to make usage uniform, the following rules have been adopted:

In an expression involving additions, subtractions, multiplications, divisions, powers, and roots, when no symbols of aggregation are involved,

(1) *All powers and roots are found first.*

(2) *All multiplications are performed next, and these may be taken in any order.*

(3) *All divisions are performed next, and these are taken in the order in which they occur from left to right.*

(4) *Finally, additions and subtractions are performed, and these may be taken in any order.*

CHAPTER II

FUNDAMENTAL OPERATIONS

34. Algebraic Expressions. Any combination of numerals, letters, and signs of operation, used for the purpose of representing numbers, is called an *algebraic expression.*

35. Polynomials ; Terms. An algebraic expression composed of parts connected by the signs + and − is called a **polynomial.** Each of the parts thus connected, together with the sign preceding it, is called a **term.**

E.g. $5\,a - 3\,xy - \frac{2}{3}\,rt + 99$ is a polynomial whose terms are $5\,a$, $-3\,xy$, $-\frac{2}{3}\,rt$, and $+99$. The sign + is understood before $5\,a$.

A polynomial of two terms is called a **binomial** ; one of three terms is called a **trinomial.** A term taken by itself is called a **monomial.**

E.g. $5\,a - 3\,xy$ is a binomial ; $5\,a - 3\,xy - \frac{2}{3}\,rt$ is a trinomial whose terms are the monomials $5\,a$, $-3\,xy$, $-\frac{2}{3}\,rt$.

According to the above definition $a + (b + c)$ may be called a binomial though it is equivalent to the trinomial $a + b + c$.

In this case a is called a **simple** term and $(b + c)$ a **compound** term. Likewise we may call $3\,t + 4\,x - 5\,(a + b)$ a trinomial having the simple terms $3\,t$, $4\,x$, and the compound term $-5(a + b)$.

36. Similar Terms. Two terms which have a factor in common are said to be **similar with respect to that factor.**

E.g. $5\,a$ and $-3\,a$ are similar with respect to a ; $-3\,xy$ and $-7\,x$ are similar with respect to x ; $5\,a$ and $-5\,b$ are similar with respect to 5 ; $7\,abc$ and $-\frac{2}{3}\,abc$ are similar with respect to abc.

ADDITION AND SUBTRACTION OF MONOMIALS

37. Adding Similar Terms. In accordance with § 28, the sum of terms which are similar with respect to a common factor is equal to the product of this common factor and the sum of its coefficients.

Notice that this rule applies at once to subtraction because, by § 20, subtraction is replaced by the addition of the subtrahend with its sign changed.

Example 1. $8\,ax^2 + 9\,ax^2 - 3\,ax^2 = [8 + 9 + (-3)]\,ax^2 = 14\,ax^2.$

Example 2. $a\sqrt{x^2+y^2} + b\sqrt{x^2+y^2} = (a+b)\sqrt{x^2+y^2}.$

Example 3. $(x+3)(x-4)+(2\,x-5)(x-4)+(3\,x-2)(x+7).$

Adding the first two terms,

$$\begin{aligned}(x+3)(x-4)+(2\,x-5)(x-4)&=(x+3+2\,x-5)(x-4)\\&=(3\,x-2)(x-4)\end{aligned}$$

Adding this sum to the third term,

$$\begin{aligned}(3\,x-2)(x-4)+(3\,x-2)(x+7)&=(x-4+x+7)(3\,x-2)\\&=(2\,x+3)(3\,x-2),\end{aligned}$$

which is the required sum.

ORAL EXERCISES

Perform the following indicated operations:

1. $5\,bc + 3\,bc + 4\,bc - 8\,bc.$
2. $4\,b^2x + 6\,b^2x - 3\,b^2x - 2\,b^2x.$
3. $7\,b^5c^4 - 9\,b^5c^4 + 3\,b^5c^4 - 18\,b^5c^4.$
4. $11\,a^6x^4 - 15\,a^6x^4 + 24\,a^6x^4 - 3\,a^6x^4.$
5. $14\,a^3x^5 - 11\,a^3x^5 - 14\,a^3x^5 + 11\,a^3x^5.$
6. $ax^2 - bx^2 + cx^2 - dx^2.$
7. $5(n-1) + 3(n-1) - 6(n-1).$
8. $4(n+1) - 8(n+1) + 11(n+1).$
9. $a(n-1) + b(n-1) - c(n-1).$
10. $3\,n(n-1) + 2\,n(n-1) - 4\,n(n-1).$
11. $n\,(n-1) - 2(n-1) + 3(n-1).$

12. $n(n+1)-4(n+1)+3(n+1)$.

13. $3\sqrt{x+1}+6\sqrt{x+1}-5\sqrt{x+1}$.

14. $8\sqrt{x-1}+3\sqrt{x-1}-8\sqrt{x-1}$.

15. $3a^n-7a^n+15a^n-20a^n$.

16. $2a^nx+3a^nx-6a^nx+5a^nx$.

17. $2a^2b^n+3a^2b^n-5a^2b^n+6a^2b^n$.

18. $5x^5y^5-10x^5y^5+15x^5y^5-20x^5y^5$.

19. $3x^{n+3}+6x^{n+3}-9x^{n+3}+12x^{n+3}$.

20. $7x^{a+6}-14x^{a+6}+42x^{a+6}+63x^{a+6}$.

WRITTEN EXERCISES

1 $7x^{4a}-14x^{4a}-21x^{4a}+28x^{4a}$.

2. $ab^n+3ab^n+5ab^n-6ab^n$.

3. $12x^2y^5+10x^2y^5-3ax^2y^5+ax^2y^5$.

4. $a(b+3)-3a(b+3)+4a(b+3)+2a(b+3)$.

5. $(a+1)(b+3)+(a-1)(b+3)-a(b+3)$.

6. $(a+b)(x-y)+(a-b)(x-y)+b(x-y)$.

7. $(a+3)(x+y)+(a-4)(x+y)+(2-a)(x+y)$.

8. $n(n-1)(n-2)+(n-1)(n-2)$.

9. $n(n-1)+n(n-1)(n-2)=n(n-1)[1+n-2]$
$=n(n-1)(n-1)=n(n-1)^2$.

10. $n(n-1)(n-2)+n(n-1)(n-2)(n-3)$.

11. $n(n-1)(n-2)(n-3)+(n-1)(n-2)(n-3)$.

In the following add the first and second terms, then this result and the last term:

12. $(a-4)(b+3)+(a+3)(b+3)+(2a-1)(b-2)$.

13. $(x+2y)(x-2y)+(x-3y)(x-2y)+(2x-y)(x-y)$.

14. $(3x-a)(y-b)+(3x+a)(y-b)+6x(y+b)$.

15. $(n+1)(x+a)+(1-4n)(x+a)+(2-3n)(x-a)$.

16. $(n-1)(n-3)+2(n-3)+n(n+1)$.

17. $(2a-3)(3a-2)+(a+4)(3a-2)+(a-4)(3a+1)$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

38. Adding Polynomials. In adding polynomials we make use of the associative and commutative laws (§§ 24, 25), in order to group the terms in a convenient manner.

Example. Add $5a^2 - 3ab - 2b^2$ and $3a^2 + 6ab - 4b^2$.

Arranging similar terms in columns, we have

$$\begin{array}{r} 5a^2 - 3ab - 2b^2 \\ \underline{3a^2 + 6ab - 4b^2} \\ 8a^2 + 3ab - 6b^2 \end{array}$$

39. Subtracting Polynomials. In like manner the terms are arranged in subtracting one polynomial from another.

Example. Subtract $4x - 2y + 6z$ from $3x + 6y - 3z$.

$$\begin{array}{r} 3x + 6y - 3z \\ \underline{4x - 2y + 6z} \\ -x + 8y - 9z \end{array}$$

The steps are:

$3x - 4x = -x$; $6y - (-2y) = 8y$; $-3z - (+6z) = -9z$.

Oral Addition of Polynomials. The sum of polynomials may also be found directly without arranging similar terms in columns as is shown in the following example:

Example. Add, $3x^3 + 7x^2 - 4x + 7$; $2x^3 + 8x - 9$; $5x^2 - 3x - 2$ and $5x^3 - 2x^2 + 7x - 3$.

Solution. We notice that the coefficients of terms containing x^3 are 3, 2, and 5. Hence the sum of these terms is $10x^3$. Similarly the sum of the terms containing x^2 is $10x^2$; the sum of the terms containing x is $8x$ and the sum of the remaining terms is -7. Hence the required sum is $10x^3 + 10x^2 + 8x - 7$.

ORAL EXERCISES

Add the following:

1. $5x^2 - 3x + 2$, $-4x^2 + 7x - 3$, $5x - 7$, $9x^2 - 3x$.
2. $2x^4 - 3x^3 - 4x^2 + 2x - 7$, $x^4 + 4x^3 - 7x^2 - 3x + 4$, $4x^3 + 8x^2 - 9$.
3. $9a^3 - 4a + 7a^2 - 3$, $5a^2 + 6a - 9$, $2a^2 - 4 + 5a$, $8a^3 - 3a^2 - 8 + 3a$.

4. Add $8x^3 - 11x - 7x^2$, $2x - 6x^2 + 10$, $-5 + 4x^3 + 9x$, and $13x^2 - 5 - 12x^3$.

5. Add $5a^3 - 2a - 12 - 10a^2$, $14 - 7a + a^2 - 9a^3$, $3a^2 - 13a^3 + 4 - 11a$, and $3 - 7a + 10a^2 + 4a^3$.

WRITTEN EXERCISES

1. Add $37a - 4b - 17c + 15d - 6f - 8h$ and $3c - 31a + 9b - 5d - h - 4f$.

2. Add $11q - 10p - 8n + 3m$, $24m - 17q + 15p - 13n$, $9n - 6m - 4q - 7p - 5n$, and $8q - 4p - 12m + 18n$.

3. From the sum of $9m^3 - 3m^2 + 4m - 7$ and $3m^2 - 4m^3 + 2m + 8$ subtract $4m^3 - 2m^2 - 4 + 8m$.

4. From the sum of $x^4 - ax^3 - a^2x^2 - a^3x + 2a^4$ and $3ax^3 + 7a^2x^2 - 5a^3x + 2a^4$ subtract $3x^4 + ax^3 - 3a^2x^2 + a^3x - a^4$.

5. From the sum of $13a - 15b - 7c - 11d$ and $7a - 6b + 8c + 3d$ subtract the sum of $6d - 5b - 7c + 2a$ and $5c - 10d - 28b + 17a$.

6. Add $(a + b - c)m + (a - b + c)n + (a - b - c)k$,

$(2a - 3b + c)m + (-3a + b + c)n + (a + 2b + 4c)k$,

and $(b - 2c)m + (2a + b - 2c)n + (-2a + 2b + c)k$.

7. From the sum of $ax^3 - bx^2 + cx - d$ and $bx^3 + ax^2 - dx + c$ subtract $(a - b)x^3 + (c - a)x^2 - (b + d)x - d + c$.

8. Add $(m^2 - 2mn + n^2)x^3 + (m^2 + 2mn + n^2)x^2 - (m + n)x + 8$,

$(m^2 + n^2)x^3 + (-2mn)x^2 + (m - n)x - 31$,

$(m^2 + mn + n^2)x^3 + (m^2 - n^2)x^2 - (2m + n)x + 4$.

9. Add $a^n + 2a^{n+1} + a^{n+2}$ and $2a^n - 4a^{n+1} + 5a^{n+2}$ and from this sum subtract $7a^{n+1} - 8a^n + a^{n+2}$.

10. Add $3x^a + 2x^{a-1} + a^{n-2}$ and $5x^a - 7x^{a-1} - 3a^{n-2}$, and from this sum subtract $-8x^a + 15x^{a-1} + 7a^{n-2}$.

REMOVAL OF PARENTHESES

40. Parentheses Preceded by a Plus Sign. Since $a+(+b)=a+b$ and $a+(-b)=a-b$, it follows that parentheses preceded by a plus sign may be removed without changing any sign within:

Thus, $a+(b-c+d-e)=a+b-c+d-e$.

41. Parentheses Preceded by a Minus Sign. Since $a-(+b)=a-b$, and $a-(-b)=a+b$, it follows that parentheses preceded by a minus sign may be removed by changing the sign of each term within.

Thus, $a-(b+c-d+e)=a-b-c+d-e$.

42. Parentheses within Parentheses. In case an expression contains signs of aggregation, one within another, these may be removed one at a time, beginning with the *innermost*, as in the following example:

$$\begin{aligned} & a-\{b+c-[d-e+f-(g-h)]\} \\ &= a-\{b+c-[d-e+f-g+h]\} \\ &= a-\{b+c-d+e-f+g-h\} \\ &= a-b-c+d-e+f-g+h. \end{aligned}$$

43. Such combinations of signs of aggregation may also be removed in order, beginning with the *outermost*, by observing *the number of minus signs which affect each term, and calling the sign of a term + if this number is even, − if this number is odd.*

Thus, in the above example, b and c are each affected by one minus sign, namely the one preceding the brace. Hence we write, $a-b-c$.

d and f are each affected by two minus signs, namely the one before the brace and the one before the bracket, while e is affected by these two and also by the one preceding it. Hence we write, $d-e+f$.

g is affected by the minus signs before the bracket, the brace, and the parenthesis, an *odd* number, while h is affected by these and also by the one preceding it, an *even* number. Hence we write, $-g+h$.

By counting in this manner as we proceed from left to right, we write the result in the final form at once, namely, $a-b-c+d-e+f-g+h$.

WRITTEN EXERCISES

In removing the signs of aggregation in the following, either of the two processes just explained may be used. The second method is shorter and should be easily followed after a little practice.

1. $7 - \{-4 - (4 - [-7]) - (5 - [4 - 5] + 2)\}.$
2. $-[-(7 - \{-4 + 9\} - 13) - (12 - 3 + \overline{-7 + 2})].$
3. $6 - (-3 - [-5 + 4] + \{7 - 3 - \overline{7 - 19}\} + 8).$
4. $5 + [-(-\{-5 - 3 + 11\} - 15) - 3] + 8.$
5. $4x - [3x - y - \{3x - y - (x - \overline{y - x}) + x\} - 3y].$
6. $3x^2 - 2y^2 - (4x^2 - \{3x^2 - \overline{y^2 - 2x^2} - 3y^2\} - y^2 + 4x^2).$
7. $7a - \{3a - [-2a - \overline{a + 3} + a] - \overline{2a - 5}\}.$
8. $l - (-2m - n - \{l - m\}) - (5l - 2n - [-3m + n]).$
9. $2d - [3d + \{2d - (e - 5d)\} - (d + 3e)].$
10. $4y - (-2y - [-3y - \{-y - \overline{y - 1}\} + 2y]).$
11. $3x - [8x - (x - 3) - \{-2x + 6 - 8\overline{x - 1}\}].$
12. $x - (x - \{-4x - [5x - \overline{2x - 5}] - [-x - \overline{x - 3}]\}).$
13. $3x - \{y - [3y + 2z] - (4x - [2y - 3z] - 3\overline{y - 2z}) + 4x\}.$
14. $x - (-x - \{-3x - [x - \overline{2x + 5}] - 4\} - [2x - \overline{x - 3}]).$
15. $a - [5b - \{a - (5c - \overline{2c - b} - 4b) + 2a - (a - \overline{2b + c})\}].$
16. $2(3b - 5a) - 7[a - 6\{2 - 5(a - b)\}].$
17. $-2\{a - 6[a - (b - c)]\} + 6\{b - (c + a)\}.$
18. $-3\{-2[-4(-a)]\} + 5\{-2[-2(-a)]\}.$
19. $-2\{-[-(x - y)]\} + \{-2[-(x - y)]\}.$
20. $a - (b - c) - [a - b - c - 2\{b + c - 3(c - a) - d\}].$
21. $2x - (3y - 4z) - \{2x - [3y + 4z - 3y - (4z + 2x)]\}.$
22. $-2(a - d) - 2[b + c + d - 3\{c + d - 4(d - a)\}].$
23. $-4(a+d) + 4(b-c) - 2[c+d+a-3\{d+a-4(b+c)\}].$
24. $a - 2b - [4a - 6b - \{3a - c + (5a - 2b - \overline{3a - c + 2b})\}].$
25. $a - [-b\{-c(-d + \overline{e - f}) + 2a - c\} + c + d].$

MULTIPLICATION OF MONOMIALS

44. Adding Exponents in Multiplication. In $2^2 \cdot 2^3 = (2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^{2+3} = 2^5$, we see that the exponents of the factors are added to obtain the exponent of the product.

Similarly, we have in general,

$$b^k \cdot b^n = b^{k+n}.$$

For by the definition of a positive integral exponent,

$$b^k = b \cdot b \cdot b \cdots \text{to } k \text{ factors}$$

and $$b^n = b \cdot b \cdot b \cdots \text{to } n \text{ factors}.$$

Then, $$\begin{aligned} b^k \cdot b^n &= (b \cdot b \cdots \text{to } k \text{ factors})(b \cdot b \cdots \text{to } n \text{ factors}) \\ &= b \cdot b \cdot b \cdots \text{to } k + n \text{ factors}. \end{aligned}$$

Hence, $$b^k \cdot b^n = b^{k+n}.$$

That is: *The product of two powers of the same base is a power of that base whose exponent is the sum of the exponents of the common base.*

45. Multiplying Monomials. In finding the product of two monomials, the factors may be *arranged* and *grouped* in any manner, according to the commutative and associative laws (§§ 26, 27).

E.g. $$\begin{aligned} (3ab^2) \times (5a^2b^3) &= 3ab^2 \cdot 5a^2b^3 \\ &= 3 \cdot 5 \cdot a \cdot a^2 \cdot b^2 \cdot b^3 \\ &= (3 \cdot 5)(a \cdot a^2)(b^2 \cdot b^3) \\ &= 15a^3b^5. \end{aligned}$$

The factors in the product are arranged so as to associate those consisting of Arabic figures and also those which are powers of the same base.

Positive and Negative Products. It is readily seen that a product is negative when it contains an *odd* number of *negative* factors; otherwise it is positive.

For by the commutative and associative laws of factors the negative factors may be grouped in *pairs*, each pair giving a *positive* product. If the number of negative factors is odd, there will be just one remaining, after the others are paired and multiplied together, and this makes the final product negative.

ORAL EXERCISES

Find the products of the following:

1. $3x^2, 2x^3$.
2. $2a^2, 4a^4$.
3. $3y^3, 3y^3$.
4. $4a^5, 2a^4$.
5. $5a^4, 4a^5$.
6. $3a^4, 4a^3$.
7. $2a^5, 2a^6$.
8. x^n, x^n.
9. $2x^n, 3x^2$.
10. $3x^n, 3x^n$.
11. x^n, x^{n+1}.
12. x^{2n}, x^n.
13. $x^{2n}, 2x^{2n}$.
14. x^{n+1}, x^n.
15. x^{n+1}, x^{n+2}.
16. x^{n-1}, x^{n+1}.
17. x^{n-1}, x^{n+3}.
18. x^{1-n}, x^{1+n}.
19. $5x^{2n}, 2x^{3n}$.
20. $6x^{3n+1}, 2x^{n+3}$.
21. $7a^{2n+1}, 3a^{n+2}$.
22. $6x^{a+3}, 2x^{2a-2}$.
23. $5a^{2n-7}, 3a^{n+8}$.
24. $4a^{4n+2}, 4a^{2n-6}$.
25. $3y^{4a}, 7y^{5a+2}$.
26. $9x^{3b}, 2x^{b+a}$.
27. $4a^{2n-a}, 3a^n$.
28. $2x^{4a-n}, 2x^{a+n}$.
29. $4x^{7n-2a}, 5x^{3n+3a}$.
30. $8x^{3a-8n}, 2x^{4a+9n}$.

WRITTEN EXERCISES

Find the products of the following:

1. $2^3 \cdot 3^4, 2^7 \cdot 3^2$.
2. $2a^2b, 3ab^2, a^3b^3$.
3. $2x^2y^3, 5x^3y^2, 2x^4y$.
4. $5xy, 2x^3y, 4xy^5, x^2y^2$.
5. $3a^5bc, ab^2c, a^2bc^4, 4ab^5c$.
6. $x^n, x^{n-1}, x^{n+1}, 2x^n$.
7. $x^{m+n-1}, x^{m-n+1}, x^{2m}$.
8. a^x, a^{3x-y}, a^{y-3x}.
9. $a^nb^m, a^{2n}b^{3m}, a^{1-3n}b^{2-4m}$.
10. $4ab^m, 2a^3b^n, 3a^6b^{2-m-n}$.
11. $2x^my^{m+n}, 3x^{m-1}y^{2n-m+2}$.
12. $a^{d-2c+2}b^{m-3n}, a^{2c-d-1}b^{2-m+3n}$.
13. $3x^{a+3b}, 2x^{a-2b}y^{c-3}, 2x^{4-2a-b}y^{2c+3}$.
14. $a^{2x-3}b^{y+1}, a^{x+3}b^{y-1}, 3a^3b^2$.
15. $3x^my^n, 2x^ny^m$.
16. $x^{a+1}, x^{b-1}y^c, y^{1-c}$.
17. $7x^{3a-4}, 3x^{6-4a}, 2x^{1-a}$.
18. $2a^{m-n}, 3a^{2m+n}, 4a^{1-m}$.
19. $3^{2-5m+3n} \cdot 2^{4a-3b}, 3^{2-3n+6m} \cdot 2^{5+3b+5a}$.
20. $(1+a)^{7-3b+a} \cdot (1-a)^{2+a-b}, (1-a)^{b-a-1} \cdot (1+a)^{3b-a-6}$.
21. $(x-3)^{4n-5a+2} \cdot (x+3)^{2a-3n-1}, (x-3)^{6a-4n-2} \cdot (x+3)^{4n-3a+1}$.
22. $(a+b)^{a+b} \cdot (a-b)^{a-b}, (a+b)^{a-b} \cdot (a-b)^{a+b}$.
23. $(x+y+1)^{x+y+1}, (x-y+1)^4, (x+y+1)^{x-y-1}$.

DIVISION OF MONOMIALS

46. Subtracting Exponents in Division. We know that $x^6 \div x^4 = x^2$, because $x^2 \times x^4 = x^6$. That is, the quotient times the divisor equals the dividend. But $x^2 = x^{6-4}$. Therefore the exponent of the quotient is equal to *the exponent of the dividend minus that of the divisor.*

In general, if a is any number, and m and k are any positive integers, of which m is the greater, then

$$a^m \div a^k = a^{m-k}.$$

For, since k and $m - k$ are both positive integers, we have $a^k \times a^{m-k} = a^{k+m-k} = a^m$. That is, a^{m-k} is the number which multiplied by a^k gives a product a^m.

Hence we have the

Rule. *The quotient of two powers of the same base is a power of that base whose exponent is the exponent of the dividend minus that of the divisor.*

Negative and Zero Exponents. Under the proper interpretation of negative numbers used as exponents, this principle also holds when $m < k$. This is considered in detail in Chapter XII. We remark here that in case $m = k$, the dividend and the divisor are equal, and the quotient is unity. Hence,

$$a^m \div a^m = a^{m-m} = a^0 = 1.$$

47. Dividing Monomials. In dividing monomials, we make use of the commutative and associative laws in order to group together as far as possible powers having a common base.

Example 1. Divide $2^4\, x^5y^6$ by $2^2\, x^2y^4$.

Solution. We divide 2^4 by 2^2, x^5 by x^2, and y^6 by y^4, and multiply the quotients.

Thus, $$2^4\, x^5y^6 \div 2^2\, x^2y^4 = 2^{4-2}\, x^{5-2}y^{6-4} = 2^2\, x^3y^2.$$

Example 2. Divide $6\, b^4c^2x^3$ by $4\, bc^2x^2$.

Solution. $$6\, b^4c^2x^3 \div 4\, bc^2x^2 = \frac{6}{4} \cdot \frac{b^4}{b} \cdot \frac{c^2}{c^2} \cdot \frac{x^3}{x^2} = \frac{3}{2} b^3x.$$

ORAL EXERCISES

Find the quotients of the following:

1. $4x^4 \div 2x^2$.
2. $6x^5 \div 3x^2$.
3. $8x^6 \div 2x^5$.
4. $8x^5 \div x^4$.
5. $10x^7 \div x^5$.
6. $-24ax^2 \div 4ax$.
7. $15x^{2n} \div 3x^n$.
8. $15x^{3n} \div 5x^{2n}$.
9. $-24x^{4a} \div 12x^{2a}$.
10. $16x^{6n} \div (-8x^{2n})$.
11. $14x^{5n} \div 7x^{4n}$.
12. $-18a^{2n+1} \div (-6a^n)$.
13. $32a^{3n+2} \div 16a^{n+2}$.
14. $-6x^{4n+3} \div x^{3n+2}$.
15. $10a^{5n+3} \div 2a^{2n+1}$.
16. $12a^{4n+7} \div (-4a^{n+6})$.
17. $a^{a-b} \div a^{b-a}$.
18. $x^{3n+2} \div x^{2n+1}$.
19. $4x^{5n+4} \div 2x^{5n-5}$.
20. $10x^{4n+5} \div 5x^{4n-6}$.
21. $12x^{7n+3} \div 6x^{7n+2}$.
22. $x^{x+y} \div x^{x-y}$.
23. $24a^{2n+3} \div 8a^{n+2}$.
24. $32a^{3n-2} \div 8a^{2n-4}$.

WRITTEN EXERCISES

Divide:

1. $2^4 \cdot 3^7 \cdot 5^2$ by $2^3 \cdot 3^4 \cdot 5$.
2. $3^7 \cdot 7^4 \cdot 13^5$ by $3^5 \cdot 7^2 \cdot 13^2$.
3. $3x^7y^2z$ by $2x^3yz$.
4. $5a^5b^7c^8$ by $5a^4b^7c^4d^2$.
5. $x^{2n}y^mz^{3m}$ by $x^ny^mz^m$.
6. $a^{3n-5}y^{2n+3}$ by $a^{n+6}y^{2n+1}$.
7. $a^{c+3d+2}b^{d-2c+6}$ by a^{c+2d-4}.
8. $3^{a+2b-7} \cdot 5^{3b-2a+4}$ by $3^{b+a-8} \cdot 5^{2b-2a+3}$.
9. $a^{3+2m\ \ 3n}b^5c^{7\ \ n}$ by $a^{2+m-4n}b^4c^{7-n}$.
10. $x^{4a+1}y^{c-a}z^{3a+c}$ by $z^{-c+3a}y^{a-c}x^{3a-2c}$.
11. $2^{3a-4} \cdot 3^{3b+6}$ by $2^{2a-5} \cdot 3^{2b+7}$.
12. $(x-2)^{3m+1} \cdot (x+2)^{2m+2}$ by $(x+2)^{1+2m} \cdot (x-2)^{1+2m}$.
13. $(x-y)^{5b-1} \cdot (x+y)^{-2b+2}$ by $(x-y)^{-2+5b} \cdot (x+y)^{-3-2b}$.
14. $(a^2-b^2)^{3+4k} \cdot (a^2+b^2)^{1-5b}$ by $(a^2-b^2)^4 \cdot (a^2+b^2)^{-2+2b}$.

MULTIPLICATION OF POLYNOMIALS

48. Rule. *The product of two polynomials is equal to the sum of the products obtained by multiplying each term of one polynomial by every term of the other.*

This follows from the distributive law of multiplication, § 28. For by this law,

$$\begin{aligned}(m+n+k)(a+b+c) &= m(a+b+c)\\ &+ n(a+b+c)\\ &+ k(a+b+c).\end{aligned}$$

Applying the same law to each part, we have the product,

$$ma + mb + mc + na + nb + nc + ka + kb + kc.$$

WRITTEN EXERCISES

Find the following indicated products:

1. $(a+b)(a^2+3ab+2b^2)$.
2. $(x-y)(3x^3-2x^2y-4xy^2+2y^3)$.
3. $(x+y)(x^2-2xy+y^2)$.
4. $(x^2+5x+6)(x^2-3x+6)$.
5. $(3x^2-2x+6)(2x^2-3x+6)$.
6. $(2x+3y)(3x+2y)(x-y)$.
7. $(a^2+2ab+b^2)(a^2-2ab+b^2)$.
8. $(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4)$.
9. $(x^2+y^2)(x^2-y^2)(x^4-y^4)$.
10. $(x^3+y^3)(x^3-y^3)(x^6+y^6)$.
11. $(16x^4+4x^2+1)(16x^4-4x^2+1)$.
12. $(x^3-4x^2+5x-1)(x^2+7x-3)$.
13. $(32x^5-16x^4+8x^3-4x^2+2x-1)(2x+1)$.
14. $(a^2b^2-3ab+6)(3+4ab-a^2b^2)$.
15. $(2x^2y^4-3x^3y+2xy^2)(xy^3-x^2y+x^3y^2)$.
16. $(x^3-y^3)(x^6+x^3y^3+y^6)$.
17. $(x^4+y^4)(x^8-x^4y^4+y^8)$.

49. Special Products. There are many special products which should be known at sight.

Example 1. $(a+b)(a-b)=a^2-b^2$.

The product of the sum and difference of two numbers equals the difference of their squares.

Example 2. $(a+b)^2=a^2+2ab+b^2$,
$(a-b)^2=a^2-2ab+b^2$.

The square of the sum of two numbers equals the sum of their squares plus twice their product.

The square of the difference of two numbers equals the sum of their squares minus twice their product.

Example 3. $(x+a)(x+b)=x^2+(a+b)x+ab$.

The product of two binomials having a common term equals the square of the common term plus the sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

Example 4. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.

The square of a polynomial equals the sum of the squares of the terms plus twice the product of each term taken with each term that succeeds it.

ORAL EXERCISES

1. $(a+b)(a-b)$.
2. $(x-2y)(x+2y)$.
3. $(3x+2y)(3x-2y)$.
4. $(x+y)^2$.
5. $(x-y)^2$.
6. $(a^2+b^2)^2$.
7. $(a^2-b^2)^2$.
8. $(2x+3y)^2$.
9. $(3x-5y)^2$.
10. $(a+b+c)^2$.
11. $(a+b-c)^2$.
12. $(a-b+c)^2$.
13. $(a+b)(a-b)(a^2+b^2)$.
14. $(x^4-y^4)(x^4+y^4)$.
15. $(x^2+x+1)(x^2+x-1)$.
16. $(x+1)(x+2)$.
17. $(x-2)(x-1)$.
18. $(x+3)(x-2)$.
19. $(x-3)(x+2)$.
20. $(x+a)(x+b)$.
21. $(x-a)(x+b)$.
22. $(x+a)(x-b)$.

MULTIPLICATION OF POLYNOMIA

48. **Rule.** *The product of two polynor* the distributive
sum of the products obtained by mu d by a monomial by
one polynomial by every term of nomial.

This follows from the distributive nomial by separating the
this law, is the product of the divisor

$(m + n + k)(a$ se monomial factors is a part of
nstituting the whole quotient. The

Applying the same found one by one as the work pro-
$ma +$ by an example.

Find *Example 1.*

$$\begin{array}{lrl|ll}
 & a^4 + a^3 - 4a^2 + 5a - 3 & & a^2 + 2a - 3, & \text{Divisor.} \\ \hline
\textit{1st part of dividend:} & \underline{a^4 + 2a^3 - 3a^2} & & a^2 - a + 1, & \text{Quotient.} \\
 & -a^3 - a^2 + 5a - 3 & & & \\
\textit{2d part of dividend:} & \underline{-a^3 - 2a^2 + 3a} & & & \\
 & a^2 + 2a - 3 & & & \\
\textit{3d part of dividend:} & \underline{a^2 + 2a - 3} & & & \\
 & 0 & & &
\end{array}$$

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

Dividend = Quotient × divisor + Remainder.

Example 2. Divide $x^3 - 7x^2 + 2x^4 + 6 + 5x$ by $2x - 1 + x^2$.

Solution. We first arrange both dividend and divisor according to the descending powers of x: [divisor.

$$\begin{array}{lr|l}
\text{Dividend or product} = & 2x^4 + x^3 - 7x^2 + 5x + 6 & \underline{x^2 + 2x - 1,} \\
\text{1st product, } 2x^2(x^2 + 2x - 1) = & \underline{2x^4 + 4x^3 - 2x^2} & 2x^2 - 3x + 1, \\
\text{Dividend minus 1st product} = & -3x^3 - 5x^2 + 5x + 6 & \text{[quotient.} \\
\text{2d product, } -3x(x^2 + 2x - 1) = & \underline{-3x^3 - 6x^2 + 3x} & \\
\text{Dividend minus 1st and 2d products} = & +x^2 + 2x + 6 & \\
\text{3d product, } 1 \cdot (x^2 + 2x - 1) = & \underline{x^2 + 2x - 1} & \\
\text{Dividend minus 1st, 2d, and 3d products} = & 7 = \text{remainder.} &
\end{array}$$

From a consideration of the preceding examples the
s of dividing by a polynomial is described as follows:

range the terms of dividend and divisor according to
(or ascending) powers of some common letter. As the
eeds, arrange each remainder in the same way.

he first term of the dividend by the first term of the
esult is the first term of the quotient.

ly the divisor by the first term of the quotient and
ct the product from the dividend.

4. *Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.*

5. *Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor.*

WRITTEN EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8+x^4y^4+y^8$ by $x^4-x^2y^2+y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3+x^2y+xy^2+y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10}-a^5b^5+b^{10}$ by a^2-ab+b^2.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b + 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

DIVISION OF POLYNOMIALS

50. Dividing by a Monomial. According to the distributive law of division, § 29, a *polynomial is divided by a monomial* by dividing each term separately by the monomial.

A **polynomial is divided by a polynomial** by separating the dividend into parts, each of which is the product of the divisor and a monomial. Each of these monomial factors is a part of the quotient, their sum constituting the whole quotient. The parts of the dividend are found one by one as the work proceeds. This is best shown by an example.

Example 1.

$$\begin{array}{lr|ll}
\text{Dividend,} & a^4 + a^3 - 4a^2 + 5a - 3 & a^2 + 2a - 3, & \text{Divisor.} \\ \cline{3-4}
\textbf{1st part of dividend:} & \underline{a^4 + 2a^3 - 3a^2 \phantom{{}+5a-3}} & a^2 - a + 1, & \textbf{Quotient.} \\
 & -a^3 - a^2 \phantom{{}+3a} + 5a - 3 & & \\
\text{2d part of dividend:} & \underline{-a^3 - 2a^2 + 3a \phantom{{}-3}} & & \\
 & a^2 + 2a - 3 & & \\
\text{3d part of dividend:} & \underline{a^2 + 2a - 3} & & \\
 & 0 & &
\end{array}$$

The three parts of the dividend are the products of the divisor and the three terms of the quotient. If after the successive subtraction of these parts of the dividend the remainder is zero, the division is exact. In case the division is not exact, there is a final remainder such that

Dividend = Quotient × divisor + Remainder.

Example 2. Divide $x^3 - 7x^2 + 2x^4 + 6 + 5x$ by $2x - 1 + x^2$.

Solution. We first arrange both dividend and divisor according to the descending powers of x: [divisor.

$$\begin{array}{lr|l}
\text{Dividend or product} = & 2x^4 + x^3 - 7x^2 + 5x + 6 & x^2 + 2x - 1, \\ \cline{3-3}
\quad\text{1st product, } 2x^2(x^2 + 2x - 1) = & \underline{2x^4 + 4x^3 - 2x^2 \phantom{{}+5x+6}} & 2x^2 - 3x + 1, \\ \cline{2-2}
\text{Dividend minus 1st product} = & -3x^3 - 5x^2 + 5x + 6 & \text{[quotient.} \\
\quad\text{2d product, } -3x(x^2 + 2x - 1) = & \underline{-3x^3 - 6x^2 + 3x \phantom{{}+6}} & \\
\text{Dividend minus 1st and 2d products} = & +x^2 + 2x + 6 & \\
\quad\text{3d product, } 1 \cdot (x^2 + 2x - 1) = & \underline{x^2 + 2x - 1} & \\
\text{Dividend minus 1st, 2d, and 3d products} = & 7 & = \text{remainder.}
\end{array}$$

51. From a consideration of the preceding examples the process of dividing by a polynomial is described as follows:

1. *Arrange the terms of dividend and divisor according to descending (or ascending) powers of some common letter. As the division proceeds, arrange each remainder in the same way.*

2. *Divide the first term of the dividend by the first term of the divisor. This result is the first term of the quotient.*

3. *Multiply the divisor by the first term of the quotient and subtract the product from the dividend.*

4. *Divide the first term of this remainder by the first term of the divisor, obtaining the second term of the quotient. Multiply the divisor by the second term of the quotient and subtract, obtaining a second remainder.*

5. *Continue in this manner until the last remainder is zero, or until a remainder is found whose first term does not contain as a factor the first term of the divisor.*

WRITTEN EXERCISES

Divide:

1. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ by $x^2 + 2xy + y^2$.
2. $x^8 + x^4y^4 + y^8$ by $x^4 - x^2y^2 + y^4$.
3. $x^5 - y^5$ by $x - y$.
4. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
5. $x^9 + y^9$ by $x^2 - xy + y^2$.
6. $x^8 - y^8$ by $x^2 - y^2$.
7. $a^6 + b^6$ by $a^2 + b^2$.
8. $x^{6a} - y^{6b}$ by $x^{2a} - y^{2b}$.
9. $a^{10} - a^5b^5 + b^{10}$ by $a^2 - ab + b^2$.
10. $2x^4 - 3x^3b + 6x^2b^2 - xb^3 + 6b^4$ by $x^2 - 2xb + 3b^2$.
11. $2x^6 - 5x^5 + 6x^4 - 6x^3 + 6x^2 - 4x + 1$ by $x^4 - x^3 + x^2 - x + 1$.
12. $26a^3b^3 + a^6 + 6b^6 - 5a^5b + 17ab^5 - 2a^4b^2 - a^2b^4$ by $a^2 - 3b^2 - 2ab$.
13. $x^4 + 2x^3 - 7x^2 - 8x + 12$ by $x^2 - 3x + 2$.
14. $4b^2 + 4ab + a^2 - 12bc - 6ac + 9c^2$ by $2b + a - 3c$.
15. $x^4 + 4xy^3 - 4xyz + 3y^4 + 2y^2z - z^2$ by $x^2 - 2xy + 3y^2 - z$.
16. $a^2b^2c + 3a^2b^3 - 3abc^3 - a^2c^3 + b^5 - 4b^3c^2 + 3ab^3c + 3bc^4 - 3a^2bc^2$ by $b^2 - c^2$.

52. Division by Detached Coefficients. The work of dividing by a polynomial may be shortened by omitting the letters and writing only the coefficients.

Example. Divide $2x^4 + x^3 - 7x^2 + 5x + 6$ by $x^2 + 2x - 1$.

Writing coefficients only,

$$\begin{array}{l|l}
2+1-7+5+6 & 1+2-1 \\ \cline{2-2}
\underline{2+4-2} & 2-3+1 \\
\multicolumn{1}{l}{\quad -3-5+5+6} & \\
\multicolumn{1}{l}{\quad \underline{-3-6+3}} & \\
\multicolumn{1}{l}{\qquad +1+2+6} & \\
\multicolumn{1}{l}{\qquad \underline{+1+2-1}} & \\
\multicolumn{1}{l}{\qquad\qquad\quad 7} &
\end{array}$$

Since dividend and divisor are arranged in descending powers of x, the quotient will be so arranged also. Since the highest power of x in the quotient is x^2 the result is $2x^2 - 3x + 1$.

This example is worked without detached coefficients on page 22.

If any terms are lacking in the dividend or divisor, zero must be written as the coefficient of each such term, as in the following example.

Example. Divide $x^4 + x^2 + 1$ by $x^2 - x + 1$.

Since the third and first powers are lacking in the dividend, we write zero in place of each.

$$\begin{array}{l|l}
1+0+1+0+1 & 1-1+1 \\ \cline{2-2}
\underline{1-1+1} & 1+1+1 \\
\multicolumn{1}{l}{\quad +1+0+0+1} & \\
\multicolumn{1}{l}{\quad \underline{+1-1+1}} & \\
\multicolumn{1}{l}{\qquad +1-1+1} & \\
\multicolumn{1}{l}{\qquad \underline{+1-1+1}} &
\end{array}$$

Hence, the quotient starts with $x^4 \div x^2 = x^2$, and is $x^2 + x + 1$.

WRITTEN EXERCISES

Divide the following by detached coefficients:

1. $x^4 + x^3 - 12x^2 + 14x - 4$ by $x^2 - 3x + 2$.
2. $2x^4 + 11x^3 - 26x^2 + 16x - 3$ by $x^2 + 7x - 3$.
3. $x^5 - x^4 - 27x^3 + 10x^2 - 30x - 200$ by $x^2 - 4x - 10$.
4. $x^5 - 1$ by $x - 1$.

53. Synthetic Division. When the divisor is of the form $x - a$, the work may be contracted still further.

Example 1. Divide $x^4 - 3x^3 + 3x^2 - 2x + 1$ by $x - 1$.

FIRST SOLUTION

$$\begin{array}{rrrrr|l}
1 & -3 & +3 & -2 & +1 & 1-1 \\ \cline{6-6}
1 & -1 & & & & 1-2+1-1 \\ \cline{1-2}
 & -2 & +3 & & & \\
 & -2 & +2 & & & \\ \cline{2-3}
 & & 1 & -2 & & \\
 & & 1 & -1 & & \\ \cline{3-4}
 & & & -1 & +1 & \\
 & & & -1 & +1 & \\ \cline{4-5}
 & & & & 0 &
\end{array}$$

SECOND SOLUTION

$$\begin{array}{rrrrr|l}
1 & -3 & +3 & -2 & +1 & 1-1 \\ \cline{6-6}
 & -1 & & & & 1-2+1-1 \\ \cline{1-2}
 & -2 & +3 & & & \\
 & & +2 & & & \\ \cline{2-3}
 & & 1 & -2 & & \\
 & & & -1 & & \\ \cline{3-4}
 & & & -1 & +1 & \\
 & & & & 1 & \\ \cline{4-5}
 & & & & 0 &
\end{array}$$

In the second solution the first term in each partial product is omitted.

In practice the first term of the divisor is also omitted, and the figures below the dividend are moved up as shown below. The sign of the second term of the divisor is changed, and the partial products are *added* instead of *subtracted*.

$$\begin{array}{rrrrr|l}
1 & -3 & +3 & -2 & +1 & +1 \\ \cline{6-6}
 & +1 & -2 & +1 & -1 & \\ \cline{1-5}
 & -2 & +1 & -1 & +0 &
\end{array}$$

Multiply the first coefficient of the dividend by $+1$, and add to the second coefficient. Multiply the sum, -2, by $+1$, and add to the third coefficient. Multiply the sum, $+1$, by $+1$, and add to the fourth coefficient. Multiply the sum, -1, by $+1$, and add to the last coefficient.

The coefficient of the highest power in the quotient is the same as that of the highest power of the dividend. The remaining coefficients of the quotient are the sums below the line in the solution. The last sum to the right is the remainder.

This method of dividing is called synthetic division.

Example 2. Divide $x^5 - 3x^4 - 4x^2 - 7x + 9$ by $x - 2$.

$$\begin{array}{rrrrrr|l}
1 & -3 & +0 & -4 & -7 & +9 & +2 \\ \cline{7-7}
 & +2 & -2 & -4 & -16 & -46 & \\ \cline{1-6}
 & -1 & -2 & -8 & -23 & -37 &
\end{array}$$

The quotient is $x^4 - x^3 - 2x^2 - 8x - 23$, and the remainder is -37.

WRITTEN EXERCISES

Divide the following, using synthetic division.

1. $x^3 + 4x^2 + x - 6$ by $x - 1$.
2. $x^3 - 8x^2 + 75$ by $x - 5$.
3. $x^4 - 3x^2 + 4x - 66$ by $x - 3$.
4. $x^7 - 1$ by $x - 1$.
5. $x^6 - 1$ by $x + 1$.
6. $x^4 + x^3 - 12x^2 + 14x - 4$ by $x - 2$.
7. $x^4 - 6x^3 + 2x^2 - 3x + 6$ by $x - 1$.
8. $x^4 - 3x^3 + 7x^2 - 4x + 3$ by $x - 4$.

54. Special Quotients. There are many special quotients which should be known at sight. The test in every case is

Quotient × *Divisor* = *Dividend.*

Example 1. $(a^2 - b^2) \div (a - b) = a + b$, and $(a^2 - b^2) \div (a + b) = a - b$.

Example 2. $(a^2 + 2ab + b^2) \div (a + b) = a + b$, and $(a^2 - 2ab + b^2) \div (a - b) = a - b$.

Example 3. $(x^2 + 5x + 6) \div (x + 2) = x + 3$, and $(x^2 + 5x + 6) \div (x + 3) = x + 2$.

ORAL EXERCISES

1. $(a^2 - b^2) \div (a + b)$.
2. $(a^2 - b^2) \div (a - b)$.
3. $(a^2 + 2ab + b^2) \div (a + b)$.
4. $(a^2 - 2ab + b^2) \div (a - b)$.
5. $(x^4 - y^4) \div (x^2 - y^2)$.
6. $(x^4 - y^4) \div (x^2 + y^2)$.
7. $(x^6 + y^6) \div (x^2 + y^2)$.
8. $(x^6 - y^6) \div (x^3 - y^3)$.
9. $(x^2 + 3x + 2) \div (x + 1)$.
10. $(x^2 + 3x + 2) \div (x + 2)$.
11. $(x^2 - 3x + 2) \div (x - 1)$.
12. $(x^2 - 3x + 2)(\div x - 2)$.
13. $(x^2 + x - 2) \div (x + 2)$.
14. $(x^2 - x - 2) \div (x - 2)$.
15. $(4x^2 - 9y^2) \div (2x + 3y)$.
16. $(x^2 + 8x + 16) \div (x + 4)$.
17. $(x^2 + 7x + 12) \div (x + 4)$.
18. $(625 - 16a^2) \div (25 - 4a)$.

CHAPTER III

FACTORING

55. Rational Integral Expressions. An algebraic expression which is rational in a certain letter (see § 15), and in which no denominator contains the letter, is said to be *rational and integral in that letter.*

56. Prime Factors. A rational and integral expression is said to be *completely factored* when it cannot be further resolved into factors which are rational and integral. Such factors are called *prime factors.*

MONOMIAL FACTORS

57. A monomial factor of an expression is evident at sight, and its removal should be the first step in every case.

That is, $$ad + bd + cd = (a + b + c)d.$$

E.g. $$4\,ax^2 + 2\,a^2x = 2\,ax(2\,x + a).$$

ORAL EXERCISES

Give the factors of each of the following:

1. $a^3 + a^2 + a.$
2. $2\,x^3 + 3\,x^2 + x.$
3. $a^2b + ab^2.$
4. $3\,a^4 + 6\,a^2 + 9\,a.$
5. $4\,x^3 + 8\,x^4 - 12\,x^5.$
6. $7\,x^2y^3 + 21\,x^4y^2.$
7. $2\,a^3b^2 + 4\,a^2b^3.$
8. $ab + ac + ad.$
9. $3\,c + 6\,b + 9\,a.$
10. $4\,a + 8\,b + 12\,c.$
11. $3\,a^2 + 9\,ab + 4\,a.$
12. $4\,x^2y^2 + 12\,x^4y.$
13. $6\,a^4x^2 - 12\,a^5x^3 + 18\,x^5.$
14. $7\,ab^3 + 14\,a^2b^2 + 21\,a^3b^3.$
15. $5\,x^3y^2 - 10\,x^2y^2 + 15\,x^2y^3.$
16. $4\,a^4x^2 - 6\,a^5x^2 + 8\,a^2x^3.$
17. $10\,x^7 - 15\,x^8 + 20\,x^9.$
18. $13\,axy + 26\,a^2x^2y^2.$

FACTORS OF BINOMIALS

58. The Difference of Two Squares.

$$a^2 - b^2 = (a - b)(a + b).$$

E.g. $4x^2 - 9z^4 = (2x + 3z^2)(2x - 3z^2)$.

ORAL EXERCISES

Factor the following:

1. $a^2 - b^2$.
2. $4x^2 - y^2$.
3. $9x^2 - y^2$.
4. $x^2 - 4y^2$.
5. $x^2 - 9y^2$.
6. $4x^2 - 4y^2$.
7. $4x^2 - 9y^2$.
8. $9x^2 - 4y^2$.
9. $x^4 - y^4$.
10. $x^2 - y^4$.
11. $x^4 - y^2$.
12. $x^6 - y^6$.
13. $x^2 - y^6$.
14. $x^6 - y^2$.
15. $x^4 - y^6$.
16. $x^6 - y^4$.
17. $25 - 4x^2$.
18. $25a^2 - 4b^2$.

59. The Difference of Two Cubes.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

E.g.
$$\begin{aligned} 8x^3 - 27y^3 &= (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2] \\ &= (2x - 3y)(4x^2 + 6xy + 9y^2). \end{aligned}$$

ORAL EXERCISES

Factor the following:

1. $a^3 - b^3$.
2. $1 - b^3$.
3. $a^3 - 1$.
4. $a^3 - 8$.
5. $8 - b^3$.
6. $a^6 - b^3$.
7. $a^6 - 1$.
8. $1 - a^6$.
9. $a^3 - b^6$.

60. The Sum of Two Cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

E.g.
$$\begin{aligned} 27x^3 + 64y^3 &= (3x + 4y)[(3x)^2 - (3x)(4y) + (4y)^2] \\ &= (3x + 4y)(9x^2 - 12xy + 16y^2). \end{aligned}$$

ORAL EXERCISES

Factor the following:

1. $a^3 + b^3$.
2. $a^3 + 1$.
3. $1 + b^3$.
4. $x^3 + b^6$.
5. $a + b$.
6. $8 + b^3$.
7. $a^3 + 8$.
8. $a^6 + 8$.
9. $8 + a^6$.

FACTORS OF TRINOMIALS

61. Trinomial Squares.

$$a^2 + 2ab + b^2 = (a+b)^2 = (a+b)(a+b),$$
$$a^2 - 2ab + b^2 = (a-b)^2 = (a-b)(a-b).$$

ORAL EXERCISES

Factor the following:

1. $x^2 + 2xy + y^2$.
2. $x^2 - 2xy + y^2$.
3. $x^2 + 4x + 4$.
4. $x^2 - 4x + 4$.
5. $x^2 + 4xy + 4y^2$.
6. $x^2 - 4xy + 4y^2$.
7. $4x^2 + 4x + 1$.
8. $4x^2 - 4x + 1$.
9. $4x^2 - 4xy + y^2$.
10. $4x^2 + 4xy + y^2$.
11. $4a^2 + 12ab + 9b^2$.
12. $4a^2 - 12ab + 9b^2$.
13. $a^2 + 6ab + 9b^2$.
14. $x^2 + 8xy + 16y^2$.
15. $9x^2 + 24xy + 16y^2$.
16. $9x^2 - 24xy + 16y^2$.

62. Trinomials of the Form $x^2 + px + q$.

$$x^2 + (a+b)x + ab = (x+a)(x+b).$$

E.g. $x^2 + 3x - 10 = (x+5)(x-2)$.

A trinomial of this form has two binomial factors, $x + a$ and $x + b$, if two numbers a and b can be found whose product is q, and whose algebraic sum is p.

ORAL EXERCISES

Factor the following:

1. $x^2 + 3x + 2$.
2. $x^2 - 3x + 2$.
3. $x^2 + 5x + 6$.
4. $x^2 - x - 2$.
5. $x^2 + x - 2$.
6. $x^2 - x - 6$.
7. $x^2 + 2x - 15$.
8. $x^2 + x - 6$.
9. $x^2 - 2x - 15$.
10. $x^2 + 8x + 15$.
11. $x^2 - 8x + 15$.
12. $x^2 - 5x - 6$.
13. $x^2 + 5x - 6$.
14. $x^2 + 7x + 12$.
15. $x^2 - 9x + 20$.
16. $x^2 - 5x - 24$.
17. $x^2 + 6x - 16$.
18. $x^2 - 7x - 30$.

63. Trinomials of the Form $mx^2 + nx + r$.

E.g. $6x^2 + 7x - 20 = (3x - 4)(2x + 5)$.

A trinomial of this form has two binomial factors of the type $ax + b$ and $cx + d$, if four numbers, a, b, c, d, can be found, such that $ac = m$, $bd = r$, and $ad + bc = n$.

64. Trinomials which Reduce to the Difference of Two Squares.

E.g.
$$x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2 + y^2)^2 - x^2y^2.$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

In this case x^2y^2 is both added to and subtracted from the expression, whereby it becomes the difference of two squares.

Example.

$$4a^8 - 16a^4b^4 + 9b^8 = 4a^8 - 12a^4b^4 + 9b^8 - 4a^4b^4.$$
$$= (2a^4 - 3b^4)^2 - 4a^4b^4.$$
$$= (2a^4 - 3b^4 + 2a^2b^2)(2a^4 - 3b^4 - 2a^2b^2).$$

WRITTEN EXERCISES

Factor the following:

1. $a^3 + b^3$.
2. $a^3 - b^3$.
3. $(a + b)^3 + c^3$.
4. $(a + b)^3 - c^3$.
5. $7ax^2 - 56a^4x^5$.
6. $a^5 - ab^4$.
7. $121x^7 - 4xy^4$.
8. $\frac{1}{8}a^3 + \frac{1}{125}b^3$.
9. $\frac{1}{8}r^3 - \frac{9}{32}rs^2$.
10. $8r^4 - 27r$.
11. $(a + b)^2 - c^2$.
12. $c^2 - (a - b)^2$.
13. $5c^2 + 7cd - 6d^2$.
14. $x^4 - 3x^2y^2 + y^4$.
15. $4x^2 - 12xy + 9y^2$.
16. $x^2 + 11xz + 30z^2$.
17. $6x^2 - 5xy - 6y^2$.
18. $3a^2x^2y^4 - 69a^2xy^2 + 336a^2$.
19. $20a^2b^2 + 23abx - 21x^2$.
20. $a^4 + 2a^2b^2 + 9b^4$.
21. $48a^3x^4y - 75ay^5$.
22. $16a^4x^3y + 54ay^4$.
23. $x^4y^2 + 2x^2yz + z^2$.
24. $a^2 + 10a - 39$.
25. $8a^2y^3 - 48a^2y^2z + 72a^2yz^2$.
26. $4m^8 - 60m^4n^4 + 81n^8$.
27. $35a^2 - 6ab - 9b^2$.
28. $(a + b)^2 - (c - d)^2$.
29. $72a^2x^2 - 19axy^2 - 40y^4$.
30. $4(a-3)^6 - 37b^2(a-3)^3 + 9b^4$.

FACTORS OF POLYNOMIALS OF FOUR TERMS

A polynomial of four terms may be readily factored if it is in one of the forms given in the next three paragraphs :

65. *It may be the cube of a binomial.*

Example 1. $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$.

Example 2. $8x^3 + 36x^2y + 54xy^2 + 27y^3$

$= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 = (2x + 3y)^3$.

66. *It may be resolvable into the difference of two squares.*

In this case three of the terms must form a trinomial square.

Example 1. $a^2 - c^2 + 2ab + b^2 = (a^2 + 2ab + b^2) - c^2$

$= (a + b)^2 - c^2 = (a + b + c)(a + b - c)$.

Example 2. $4x^2 + z^6 - 4x^4 - 1 = z^6 - (4x^4 - 4x^2 + 1)$

$= z^6 - (2x^2 - 1)^2 = (z^3 + 2x^2 - 1)(z^3 - 2x^2 + 1)$.

67. *A binomial factor may be found by grouping the terms.*

In this case the terms are grouped by twos as in the following examples.

Example 1. $ax + ay + bx^2 + bxy = (ax + ay) + (bx^2 + bxy)$

$= a(x + y) + bx(x + y) = (a + bx)(x + y)$.

Example 2. $ax + bx + a^2 - b^2 = (ax + bx) + (a^2 - b^2)$

$= x(a + b) + (a - b)(a + b) = (x + a - b)(a + b)$.

WRITTEN EXERCISES

Factor the following polynomials :

1. $x^3 + 3x^2y + 3xy^2 + y^3$.
2. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.
3. $4a^4 - 4a^2b^2 + b^4 - 16x^2$.
4. $2ad + 3bc + 2ac + 3bd$.
5. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.
6. $36a^4 - 24a^3 + 24a - 16$.
7. $mnx^2 - mrx - rn^2x + r^2n$.
8. $a^2b^2 - a^2bc^2n - abn + an^2c^2$.
9. $2y^2 + 4by + 3cy + 6bc$.
10. $bcyz + c^2z^2 + bdy + cdz$.
11. $5a^2c + 12cd - 6ad - 10ac^2$.
12. $a^2 - b^2x^2 + acx^2 - bcx^3$.
13. $b^3c^2 - c^2y^3 - b^3y^2 + y^5$.
14. $m^{a+b} + m^an^a + m^bn^b + n^{a+b}$.

FACTORS FOUND BY GROUPING

68. The discovery of factors by the proper grouping of terms is of wide application. Polynomials of five, six, or more terms may frequently be thus resolved into factors.

Example 1. $a^2 + 2ab + b^2 + 5a + 5b = (a+b)^2 + (5a + 5b)$

$= (a+b)(a+b) + 5(a+b) = (a+b+5)(a+b).$

Example 2. $x^2 - 7x + 6 - ax + 6a = x^2 - 7x + 6 - (ax - 6a)$

$= (x-1)(x-6) - a(x-6) = (x-6)(x-1-a).$

Example 3. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$

$= (a^2 - 2ab + b^2) - (x^2 - 2xy + y^2).$

$= (a-b)^2 - (x-y)^2 = (a-b+x-y)(a-b-x+y).$

Example 4. $ax^2 + ax - 6a + x^2 + 7x + 12$

$= a(x^2 + x - 6) + (x^2 + 7x + 12)$

$= a(x+3)(x-2) + (x+3)(x+4)$

$= (x+3)[a(x-2) + x + 4] = (x+3)(ax - 2a + x + 4).$

In some cases the grouping is effected only after a term has been separated into two parts.

Example 5. $2a^3 + 3a^2 + 3a + 1 = a^3 + (a^3 + 3a^2 + 3a + 1)$

$= a^3 + (a+1)^3 = (a + a + 1)[a^2 - a(a+1) + (a+1)^2]$

$= (2a+1)(a^2 + a + 1).$

As soon as the term $2a^3$ is separated into two terms the expression is shown to be the sum of two cubes.

Again, the grouping may be effective after a term has been both added and subtracted:

Example 6. $x^8 + x^4y^4 + y^8 = x^8 + 2x^4y^4 + y^8 - x^4y^4$

$= (x^4 + y^4 - x^2y^2)(x^4 + y^4 + x^2y^2)$

$= (x^4 + y^4 - x^2y^2)(x^2 + y^2 - xy)(x^2 + y^2 + xy).$

Example 7. $a^4 + b^4 = (a^4 + 2a^2b^2 + b^4) - 2a^2b^2$

$= (a^2 + b^2)^2 - (ab\sqrt{2})^2$

$= (a^2 + b^2 + ab\sqrt{2})(a^2 + b^2 - ab\sqrt{2}).$

In this case the factors are irrational as to one coefficient. Such factors are often useful in higher mathematical work.

WRITTEN EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - ax + ay$.
2. $a^2 - ab + b^2 + a^3 + b^3$.
3. $a^3 - b^3 - a^2 - ab - b^2$.
4. $a^2 - 2ab + b^2 - x^2 + 2xy - y^2$.
5. $a^4 + 2a^3b - a^2c^2 + a^2b^2 - 2abc^2 - b^2c^2$.
6. $x^4 - y^4 + ax^2 + ay^2 - x^2 - y^2$.
7. $a^4 + a^2b^2 + b^4 + a^3 + b^3$.

In example 7 group the first three and the last two terms and then add and subtract a^2x^2.

8. $a^3 - 1 + 3x - 3x^2 + x^3$. Group the last four terms.
9. $x^3 + x^2 + 3x + y^3 - y^2 + 3y$.

Group in pairs, the 1st and 4th, 2d and 5th, 3d and 6th terms.

10. $x^4 + x^3y - xy^3 - y^4 + x^3 - y^3$.
11. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 - x^4$.
12. $x^4 + 4x^2z - 4y^2 + 4yw + 4z^2 - w^2$.
13. $2a^2 - 12b^2 + 3bd - 5ab - 9bc - 6ac + 2ad$.

Group the terms: $2a^2 - 5ab - 12b^2$.

14. $a^2 + ab - 4ac - 2b^2 + 4bc + 3ad - 3bd$.
15. $a^3 + 2 - 3a$.

Add and subtract 1 and group thus: $(a^3 - 1) + (3 - 3a)$.

16. $4a^2 + a - 8ax - x + 4x^2$.
17. $3a^2 - 8ab + 4b^2 + 2ac - 4bc$.
18. $a^6 + 2a^3b^3 + b^6 - 2a^4b - 2ab^4$.
19. $a^3 - 3a^2 + 4$.

Group thus: $(a^3 - 2a^2) + (4 - a^2)$.

20. $a^2c - ac^2 - a^2b + ab^2 - b^2c + bc^2$.
21. $a^2b - a^2c + b^2c - ab^2 + ac^2 - bc^2$.
22. $3x^3 - x^2 - 4x + 2$. Add and subtract $-2x^2$.
23. $2x^3 - 11x^2 + 18x - 9$. Add and subtract $9x^2$.
24. $x^8 + y^8$. Add and subtract $2x^4y^4$.

FACTORS FOUND BY THE REMAINDER THEOREM

69. It is possible to determine in advance whether a polynomial in x is divisible by a binomial of the form $x-a$.

E.g. In dividing $x^4-4x^3+7x^2-7x+2$ by $x-2$, the quotient is found to be x^3-2x^2+3x-1.

Since *Quotient* × *Divisor* = *Dividend*, we have

$$(x-2)(x^3-2x^2+3x-1)=x^4-4x^3+7x^2-7x+2.$$

As this is an *identity*, it holds for all values of x. For $x=2$ the factor $(x-2)$ is zero, and hence the left member is zero.

Hence for $x=2$ the right member must also be zero. This is indeed the case, viz.:

$$2^4-4\cdot 2^3+7\cdot 2^2-7\cdot 2+2=16-32+28-14+2=0.$$

Hence, if $x-2$ is a factor of $x^4-4x^3+7x^2-7x+2$, the latter must reduce to zero for $x=2$.

70. The Remainder Theorem. In general let D represent any polynomial in x. Suppose D has been divided by $x-a$ until the remainder no longer contains x. Then, calling the quotient Q and the remainder R, we have the identity

$$D=Q(x-a)+R, \tag{1}$$

which holds for all values of x.

The substitution of a for x in (1) does not affect R, reduces $Q(x-a)$ to zero, and may or may not reduce D to zero.

(1) If $x=a$ reduces D to zero, then $0\equiv 0+R$. Hence R is zero, and the division is exact. That is, $x-a$ is a factor of D.

(2) If $x=a$ does *not* reduce D to zero, then R is not zero, and the division is not exact. That is, $x-a$ is not a factor of D.

Hence: *If a polynomial in x reduces to zero when a particular number a is substituted for x, then $x-a$ is a factor of the polynomial, and if the substitution of a for x does not reduce the polynomial to zero, then $x-a$ is not a factor.*

This principle is called the **Remainder theorem.**

In applying the remainder theorem the trial divisor must always be written in the form $x-a$.

Example 1. Factor $x^4 + 6x^3 + 3x^2 + x + 3$.

If there is a factor of the form $x - a$, then the only possible values of a are the various divisors of 3, namely $+1$, -1, $+3$, -3.

To test the factor $x + 1$, we write it in the form $x - (-1)$ where $a = -1$. Substituting -1 for x in the polynomial, we have

$$1 - 6 + 3 - 1 + 3 = 0.$$

Hence, $x + 1$ is a factor.

On substituting $+1$, $+3$, -3 for x successively, no one reduces the polynomial to zero. Hence, $x - 1$, $x - 3$, $x + 3$ are not factors.

Example 2. Factor $3x^3 - x^2 - 4x + 2$.

If $x - a$ is a factor, then a must be a factor of $+2$. We therefore substitute, $+2$, -2, $+1$, -1 and find the expression becomes zero when $+1$ is substituted for x. Hence, $x - 1$ is a factor. The other factor is found by division to be $3x^2 + 2x - 2$, which is prime.

Hence,

$$3x^3 - x^2 - 4x + 2 = (x - 1)(3x^2 + 2x - 2).$$

ORAL EXERCISES

1. Is $x - 1$ a factor of $x^3 - 4x^2 + 5x - 2$?
2. Is $x - 1$ a factor of $x^3 + 2x^2 - 7x + 3$?
3. Is $x - 2$ a factor of $x^3 + x^2 - x - 10$?
4. Is $x - 2$ a factor of $x^3 - 2x^2 + 4x - 8$?
5. Is $x + 2$ a factor of $x^3 - 2x^2 + 4x - 8$?
6. Is $x - 3$ a factor of $x^3 - 3x^2 - 4x + 12$?
7. Is $x - 3$ a factor of $x^3 + 3x^2 - 15x - 9$?
8. Is $x + 2$ a factor of $x^3 + 3x^2 - 2x - 10$?
9. Is $x + 3$ a factor of $x^3 + 4x^2 + 2x - 3$?
10. Is $x + 3$ a factor of $x^3 - 4x^2 - 2x - 6$?
11. Is $x + 1$ a factor of $x^4 + 3x^3 - 4x^2 + x + 7$?
12. Is $x + 1$ a factor of $x^5 - 2x^3 + 4x + 2$?
13. Is $x + 2$ a factor of $x^5 + 4x^2 + 16$?
14. Is $x + 2$ a factor of $x^5 + 5x^2 + 10$?

WRITTEN EXERCISES

Factor by means of the remainder theorem:

1. $3x^3 - 2x^2 + 5x - 6.$
2. $2x^3 + 3x^2 - 3x - 4.$
3. $2x^3 + x^2 - 12x + 9.$
4. $x^3 + 9x^2 + 10x + 2.$
5. $a^3 - 3a + 2.$
6. $m^3 + 5m^2 + 7m + 3.$
7. $x^4 + 3x^3 - 3x^2 - 7x + 6.$
8. $3r^3 + 5r^2 - 7r - 1.$
9. $2z^3 + 7z^2 + 4z + 3.$
10. $a^3 - 6a^2 + 11a - 6.$

71. **Applying the Remainder Theorem.** By use of the remainder theorem we may find under what conditions $x + y$ and $x - y$ are factors of $x^n + y^n$ and $x^n - y^n$.

1. Is $x + y$ a factor of $x^n + y^n$?

If we substitute $-y$ for x we have $(-y)^n + y^n$. This is zero only when $(-y)^n = -y^n$; that is, when n is an *odd integer*.

Hence $x + y$ is a factor of $x^5 + y^5$, but *not* of $x^6 + y^6$.

2. Is $x + y$ a factor of $x^n - y^n$?

Here we have $(-y)^n - y^n$. This reduces to zero only when $(-y)^n = +y^n$; that is, when n is an *even integer*.

Hence $x + y$ is a factor of $x^6 - y^6$, but *not* of $x^7 - y^7$.

3. Is $x - y$ a factor of $x^n + y^n$?

Since $y^n + y^n$ is never zero, $x - y$ is not a factor of $x^n + y^n$.
E.g. $x - y$ is not a factor of $x^3 + y^3$, nor of $x^4 + y^4$.

4. Is $x - y$ a factor of $x^n - y^n$?

Since $y^n - y^n = 0$, $x - y$ is a factor of $x^n - y^n$, for all integral values of n.
E.g. $x - y$ is a factor of $x^3 - y^3$ and also of $x^4 - y^4$.

Summary. From the foregoing examples, we conclude that:

(1) *$x + y$ is a factor of $x^n - y^n$ if n is* ***even*** *but not if n is* ***odd.***
(2) *$x - y$ is a factor of $x^n - y^n$ whether n is* ***even*** *or* ***odd.***
(3) *$x + y$ is a factor of $x^n + y^n$ if n is* ***odd*** *but not if n is* ***even.***
(4) *$x - y$ is not a factor of $x^n + y^n$ in* ***any*** *case.*

MISCELLANEOUS EXERCISES IN FACTORING

1. $20a^3x^3y - 45a^3xy^3$. **4.** $16x^2 - 72xy + 81y^2$.

2. $24am^5n^2 - 375am^2n^5$. **5.** $162a^3b + 252a^2b^2 + 98ab^3$.

3. $432ar^4s + 54ars^4$. **6.** $48x^5y - 12x^3y - 12x^2y + 3y$.

7. $12a^2bx^2 + 8ab^2x^2 + 18a^2bxy + 12ab^2xy$.

8. $18x^3y - 39x^2y^2 + 18xy^3$. **16.** $a^8 - y^8$. **17.** $a^{16} - y^{16}$.

9. $4x^2 - 9xy + 6x - 9y + 4x + 6$. **18.** $a^8 + a^4y^4 + y^8$.

10. $6x^2 - 13xy + 6y^2 - 3x + 2y$. **19.** $a^3 + a - 2$.

11. $6x^4 - 15x^2y^2 + 9y^4$. **20.** $a^8 - 18a^4y^4 + y^8$.

12. $16x^4 + 24x^2y^2 + 8y^4$. Group as follows:

13. $15x^4 + 24x^2y^2 + 9y^4$. $a^8 - 2a^4y^4 + y^4 - 16a^4y^4$.

14. $a^6 + y^6$. **15.** $a^{12} + y^{12}$. **21.** $a^{16} - 6a^8y^8 + y^{16}$.

22. $x^3 + 4x^2 + 2x - 1$. **25.** $a^3 + a^2 + a + 1$.

23. $3x^3 + 2x^2 - 7x + 2$. **26.** $a^3 + 9a^2 + 16a + 4$.

24. $a^8 - 3a^4y^4 + y^8$. **27.** $2x^4 + x^3y + 2x^2y^2 + xy^3$.

28. $(x - 2)^3 - (y - z)^3$.

29. $a^6 + b^6 + 2ab(a^4 - a^2b^2 + b^4)$.

30. $8a^3 + 6ab(2a - 3b) - 27b^3$.

31. $a(x^3 + y^3) - ax(x^2 - y^2) - y^2(x + y)$.

32. $a^3 - b^3 + 3b^2c - 3bc^2 + c^3$.

33. $a^4 + 2a^3b - 2ab^2c - b^2c^2$.

34. $a^4 + 2a^3b + a^2b^2 - a^4b^2 - 2a^2b^2c - b^2c^2$.

35. $6(x + y)^2 + 5(x^2 - y^2) - 6(x - y)^2$.

36. $9(x - a)^2 - 24(x - a)(x + a) + 16(x + a)^2$.

37. $12(c + d)^2 - 7(c + d)(c - d) - 12(c - d)^2$.

38. $(a^2 + 5a - 3)^2 - 25(a^2 + 5a - 3) + 150$.

39. $2x^3 + x^2 - 6x + 3$.

40. $12x^4 + 14x^3 - 3x^2 - 4x + 1$.

41. $6x^3 - x^2 - 20x + 12$.

42. $x^5 - 4x^4 - 40x^3 - 58x^2 - x + 6$.

ORAL EXERCISES

Find the L. C. M. of each the following sets of expressions. Give each result in the factored form.

1. $(x-2)(x-3), \quad (x-4)(x-3).$
2. $4(a-b)(b-c), \quad 2(b-c)(d-a).$
3. $3\,a^2(x+2)(x-1), \quad 6\,a^3(x-1)(x-2).$
4. $5(a+2\,b)(a-2\,b), \quad 3(a+2\,b)(a-4\,b).$
5. $12(x^2-y^2), \quad x-y, \quad x+y.$
6. $3(x^2-4\,xy+4\,y^2), \quad 3(x-2\,y).$
7. $(2\,x-1)(x+4), \quad 2(x+4), \quad 4(2\,x-1).$
8. $4\,a^2+12\,ab+9\,b^2, \quad 2\,a+3\,b.$
9. $9\,x^2+24\,xy+16\,y^2, \quad 9\,x^2-16\,y^2.$
10. $x^2-10\,xy+25\,y^2, \quad x^2-25\,y^2.$

WRITTEN EXERCISES

Find the H. C. F. and also the L. C. M of each of the following sets:

1. $x^2+y^2, \quad x^6+y^6.$ 2. $x^2+xy+y^2, \quad x^3-y^3.$
3. $x^2-5\,x-6, \quad x^2-2\,x-3, \quad x^2+19\,x+18.$
4. $x^4-6\,x^2+1, \quad x^3+x^2-3\,x+1, \quad x^3+3\,x^2+x-1.$
5. $162\,a^3b+252\,a^2b^2+9\,ab^3, \quad 54\,a^3+42\,a^2b.$
6. $2\,x^3+x^2-8\,x+3, \quad x^2+2\,x-1.$
7. $3\,r^3+5\,r^2-7\,r-1, \quad 3\,r^2+8\,r+1.$
8. $a^3-3\,a^2+4, \quad ax-ab-2\,x+2\,b.$
9. $a^6+2\,a^3b^3+b^6-2\,a^4b-2\,ab^4, \quad a^3-2\,ab+b^3.$
10. $8\,a^3-36\,a^2b+54\,ab^2-27\,b^3, \quad 4\,a^2-9\,b^2.$
11. $2\,y^2+4\,by+3\,cy+6\,bc, \quad y^2-3\,by-10\,b^2.$
12. $x^{16}-y^{16}, \quad x^8-y^8, \quad x^4-y^1.$
13. $m^3+8\,m^2+7\,m, \quad m^3+3\,m^2-m-3, \quad m^3-7\,m-6.$

CHAPTER IV

FRACTIONS

74. An algebraic fraction is the indicated quotient of two algebraic expressions.

Thus $\frac{n}{d}$ means *n divided by d.*

In arithmetic a fraction such as $\frac{2}{3}$ is usually regarded as 2 of the 3 equal parts of a unit.

However, a fraction such as $\frac{5}{3\frac{1}{2}}$ cannot be regarded in this way, since a unit cannot be divided into $3\frac{1}{2}$ equal parts. $\frac{5}{3\frac{1}{2}}$ indicates that 5 is to be divided by $3\frac{1}{2}$; *i.e.* $\frac{5}{3\frac{1}{2}} = 5 \div 3\frac{1}{2}$.

Since a fraction is a quotient, it follows from the definition of division that if a fraction is multiplied by its denominator the product is the numerater.

That is, $$b \times \frac{a}{b} = a.$$

75. Fundamental Theorem on Fractions. *Both terms of a fraction may be divided or both may be multiplied by the same number without changing the value of the fraction.*

Proof. If a, b, and k are any numbers, we are to prove that

$$\frac{ak}{bk} = \frac{a}{b}.$$

From § 74, we have $$a = \frac{a}{b} \times b.$$

Multiplying both sides by k, $$ak = \frac{a}{b} \times bk.$$

Dividing both sides by bk, $$\frac{ak}{bk} = \frac{a}{b}. \quad (1)$$

Reading (1) in the reverse order, we have

$$\frac{a}{b} = \frac{ak}{bk}. \quad (2)$$

76. Reduction of Fractions. The form of a fraction may be modified in various ways without changing its value. Any such transformation is called a *reduction of the fraction.*

The most important reductions are the following:

(*A*) *Reduction by manipulation of signs.*

E.g. $\frac{n}{d} = -\frac{-n}{d} = -\frac{n}{-d} = \frac{-n}{-d}$; $\frac{b-a}{c-d} = -\frac{a-b}{c-d} = \frac{a-b}{d-c}$.

(*B*) *Reduction to lowest terms.*

E.g. $\frac{x^4+x^2+1}{x^6-1} = \frac{\cancel{(x^2+x+1)}\cancel{(x^2-x+1)}}{(x-1)\cancel{(x^2+x+1)}(x+1)\cancel{(x^2-x+1)}} = \frac{1}{(x-1)(x+1)}$.

(*C*) *Reduction to integral or mixed expressions.*

E.g. $\frac{2x^3+x^2+x+2}{x^2+1} = 2x+1+\frac{-x+1}{x^2+1} = 2x+1-\frac{x-1}{x^2+1}$.

(*D*) *Reduction to fractions having a common denominator.*

E.g. $\frac{2}{x+3}$ and $\frac{3}{x+2}$ become respectively $\frac{2(x+2)}{(x+3)(x+2)}$ and $\frac{3(x+3)}{(x+3)(x+2)}$; $a+1$ and $\frac{1}{a-1}$ become respectively $\frac{a^2-1}{a-1}$ and $\frac{1}{a-1}$.

77. These reductions are useful in connection with the various operations upon fractions. They depend upon the principles indicated below.

Reduction (*A*) is simply an application of the law of signs in division, § 22. It is often needed in connection with reduction (*D*). See p. 43.

Reduction (*B*) depends upon the theorem, § 75, $\frac{ak}{bk} = \frac{a}{b}$, by which *a common factor may be removed from both terms of a fraction.* This reduction is *complete* when the numerator and denominator have been divided by their H. C. F.

Reduction (*C*) is merely the process of performing the indicated division.

Reduction (*D*) depends upon the theorem of § 75, $\frac{a}{b} = \frac{ka}{kb}$, by which *a common factor is introduced into the terms of a fraction.*

A fraction is thus reduced to another fraction whose denominator is *any required multiple* of the given denominator.

Niels Henrick Abel (1802–1829) was a distinguished Norwegian mathematician.

Though he lived only twenty-seven years, Abel made contributions of far-reaching importance in mathematics. Not least noteworthy of these was his proof that literal equations of degree higher than the fourth cannot, in general, be solved by means of radicals.

His complete works were published in 1839.

ORAL EXERCISES

Reduce the following fractions to lowest terms.

1. $\frac{x^2+5x+6}{x^2+6x+9}$.
2. $\frac{x^2+3x+2}{x^2-4}$.
3. $\frac{a^2-b^2}{a^2+2ab+b^2}$.
4. $\frac{a^2+4ab+4b^2}{a^2-4b^2}$.
5. $\frac{a^3-b^3}{a^2-b^2}$.
6. $\frac{a^3-b^3}{a^2-2ab+b^2}$.
7. $\frac{a^3-b^3}{(a^2+ab+b^2)c}$.
8. $\frac{(x+4)^2}{x^2+7x+12}$.
9. $\frac{(x-3)^2}{x-5x+6}$.

WRITTEN EXERCISES

Reduce the following so that the letters in each factor shall occur in alphabetical order, and no negative sign shall stand before a numerator or denominator, or before the first term of any factor.

1. $\frac{n-m}{b-a}$.
2. $-\frac{(b-a)(c-d)}{x(s-r-t)}$.
3. $\frac{-(x-y)}{(b-a)(c-d)}$.
4. $\frac{-(x-y)(z-y)}{-(b-a)(c-d)}$.
5. $\frac{r-s}{(a-b)(c-b)(c-a)}$.
6. $\frac{-a(c+b)}{b(c-a)}$.
7. $\frac{-(c-a)(d-c)}{(a-b)(b-c)}$.
8. $\frac{(b-a)(c-b)(c-a)}{(y-x)(y-z)(z-x)}$.
9. $-\frac{-1}{(a-b)(b-c)(c-a)}$.
10. $\frac{(c-b-a)(b-a-c)}{3(a-c)(b-c)(c-a)}$.

Reduce each of the following to lowest terms:

11. $\frac{a^4-b^4}{a^6-b^6}$.
12. $\frac{c^2-(a-b)^2}{(a+c)^2-b^2}$.
13. $\frac{7ax^2-56a^4x^5}{28x^2(1-64a^6x^6)}$.
14. $\frac{m^3+5m^2+7m+3}{m^2+4m+3}$.
15. $\frac{a^3-7a+6}{a^3-7a^2+14a-8}$.
16. $\frac{x^3+2x^2+2x+1}{x^4+x^3-x^2-2x-2}$.
17. $\frac{2x^3-x^2-8x-3}{2x^3-3x^2-7x+3}$.
18. $\frac{4x^3+8x^2-3x+5}{6x^3-5x^2+4x-1}$.

Reduce each of the following to an integral or mixed expression:

19. $\dfrac{x^4+1}{x+1}$.
21. $\dfrac{x^4}{x-1}$.
23. $\dfrac{c^5}{c^3+c^2-c+1}$.

20. $\dfrac{x^5+1}{x+1}$.
22. $\dfrac{a^3}{a^2+a+1}$.
24. $\dfrac{x^2-x+1}{x^2+x+1}$.

25. $\dfrac{a^4+a^2b^2+b^4}{a-b.}$
27. $\dfrac{x^3-x^2-x+1}{x^3+x^2+x-1}$.

26. $\dfrac{3a^3-3a^2+3a-1}{a-2}$.
28. $\dfrac{4m^4-3m^3+3}{2m^2-2m+1}$.

Reduce each of the following sets of expressions to equivalent fractions having the lowest common denominator:

29. $\dfrac{1}{x^4-3x^2y^2+y^4}, \dfrac{1}{x^2-xy-y^2}, \dfrac{1}{x^2+xy-y^2}$.

30. $\dfrac{x^2+y^2}{x^3+y^3+x^2-xy+y^2}, \dfrac{x+y-1}{x^2-xy+y^2}, \dfrac{x^2+xy+y^2}{x+y+1}$.

31. $\dfrac{x}{(a-b)(c-b)(c-a)}, \dfrac{y}{(a-b)(b-c)(a-c)},$

32. $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}, d.$ $\left[\dfrac{z}{(b-a)(b-c)(a-c)}\right.$

33. $\dfrac{b-c}{(a-c)(a-b)}, \dfrac{a-b}{(c-a)(b-c)}, \dfrac{c-a}{(b-a)(c-b)}$.

34. $\dfrac{m-n}{a^3-6a^2+11a-6}, \dfrac{a+2}{a^2-4a+3}, \dfrac{a+3}{a^2-3a+2}$.

If *a*, *b*, *m* are positive numbers, arrange each of the following sets in decreasing order. Verify the results by substituting convenient numbers for *a*, *b*, *m*.

Suggestion. Reduce the fractions in each set to equivalent fractions having a common denominator.

35. $\dfrac{a}{a+1}, \dfrac{2a}{a+2}, \dfrac{3a}{a+3}$.
36. $\dfrac{m}{2m+1}, \dfrac{2m}{3m+2}, \dfrac{3m}{4m+3}$.

37. $\dfrac{a+3b}{a+4b}, \dfrac{a+b}{a+2b}, \dfrac{a+4b}{a+5b}$.

ADDITION AND SUBTRACTION OF FRACTIONS

78. Common Denominator. Fractions which have a common denominator are added or subtracted in accordance with the distributive law for division, § 29.

That is,
$$\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}.$$

In order to add or subtract fractions not having a common denominator, they should first be reduced to *equivalent fractions having a common denominator.*

In manipulating fractions it is advantageous to keep their terms *in the factored form* as long as possible. When several fractions are to be combined, it is sometimes best to take only part of them at a time.

Example.
$$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}+\frac{1}{(3-x)(4-x)}.$$

Taking the first two together, we have
$$\frac{1}{(x-1)(x-2)}-\frac{1}{(2-x)(x-3)}=\frac{1}{(x-1)(x-2)}+\frac{1}{(x-2)(x-3)}$$
$$=\frac{2x-4}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)(x-3)}.$$

Taking this result with the third fraction,
$$\frac{2}{(x-1)(x-3)}+\frac{1}{(x-3)(x-4)}=\frac{3x-9}{(x-1)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

If all are taken at once, the work should be carried out as follows:

The numerator of the sum is
$$(x-3)(x-4)+(x-1)(x-4)+(x-1)(x-2).$$

Adding the first two terms with respect to $(x-4)$, we have
$$(x-3)(x-4)+(x-1)(x-4)=(2x-4)(x-4)=2(x-2)(x-4).$$

Again adding with respect to $x-2$ we have
$$2(x-2)(x-4)+(x-1)(x-2)=(3x-9)(x-2)=3(x-3)(x-2).$$

Hence, the sum is
$$\frac{3(x-3)(x-2)}{(x-1)(x-2)(x-3)(x-4)}=\frac{3}{(x-1)(x-4)}.$$

WRITTEN EXERCISES

Perform the following indicated additions and subtractions:

1. $\frac{2}{x-3}+\frac{3}{x-4}-\frac{4}{x-5}$.

2. $\frac{1}{x-1}-\frac{4}{x-2}+\frac{7}{x-3}$.

3. $\frac{2}{(x+1)^2}+\frac{3}{x+1}+\frac{4}{x-2}$.

4. $\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$.

5. $\frac{5x+6}{x^2+x+1}-\frac{3x-4}{x^2-x+1}$.

6. $\frac{3}{4(x+3)}-\frac{5}{8(x+5)}-\frac{1}{8(x+1)}$.

7. $\frac{1}{12(x+1)}-\frac{7}{3(x-2)}+\frac{13}{4(x-3)}$.

8. $\frac{1}{5(x+2)}+\frac{4x-8}{5(x^2+1)}$.

9. $\frac{2}{(x-2)^2}-\frac{1}{x-2}+\frac{1}{x+1}$.

10. $\frac{2}{(x-2)^2}+\frac{1}{5(x-2)}-\frac{x+2}{5(x^2+1)}$.

11. $\frac{1}{(x-1)^2}+\frac{1}{x-1}-\frac{1}{x^2-1}$.

12. $\frac{1}{2(1-3x)^3}+\frac{3}{8(1-3x)^2}+\frac{3}{32(1-3x)}+\frac{1}{32(1+x)}$.

13. $\frac{1}{(1-a)(2-a)}-\frac{1}{(2-a)(a-3)}+\frac{2}{(3-a)(a-1)}$.

14. $\frac{xy}{(z-y)(x-z)}-\frac{yz}{(x-z)(x-y)}-\frac{xz}{(y-x)(y-z)}$.

15. $\frac{1}{a-1}-\frac{2a-5}{a^2-2a+1}-\frac{5a^2-3a-2}{(a-1)^3}$.

16. $\frac{1}{m^2+m+1}-\frac{1}{m^2-m+1}+\frac{2m+2}{m^4+m^2+1}$.

17. $\frac{1}{b^2-3b+2}+\frac{1}{b^2-5b+6}-\frac{2}{b^2-4b+3}$.

18. $\frac{r+s}{(r-t)(s-t)}-\frac{s+t}{(r-s)(t-r)}-\frac{r+t}{(t-s)(s-r)}$.

19. $\frac{p^2+q^2}{(p-q)(p+r)}+\frac{q^2-pr}{(q-r)(q-p)}+\frac{r^2+pq}{(r-q)(r+p)}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

79. Theorem. *The product of two fractions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.*

That is, $$\frac{a}{b} \times \frac{n}{d} = \frac{an}{bd}.$$

For, let $$x = \frac{a}{b} \times \frac{n}{d}.$$

Multiplying both sides by bd, $$bdx = bd\left(\frac{a}{b} \times \frac{n}{d}\right).$$

By the commutative law of factors (§ 26), $$bdx = b \times \frac{a}{b} \times d \times \frac{n}{d}.$$

Canceling, $$bdx = b \times \frac{a}{b} \times d \times \frac{n}{d} = an.$$

Dividing both sides by bd, $$x = \frac{an}{bd}.$$

Therefore, $$\frac{a}{b} \times \frac{n}{d} = \frac{an}{bd}.$$

80. Power of a Fraction. It follows that *a fraction is raised to any power by raising the numerator and denominator separately to that power.*

For $$\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}, \quad \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}, \text{ etc.}$$

81. Reciprocal of a Fraction. A fraction multiplied by itself inverted equals $+1$.

For $$\frac{n}{d} \times \frac{d}{n} = \frac{nd}{nd} = +1 \quad \text{and} \quad -\frac{n}{d} \times \left(-\frac{d}{n}\right) = \frac{nd}{nd} = +1.$$

If the product of two numbers is 1, each is called the *reciprocal* of the other. Hence the reciprocal of a fraction is the **fraction inverted.**

Also, since from $ab = 1$, we have $a = \frac{1}{b}$, and $b = \frac{1}{a}$, it follows that if two numbers are reciprocals of each other, then either is the quotient obtained by dividing 1 by the other.

82. Theorem. *Dividing by any number is equivalent to multiplying by its reciprocal.*

For $$n \div d \text{ or } \frac{n}{d} = n \times \frac{1}{d}.$$

83. Theorem. *Dividing a number by a fraction is equivalent to multiplying by the fraction inverted.*

For by § 81, the reciprocal of the fraction is the fraction inverted.

84. Theorem. *A fraction is divided by an integer by multiplying its denominator or dividing its numerator by that integer.*

For $$(1)\quad \frac{n}{d} \div a = \frac{n}{d} \times \frac{1}{a} \qquad \text{by § 82,}$$

but $$\frac{n}{d} \times \frac{1}{a} = \frac{n}{ad} \qquad \text{by § 79,}$$

and $$(2)\quad \frac{n}{ad} = \frac{n \div a}{ad \div a} = \frac{n \div a}{d} \qquad \text{by § 75.}$$

In multiplying and dividing fractions, their terms should at once be put into factored forms.

85. Preliminary Reductions. When mixed expressions or sums of fractions are to be multiplied or divided, these operations are indicated by means of parentheses, and the additions or subtractions within the parentheses should be performed first.

Example 1. $\left(1 + \frac{a}{b}\right) \times \left(1 - \frac{b}{a}\right) = \frac{b + a}{b} \times \frac{a - b}{a} = \frac{a^2 - b^2}{ab}.$

Example 2. Simplify

$$\left[\left(1 - a + \frac{2a^2}{1 + a}\right) \div \left(\frac{1}{1 + a} - \frac{1}{1 - a}\right)\right] \times \frac{3a^3}{a^4 - 1}.$$

Performing the indicated operations within the parentheses, we have

$$\left[\frac{1 - a^2 + 2a^2}{1 + a} \div \frac{1 - a - 1 - a}{1 - a^2}\right] \times \frac{3a^3}{a^4 - 1} = \left[\frac{1 + a^2}{1 + a} \div \frac{2a}{a^2 - 1}\right] \times \frac{3a^3}{a^4 - 1}$$

$$= \frac{1 + a^2}{1 + a} \times \frac{a^2 - 1}{2a} \times \frac{3a^3}{(a^2 - 1)(a^2 + 1)} = \frac{3a^2}{2(a + 1)}.$$

WRITTEN EXERCISES

Perform the following indicated operations and reduce each result to its simplest form.

1. $$\frac{x^4+x^2y^2+y^4}{x^3-y^3}\times\frac{x^2-y^2}{x^3+y^3}.$$

2. $$\frac{a^2-b^2x^2+acx^2-bcx^3}{(6\,a^2+3\,a-4)(6\,a^2-3\,a+4)}\div\frac{-\,ay^2-bxy^2-cx^2y^2}{20\,x^2-15\,ax^2-30\,a^2x^2}.$$

3. $$\frac{20\,r^2s^2+23\,rst-21\,t^2}{8\,m^2n^3-48\,m^2n^2y+72\,m^2ny^2}\times\frac{12\,mn^3-28\,mn^2y-24\,mny^2}{10\,r^2s^2+24\,rst-18\,t^2}$$ $$\left[\div\frac{3\,n+2\,y}{2\,s+3\,t}.\right.$$

4. $$\left(\frac{a}{b}-\frac{b}{a}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}-\frac{a}{b}-\frac{b}{a}+1\right)\div\frac{a^5+b^5}{a-b}.$$

5. $$\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)\left(1-\frac{x-y}{x+y}\right)\left(2+\frac{2\,y}{x-y}\right).$$

6. $$\left(\frac{a}{a-b}-\frac{b}{a+b}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\div\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$$

7. $$\left(1+\frac{b}{a-b}\right)\left(1-\frac{b}{a+b}\right)\div\left(1+\frac{b^2}{a^2-b^2}\right).$$

8. $$\left(\frac{m+n}{m-n}-\frac{m-n}{m+n}\right)\left(m+n+\frac{2\,n^2}{m-n}\right)\div\left(\frac{m+n}{m-n}+\frac{m-n}{m+n}\right).$$

9. $$\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\cdot\left(x^2+y^2+\frac{2\,x^2y^2+2\,y^4}{x^2-y^2}\right)\div\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right).$$

10. $$\left(\frac{x+y+z}{x+y}+\frac{z^2}{(x+y)^2}\right)\cdot\left(\frac{(x+y)^3}{(x+y)^3-z^3}\right)\cdot\left(1+\frac{z}{x+y}\right).$$

11. $$\frac{a^2+ab+b^2}{a^2-ab+b^2}\cdot\frac{a+b}{a^3-b^3}\cdot\left(a^2+\frac{b^3-a^2b}{a+b}\right).$$

12. $$\frac{m^2+mn}{m^2+n^2}\cdot\frac{m^3-mn^2-m^2+n^2}{m^3n-n^4}\cdot\frac{m^2n^2+mn^3+n^4}{m^4-2\,m^3+m^2}.$$

13. $$\left(xy^9+x^9y-\frac{2x^{11}y^3}{x^2y^2}\right)\div\left[\frac{x^2+y^2}{x^2}\cdot\left(\frac{1}{y^2}-\frac{1}{z^2}\right)-\frac{x^2+y^2}{y^2}\cdot\left(\frac{1}{x^2}-\frac{1}{z^2}\right)\right].$$

COMPLEX FRACTIONS

86. A Complex Fraction is one which contains a fraction either in its numerator or in its denominator or in both.

Since every fraction is an indicated division, any complex fraction may be simplified by performing the division.

It is usually better, however, to remove all the minor denominators at once *by multiplying both terms of the complex fraction by the least common multiple of all the minor denominators according to the fundamental theorem* § 75.

For example,
$$\frac{\frac{x}{3}+\frac{x}{2}}{\frac{2x^2}{3}-\frac{3}{2}}=\frac{\left(\frac{x}{3}+\frac{x}{2}\right)\cdot 6}{\left(\frac{2x^2}{3}-\frac{3}{2}\right)\cdot 6}=\frac{2x+3x}{4x^2-9}=\frac{5x}{4x^2-9}.$$

Again in $\dfrac{\frac{2}{x-1}-\frac{1}{x-2}}{\frac{3}{x-4}-\frac{4}{x-1}}$ multiply both terms by $(x-1)(x-2)(x-4)$,

obtaining $\dfrac{2(x-2)(x-4)-(x-1)(x-4)}{3(x-1)(x-2)-4(x-2)(x-4)}$

$$=\frac{2x^2-12x+16-x^2+5x-4}{3x^2-9x+6-4x^2+24x-32}=-\frac{x^2-7x+12}{x^2-15x+26}.$$

A complex fraction may contain another complex fraction in one of its terms.

E.g. $\dfrac{1}{a+\dfrac{a+1}{a+\frac{1}{a-1}}}$ has the complex fraction $\dfrac{a+1}{a+\frac{1}{a-1}}$

in its denominator. This latter fraction is first reduced by multiplying its numerator and denominator by $a-1$, giving

$$\frac{1}{a+\dfrac{a+1}{a+\frac{1}{a-1}}}=\frac{1}{a+\dfrac{a^2-1}{a^2-a+1}}=\frac{a^2-a+1}{a^3+a-1}.$$

WRITTEN EXERCISES

Simplify each of the following:

1. $$\frac{\dfrac{m^2+mn}{m^2-n^2}}{\dfrac{m}{m-n}-\dfrac{n}{m+n}}.$$

2. $$\frac{\dfrac{a^4-b^4}{a^2-2ab+b^2}}{\dfrac{a^2+ab}{a-b}}.$$

3. $$\frac{\dfrac{x^5-3x^4y+3x^3y^2-x^2y^3}{x^3y-y^4}}{\dfrac{x^5-2x^4y+x^3y^2}{x^2y^2+xy^3+y^4}}.$$

4. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2-x^2}}{\dfrac{1}{a+x}-\dfrac{1}{a-x}-\dfrac{2a}{a^2-x^2}}.$$

5. $$\frac{\dfrac{1}{a+x}+\dfrac{1}{a-x}+\dfrac{2a}{a^2+x^2}}{\dfrac{1}{a-x}-\dfrac{1}{a+x}+\dfrac{2x}{a^2+x^2}}.$$

6. $$\frac{m^2-mn+n^2-\dfrac{m^3-n^3}{m+n}}{m^2+mn+n^2+\dfrac{m^3+n^3}{m-n}}.$$

7. $$\frac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

8. $$\frac{\dfrac{a-\dfrac{1}{a^2}}{a-2+\dfrac{1}{a}}}{\dfrac{a^2+1+\dfrac{1}{a^2}}{a-2+\dfrac{2}{a}-\dfrac{1}{a^2}}}.$$

9. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}.$$

10. $$\frac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}.$$

11. $$\frac{3}{3+\dfrac{3}{3+\dfrac{3}{3+\dfrac{3}{x}}}}.$$

12. $$\frac{\dfrac{a}{a+b}+\dfrac{b}{a-b}+\dfrac{2ab}{a^2-b^2}}{\dfrac{1}{(a+b)^2}+\dfrac{1}{(a-b)^2}}.$$

CHAPTER V

EQUATIONS OF THE FIRST DEGREE IN ONE UNKNOWN

87. **Rational Equations.** An equation *is rational in a given letter* if every term in the equation is rational with respect to that letter. See § 15.

E.g. $3x + 2 = x - 4$ is rational in x,

as is also $\frac{2}{x} + 5 = \frac{3}{x}$.

But $\sqrt{x} + 2 = x + 3$ is irrational in x.

88. **Integral Equations.** An equation is *integral in a given letter* if every term is rational and integral in that letter; it is fractional if it contains the letter in any denominator.

E.g. $x - 2 = 2x + 3$ is integral in x,

while $\frac{2}{x} + 5 = \frac{3}{x}$ is fractional in x.

89. **Degree of an Equation.** The *degree* of a rational, integral equation in a given letter is the highest exponent of that letter in the equation.

E.g. $2x - 3 = 5$ is of the first degree in x,

and $x^2 + 3x = 4$ is of the second degree in x.

In determining the degree of an equation according to this definition it is necessary that all indicated multiplications be performed as far as possible.

E.g. $(x - 2)(x - 3) = 0$ is of the second degree in x, since it reduces to $x^2 - 5x + 6 = 0$.

90. **Substitution.** When in an algebraic expression a letter is replaced by another number symbol, this is called a *substitution on that letter.*

E g. In the expression $2a + 5$, if a is replaced by 3, giving $2 \cdot 3 + 5$, this is a substitution on the letter a.

91. Satisfying an Equation. An equation containing a single letter is said to be *satisfied* by any substitution on that letter which reduces both members of the equation to the same number.

E.g. $4x + 8 = 24$ is satisfied by $x = 4$, since $4 \cdot 4 + 8 = 24$.

92. Identity. An equation in a single letter which is satisfied by *every substitution* on that letter is called an *identity.*

E.g. $2(x + 3) = 2x + 6$ is an identity.

If an equation is an identity, *both members reduce to the same expression* when all indicated operations are performed.

E.g. In $2(x + 3) = 2x + 6$, both members are $2x + 6$ when the operations indicated are performed.

93. Equation of Condition. An equation which is not an identity is called an *equation of condition* or simply an *equation.*

The members of an equation of condition *cannot be reduced to the same expression* by performing the indicated operations.

E.g. $3(x - 2) = 4(x - 3)$ cannot be so reduced. This equation is satisfied by $x = 6$, and by no other value of x.

94. Root of an Equation. A value of the unknown which satisfies an equation is called a *root* or *solution* of the equation.

E.g. $x = 6$ is a root of the equation $3(x - 2) = 4(x - 3)$.

ORAL EXERCISES

1. What is the value of $3x^2 - 2x + 1$ if 2 is substituted for x?

2. Is $x^2 + 4x + 4 = (x + 2)^2$ an identity or simply an equation?

3. What is the degree of the equation $(x - 1)(x - 2)(x - 3) = 0$?

4. Is the equation $\frac{5x - 1}{4} + \frac{2x^2 + 7}{6} = 0$ integral or fractional in x? Why?

5. Is $\frac{2x + 5}{x - 1} + \frac{3}{x} + 16 = 0$ integral or fractional in x?

WRITTEN EXERCISES

1. Find whether $4(x-1)=3(x^2-1)$ is satisfied by $x=1$; by $x=2$; by $x=\frac{1}{3}$.

2. Is $3a+4b=12$ satisfied by $a=0$, $b=3$? by $a=4$, $b=0$? by $a=2$, $b=2$?

3. Is $x^3+3x^2+3x+1=(x+1)^3$ satisfied by $x=1$? by $x=2$? by $x=-1$? by $x=-2$?

What kind of an equation is this?

4. Find by performing the indicated operations whether or not the following is an identity:

$$12(x+y)^2+17(x+y)-7=(3x+3y-1)(4x+4y+7).$$

5. Find whether the following is an identity:

$$2(a+b)^2+5(a+b)+8ab=(2a+2b+1)(a+b+1).$$

6. Is $x^2-16=(x-4)(x+5)(x+6)$ satisfied by $x=2$? by $x=3$? by $x=4$?

SOLUTION OF EQUATIONS

95. Equivalent Equations. Two equations are said to be equivalent *if every root of either of them is also a root of the other.*

E.g. $3x+6=12$ and $x+2=4$ are both satisfied by $x=2$ and by no other values of x. Hence, these equations are equivalent.

96. Axioms. The process of solving equations is based on the following *axioms:*

Axiom 1. *If equals are added to equals, the sums are equal.*

Axiom 2. *If equals are subtracted from equals, the remainders are equal.*

Axiom 3. *If equals are multiplied by equals, the products are equal.*

Axiom 4. *If equals are divided by equals, the quotients are equal.*

Axiom 5. *If two algebraic expressions are equal to the same expression, they are equal to each other.*

Axiom 6. *Any quantity may be substituted for its equal.*

97. Deriving Equivalent Equations. From these axioms it follows that an equation may be changed into an equivalent equation by any one of the following processes:

(1) *Adding the same number to both members.*

(2) *Subtracting the same number from both members.*

(3) *Multiplying both members by any known number not zero.*

(4) *Dividing both members by any known number not zero.*

(5) *Changing the form of either member in any way which leaves its value unaltered.*

98. Multiplication by zero is always *a possible operation*, but the equation resulting from multiplying both members by zero is not equivalent to the original equation.

Thus, $x + 4 = 8$ is satisfied by $x = 4$, and by no other number. But $(x + 4) \cdot 0 = 8 \cdot 0$ reduces to $0 = 0$ no matter what value is given to x.

Hence, in solving an equation, *its members must never be multiplied by an expression which equals zero.*

99. Division by zero is always an *impossible operation* or an *indeterminate operation* (see § 31), and hence *the members of an equation should never be divided by any expression which equals zero.*

100. Solving an Equation. The process of solving an equation consists in deriving successive equivalent equations, each simpler than the preceding, until finally one is reached in which the unknown stands alone on one side, and does not occur on the other side.

Example 1.

$$\text{Solve } 2(4x + 5) + 18x + 49 = 7(6 + 6x) + 2x + 8. \qquad (1)$$

Solution. Performing indicated operations, that is, changing the form of each member but not its value,

$$8x + 10 + 18x + 49 = 42 + 42x + 2x + 8. \qquad (2)$$

Performing further indicated operations,

$$26x + 59 = 44x + 50. \qquad (3)$$

Subtracting $26x$ and also 50 from each member,

$$9 = 18x. \qquad (4)$$

Dividing both members by 18,

$$x = \tfrac{9}{18} = \tfrac{1}{2}. \qquad (5)$$

Example 2. Solve, $(3x-2)(x+4)-7=(x-1)(2x-3)+x^2+12.$

Solution. Performing indicated operations,

$$\cancel{3x^2}+10x-8-7=\cancel{2x^2}-5x+3+\cancel{x^2}+12.$$

Subtracting $3x^2$ from both members and adding $5x$ and 15 to both members, we have

$$15x=30.$$

Dividing by 15, $x=2.$

Check. Substituting $x=2$,

$$(3\cdot2-2)(2+4)-7=(2-1)(2\cdot2-3)+2^2+12$$
$$24-7=1+4+12$$
$$17=17.$$

It should be noted that, in case an integral equation is multiplied by an expression containing the unknown, any value of the unknown which makes the multiplier zero is a root of the resulting equation.

Thus, if $x+4=8$ is multiplied by $x-2$ giving $(x+4)(x-2)=8(x-2)$, a new root $x=2$ is introduced.

Similarly, if a factor containing the unknown is removed from both members of an equation, then any value which makes this factor zero is a root of the original equation but may not be a root of the resulting equation.

Hence, multiplying by an expression containing the unknown introduces new roots, while dividing by such an expression removes roots.

ORAL EXERCISES

Solve the following equations:

1. $x+7=15.$
2. $x-3=-8.$
3. $13x=48+x.$
4. $5x=3+4x.$
5. $10y-4=9y+6.$
6. $42w=20w+66.$
7. $10m+18=2m+50.$
8. $25x-2=8x+66.$
9. $20x-30=-7x-4.$
10. $-8x=-33+3x.$
11. $-4a+45=7-2a.$
12. $-3y+6=-4y-2.$
13. $-2x-3=2x+5.$
14. $5x-2=-7x+5.$
15. $-5z-4=-3z+2.$
16. $-x-8=-11x+2.$
17. $4x+7=-2x-5.$
18. $2(x-3)=3(x+2).$
19. $3(x-2)=4(x-1).$
20. $-2(2-x)=5(x+1).$

WRITTEN EXERCISES

Solve the following equations:

1. $7(x+6)+10x = 5x + 66.$
2. $6x + 8(x+1) = 89 - 3(2x+7).$
3. $4(3x+2) = 18x + 36 - 5(9x+3).$
4. $42(1+x) - 2(2x+5) = 16x - 34.$
5. $17(x-1) + 20 = -3(2x-1).$
6. $(2a+3)(3a-2) = a^2 + a(5a+3).$
7. $6(b-4)^2 = -5 - (3-2b)^2 - 5(2+b)(7-2b).$
8. $(y-3)^2 + (y-4)^2 - (y-2)^2 - (y^2-9) = 0.$
9. $(x-3)(3x+4) - (x-4)(x-2) = (2x+1)(x-6).$
10. $2(3r-2)(4r+1) + (r-4)^3 = (r+4)^3 - 2.$
11. $a^3 - c + b^3c + abc = b.$ (Solve for c).

In the next three examples solve for y:

12. $(b-2)^2(b-y) - 3by + (2b+1)(b-1) = 3 - 2b,$
13. $ny(y+n) - (y+m)(y+n)(m+n) + my(y+m) = 0.$
14. $(m+n)(n+b-y) + (n-m)(b-y) = n(m+b).$

In the next eight examples solve for x:

15. $2(12-x) + 3(5x-4) + 2(16-x) = 12(3+x).$
16. $(b-a)x - (a+b)x + 4a^2 = 0.$
17. $(x-a)(b-c) + (b-a)(x-c) - (a-c)(x-b) = 0.$
18. $(x-3)(x-7) - (x-5)(x-2) + 12 = 2(x-1).$
19. $(a+b)^2 + (x-b)(x-a) - (x+a)(x+b) = 0.$
20. $\frac{7}{12}(5x-1) + \frac{5}{16}(2-3x) + \frac{1}{3}(4+x) = \frac{3}{8}(1+2x) - \frac{9}{16}.$
21. $a(x-b) - (a+b)(x+b-a) = b(x-a) + a^2 - b^2.$
22. $(l-m)(x-n) + 2l(m+n) = (l+m)(x+n).$

Solve each of the following equations for each letter in terms of the others:

23. $l(W+w') = l'W'.$
24. $(v-n)d = (v-n_1)d_1.$
25. $m_2s_2(t_2-t) = (m+m_1)(t-t_1).$
26. $(m+m_1)(t_1-t) = lm_2 + m_2t.$

PROBLEMS

1. What number must be added to each of the numbers 2, 26, and 10 in order that the product of the first two sums may equal the square of the last sum?

2. What number must be subtracted from each of the numbers 9, 12, and 18 in order that the product of the first two remainders may equal the square of the last remainder?

3. What number must be added to each of the numbers a, b, c in order that the product of the first two sums may equal the square of the last?

Note that problem 1 is a special case of 3.

4. What number must be added to each of the numbers 4, 2, 3, and 1 in order that the product of the first two sums may equal the product of the last two?

5. What number must be added to each of the numbers a, b, c, d in order that the product of the first two sums may equal the product of the last two?

6. What number must be added to each of the numbers 3, 1, 5, 2 in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

7. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may equal the sum of the squares of the last two?

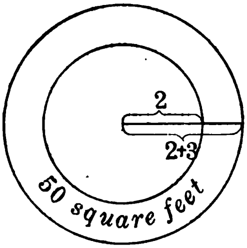
Note that problem 6 is a special case of problem 7.

8. What number must be added to each of the numbers 3, 6, 9, 7 in order that the sum of the squares of the first two sums may be 4 more than twice the product of the last two?

9. What number must be added to each of the numbers a, b, c, d in order that the sum of the squares of the first two sums may be k more than twice the product of the last two?

10. The radius of a circle is increased by 3 feet, thereby increasing the area of the circle by 50 square feet. Find the radius of the original circle.

The area of a circle is πr^2. Use $3\frac{1}{7}$ for π. Hence $(r + 3)^2 \cdot \frac{22}{7} = \frac{22}{7} r^2 + 50$.



11. The radius of a circle is decreased by two feet, thereby decreasing the area by 36 square feet. Find the radius of the original circle.

12. The radius of a circle is increased by a feet, thereby increasing the area by b square feet. Find the radius of the original circle.

13. The radius of a circle is decreased by a feet, thereby decreasing the area by b square feet. Find the radius of the original circle.

14. Each side of a square is increased by 2 feet, thereby increasing its area by 12 square feet. Find the side of the original square.

15. Each side of a square is decreased by 3 feet, thereby decreasing its area by 15 square feet. Find the side of the original square.

16. Each side of a square is increased by a feet, thereby increasing its area by b square feet. Find the side of the original square.

17. Two opposite sides of a square are each increased by 5 feet and the other two sides by 7 feet, thereby producing a rectangle whose area is 90 square feet greater than that of the square. Find the side of the square.

18. Two opposite sides of a square are each decreased by 2 feet, the other two sides by 3 feet, thereby producing a rectangle whose area is 60 square feet less than that of the square. Find the side of the square.

19. Two opposite sides of a square are each increased by a feet and the other two by b feet, thereby producing a rectangle whose area is c square feet greater than that of the square. Find the side of the square.

20. A messenger starts for a distant point at 4 A.M., going 5 miles per hour. Four hours later another starts from the same place, going in the same direction at the rate of 9 miles per hour. When will they be together? When will they be 8 miles apart? How far apart will they be at 2 P.M.?

21. One object moves with a velocity of v_1 feet per second and another along the same path in the same direction with a velocity of v_2 feet. How long will it require the latter to gain n feet on the former?

Let $t =$ the required number of seconds. Then the first object moves v_1t feet and the second object moves v_2t feet in the required time.

Hence, $$v_2t = v_1t + n,$$

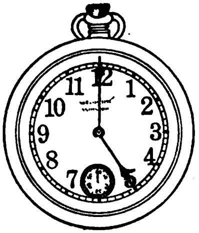
and $$t = \frac{n}{v_2 - v_1}.$$

Discussion. If $v_2 > v_1$ and $n > 0$, the value of t is positive, *i.e.* the objects will be in the required position some time *after* the time of starting.

If $v_2 < v_1$ and $n > 0$, the value of t is negative, which may be taken to mean that if the objects had been moving before the instant taken in the problem as the time of starting, then they would have been in the required position some time *earlier*.

If $v_2 = v_1$ and $n \neq 0$, the solution is impossible. See § 31. This means that the objects will never be in the required position.

22. State and solve a problem which is a special case of 21 under each of the conditions mentioned in the discussion.

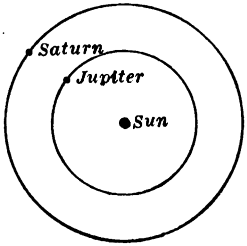


23. At what time after 5 o'clock are the hands of a clock first in a straight line?

24. How many minutes after 8 o'clock will the minute hand be 12 minute spaces behind the hour hand?

Two planets are said to be in conjunction when they are in a straight line with the sun. In the following problem conjunction means that the planets are on the *same side* of the sun.

25. Saturn completes its journey about the sun in 29 years and Jupiter in 12 years. How many years elapse from conjunction to conjunction?



26. An object moves in a fixed path at the rate of v_1 feet per second, and another which starts a seconds later moves in the same path at the rate of v_2 feet per second. In how many seconds will the latter overtake the former?

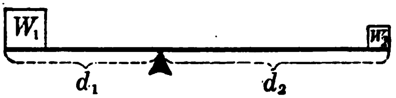
27. In problem 26 how long before they will be d feet apart?

If d is zero in problem 27, then it is the same problem as 26. If d is not zero and a is zero, it is the same as problem 21. Solve both of these problems, using the formula obtained in problem 27.

28. A, who weighs 75 pounds, sits 7 feet from the fulcrum. If B weighs 105 pounds, at what distance from the fulcrum should he sit in order to make a balance?

Suggestion. If the distances from the ends of a lever to the fulcrum are d_1 and d_2 and if the weights on the lever at its ends are w_1 and w_2, then

$$d_1 w_1 = d_2 w_2.$$



29. A and B together weigh $212\frac{1}{2}$ pounds. They balance when A is 6 feet, and B is $6\frac{3}{4}$ feet, from the fulcrum. Find the weight of each.

30. A lever 9 feet long carries weights of 17 and 32 pounds respectively at its ends. Where should the fulcrum be placed so as to make the lever balance?

31. A lever of unknown length is balanced when weights of 30 and 45 pounds respectively, are placed on it at opposite ends. Find the length of the lever, if the smaller weight is two feet farther from the fulcrum than the greater.

Suggestion. Let x be the distance from the greater weight to the fulcrum.

32. A beam carries 3 weights, one at each end weighing 100 and 120 pounds respectively, and the third weighing 150 pounds 2 feet from its center, where the fulcrum is. What is the length of the beam if this arrangement makes it balance?

33. A beam whose fulcrum is at its center is made to balance when weights of 60 and 80 pounds are placed at one end and 2 feet from that end respectively, and weights of 50 and 100 pounds are placed at the other end and 3 feet from it respectively. Find the length of the beam.

34. A man weighing 190 pounds is trying to pry up a rock by use of a plank 12 feet long and a block on which the plank rests as a fulcrum. Find the weight of the stone if he can just lift it when the fulcrum is 2 feet from the stone. How heavy a stone could he lift by putting the fulcrum 3 feet away from it? Four feet away from it?

35. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain 1 lap on the slower?

36. At what times between 12 o'clock and 6 o'clock are the hands of a watch together? (Find the time required to gain one circuit, two circuits, etc.)

37. The planet Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Starting in conjunction, how long before they will again be in this position?

CHAPTER VI

FRACTIONAL EQUATIONS IN ONE UNKNOWN

101. Clearing of Fractions. When an equation contains fractions, the first step in the solution is usually to *clear it of fractions.* This may be done by multiplying both members of the equation by such an expression as *will cancel all the denominators,* namely, by the least common multiple of all the denominators.

E.g. In the equation $\frac{x}{x+1} = \frac{2}{x-1} + \frac{x+3}{x-1}$, if we multiply both members by $x^2 - 1$, all the denominators can be canceled.

Thus, $$\frac{x\overset{x-1}{\cancel{(x^2-1)}}}{\cancel{x+1}} = \frac{2\overset{x+1}{\cancel{(x^2-1)}}}{\cancel{x-1}} + \frac{(x+3)\overset{x+1}{\cancel{(x^2-1)}}}{\cancel{x-1}}.$$

Hence, $$x(x-1) = 2(x+1) + (x+3)(x+1),$$

or $$\cancel{x^2} - x = 2x + 2 + \cancel{x^2} + 4x + 3.$$

Transposing, $$-7x = 5.$$

Hence, $$x = -\tfrac{5}{7}.$$

Check. Substituting $x = -\frac{5}{7}$, we get
$$-\tfrac{5}{2} = -\tfrac{14}{12} - \tfrac{16}{12} = -\tfrac{30}{12} = -\tfrac{5}{2}.$$

102. Checking Solutions. It does not always follow that the value of the unknown found by clearing of fractions is a *solution of the given equation.*

E.g. In the equation $\frac{x}{x-1} = \frac{1}{x-1} + \frac{x}{x+1}$, if we multiply both members by $x^2 - 1$, and cancel the denominators, we get
$$x(x+1) = (x+1) + x(x-1),$$

or $$\cancel{x^2} + x = x + 1 + \cancel{x^2} - x.$$

Hence, $$x = 1.$$

But $x = 1$, will not satisfy the given equation since it reduces the left member and also the first term in the right member to $\frac{1}{0}$, which is an impossible operation. (See § 31.) This equation, therefore, *has no solution.*

103. *It is therefore necessary to examine any result, obtained in solving an equation by a process which involves clearing of fractions, to see that no denominator in the given equation is reduced to zero when this result is substituted.*

We have, then, the following

Rule: To solve an equation containing fractions:

(1) *Reduce all fractions to their lowest terms.*

(2) *Multiply both members by the lowest common multiple of the denominators.*

(3) *Reject any root of the resulting equation which reduces any denominator of the given equation to zero. Any other solution thus obtained will satisfy the given equation.*

104. Fractional Equations Solved as Linear Equations. If when an equation is cleared of fractions, all terms cancel except those of the first degree, then the resulting equation is of the first degree, and may be solved by the methods heretofore used for such equations.

E.g. The two illustrations given on page 63 lead to first degree equations, since the terms containing x^2 cancel after clearing of fractions.

WRITTEN EXERCISES

Solve the following equations, rejecting all solutions which reduce any denominator to zero:

1. $\frac{4}{x+1}-\frac{3}{x-1}=\frac{-2}{x^2-1}$.
2. $\frac{1}{x}+\frac{2}{x-1}=\frac{8}{x(x-1)}$.
3. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}$.
4. $\frac{3}{x-3}-\frac{2}{x-2}=\frac{8}{4x^2-20x+24}$.
5. $\frac{1-2x}{3-4x}-\frac{5-6x}{7-8x}=\frac{8}{3}\cdot\frac{1-3x^2}{21-52x+32x^2}$.
6. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=14\cdot\frac{(2x-3)^2}{25-49x^2}$.

7. $$\frac{x-9}{x-5}-\frac{x-7}{x-2}-\frac{x-9}{x-4}=\frac{x-8}{x-5}-\frac{x-7}{x-4}-\frac{x-8}{x-2}.$$

Suggestion. First transpose and unite the fractions having a common denominator.

8. $$\frac{3x-4}{x+5}-\frac{4x-1}{x+4}=-\frac{x^2+44}{x^2+9x+20}.$$

9. $$\frac{1}{x-1}-\frac{2}{2x+1}=\frac{1}{x-2}-\frac{2}{2x-1}.$$

10. $$\frac{x-1}{x-2}+\frac{x-2}{3-x}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$$

11. $$\frac{7x+3}{5}-2=\frac{21x+9}{15}+\frac{17x-3}{3x+11}.$$

12. $$\frac{x-2}{x-3}-\frac{x-3}{x-2}=\frac{4x-9}{x^2-5x+6}.$$

13. $$\frac{x+1}{x+2}-\frac{x-1}{x+3}=\frac{2x+7}{x^2+5x+6}.$$

14. $$\frac{x-1}{x+3}-\frac{x+1}{x-3}=\frac{2x+1}{x^2-9}.$$

15. $$\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}.$$

16. $$\frac{1}{a-b}+\frac{a-b}{x}=\frac{1}{a+b}+\frac{a+b}{x}.$$

17. $$\frac{p^2x}{3(m+n)^2-px(m+n)}=\frac{p}{2(m+n)}.$$

18. $$\frac{m-q}{x-n}+\frac{n-p}{x-q}=\frac{m-q}{x-p}+\frac{n-p}{x-m}.$$

Transpose and add the fractions with the same numerators.

19. $$\frac{x+m}{x-m}-\frac{x-m}{x+m}=\frac{x+a}{x^2-m^2}.$$

20. $$\frac{3x-2}{2x+3}=\frac{x^2+15x+31}{2x^2+5x+3}+\frac{x-1}{x+1}.$$

21. $\dfrac{2x-3}{2x+2} - \dfrac{x-8}{5x+2} = \dfrac{x+2}{2x+2} + \dfrac{3}{10}.$

22. $\dfrac{18x^2+7x-3}{9x^2-1} - \dfrac{3x+1}{3x-1} = 1.$

23. $\dfrac{7x^2+11x+4}{6x^2+13x+5} + \dfrac{x+3}{2x+1} = \dfrac{5x+11}{3x+5}.$

24. $\dfrac{3x+1}{5x-7} - \dfrac{x-3}{2x-7} = \dfrac{x^2-10x+11}{10x^2-49x+49}.$

25. $a^2b - \dfrac{a+x}{b} = ab^2 - \dfrac{b+x}{a}.$

PROBLEMS LEADING TO FRACTIONAL EQUATIONS

1. Find a number such that, if it is added to each term of the fraction $\frac{3}{8}$ and subtracted from each term of the fraction $\frac{13}{24}$, the results will be equal.

2. Find a number such that, if it is added to each term of the fraction $\dfrac{a}{b}$ and subtracted from each term of the fraction $\dfrac{m}{n}$, the results will be equal.

3. Find a number of two digits in which the tens' digit is 5 greater than the units' digit, and such that, if the number is divided by the sum of its digits, the quotient is 7 and the remainder is 3.

4. In a number of three digits the tens' digit is 3 greater than the units' digit, and the hundreds' digit is 2 greater than the tens' digit. If the number is divided by the sum of the digits, the quotient is 50, and the remainder is 13. Find the number.

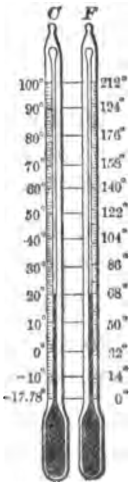
5. A pipe can fill a cistern in 6 hours, another in 9 hours, and a third can empty the cistern in 12 hours. How long will it take to fill the cistern when all three pipes are running?

6. A man can do a piece of work in 16 days, another in 18 days, and a third in 15 days. How many days will it require all to do it when working together?

7. *A* can do a piece of work in *a* days, *B* can do it in *b* days, *C* in *c* days, and *D* in *d* days. How long will it require all to do it when working together?

PROBLEMS ON THERMOMETER READINGS

There are two kinds of thermometers in use in this country, called the Fahrenheit and Centigrade, the former for common purposes, and the latter for scientific records and investigations. Hence, it frequently becomes necessary to translate readings from one kind to the other.



The freezing and boiling points are two fixed temperatures by means of which the computations are made. On the Centigrade these are marked 0° and 100° respectively and on the Fahrenheit they are marked 32° and 212° respectively. See the cut. Hence between the two fixed points there are 100 degrees Centigrade and 180 degrees Fahrenheit.

That is, 100 degree spaces on the Centigrade correspond to 180 degree spaces on the Fahrenheit.

Hence a change of 1° *Centigrade corresponds to* a change of $\frac{9}{5}$° *Fahrenheit*, and a change of 1° *Fahrenheit corresponds to* a change of $\frac{5}{9}$° *Centigrade.*

All problems comparing the two thermometers are solved by reference to these fundamental relations.

8. If the temperature falls 15 degrees Centigrade, how many degrees Fahrenheit does it fall?

9. If the temperature rises 18 degrees Fahrenheit, how many degrees Centigrade does it rise?

10. Translate $+ 25°$ Centigrade into Fahrenheit reading.

25° Centigrade equals $\frac{9}{5} \cdot 25° = 45°$ Fahrenheit.

45° above the freezing point $= 45° + 32°$ above 0° Fahrenheit. Hence, calling the Fahrenheit reading F, we have $F = 32 + \frac{9}{5} \cdot 25$.

11. Translate $+ 14°$ Centigrade into Fahrenheit reading.

Reasoning as before, $F = 32 + \frac{9}{5} \cdot 14$.

From the two preceding problems we have the formula

$$\mathbf{F = 32 + \tfrac{9}{5} C.} \qquad (1)$$

Translate this into words, understanding that F and C stand for readings on the respective thermometers.

12. Solve the above equation for C in terms of F and find

$$\mathbf{C = \tfrac{5}{9}(F - 32).} \qquad (2)$$

13. By substituting in the equation $F = 32 + \frac{9}{5} C$, find the Fahrenheit reading when the Centigrade is 60°; 100°; $-40°$ or the freezing point of mercury.

14. In like manner find the Centigrade reading when the Fahrenheit is $-40°$; $-20°$; 98° or blood heat; 212° or the boiling point.

15. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 102°?

16. What is the temperature Fahrenheit when the sum of the Centigrade and Fahrenheit readings is zero?

17. What is the temperature Centigrade when the sum of the Centigrade and Fahrenheit readings is 140°?

18. What is the temperature in each reading when the Fahrenheit is 50° higher than the Centigrade?

CHAPTER VII

EQUATIONS OF THE FIRST DEGREE IN TWO OR MORE UNKNOWNS

105. The degree of an integral equation in two or more letters is the sum of the exponents of those letters in that one of its terms in which this sum is greatest.

E.g. $y = 2x + 1$ is of the *first* degree in x and y. $y^2 = 2x + y$ and $y = 2xy + 3$ are each of the *second* degree in x and y.

106. An Indeterminate Equation. If a single equation contains two unknowns, an unlimited number of pairs of values may be found which satisfy the equation.

Thus, in the equation $y = 2x + 1$, if $x = 1$, $y = 2 \cdot 1 + 1 = 3$; and if $x = -2$, $y = 2(-2) + 1 = -3$.

Hence, $x = 1$, $y = 3$; and $x = -2$, $y = -3$ are pairs of numbers which satisfy this equation.

For this reason a single equation in two unknowns is called *an indeterminate equation*, and the unknowns are sometimes called *variables*. A solution of such an equation is any pair of numbers which satisfy it.

Solutions of the equation $y = 2x + 1$ may be found by assigning any value to x and computing the corresponding value of y. Thus, we may take $x = 0, 1, 2, 3, 4, 5, 6, 7$, etc., and find
$y = 1, 3, 5, 7, 9, 11, 13, 15$, etc.
or we may take $x = -1, -2, -3, -4, -5, -6, -7$, etc., and find
$y = -1, -3, -5, -7, -9, -11, -13$, etc.

107. Two Indeterminate Equations. In the case of two equations, each in two unknowns, there are in general *only a limited number of values* which satisfy both equations.

E.g. The equations $y = 2x + 1$ and $y = 5x - 2$ are both satisfied by $x = 1$, $y = 3$, and by *no other pair* of values of x and y.

INDEPENDENT AND DEPENDENT EQUATIONS

108. Simultaneous Equations. Two equations in two unknowns which are satisfied by the same pair of values of the unknowns are called *simultaneous, or consistent, equations.*

E.g. $x + y = 5$ and $x - y = 3$ are simultaneous, because both are satisfied by $x = 4$, $y = 1$.

109. Contradictory Equations. Two equations in two unknowns, which cannot be satisfied by the same pair of values of the unknowns, are called *contradictory, or inconsistent, equations.*

E.g. $x + y = 5$ and $x + y = 2$ are contradictory, since no pair of values of x and y can make their sum both 5 and 2 at the same time.

110. Dependent Equations. Two equations in two unknowns are said to be *dependent* if one can be derived from the other by any of the processes used in deriving equivalent equations (§ 97). In this case *every pair of values which satisfies one will also satisfy the other.*

E.g. $x + y = 5$ and $2x + 2y = 10$ are dependent, since the second may be derived from the first by multiplying both members by 2.

111. Independent Equations. Two equations in two unknowns are *independent* if neither can be derived from the other.

E.g. $x + y = 5$ and $x - y = 3$ are independent equations.

If two equations of the first degree in x and y are not only simultaneous, but also independent, they have one and only one pair of values of x and y which together satisfy both.

E.g. $y = 2x + 1$ and $y = 5x - 2$ are simultaneous and independent, and they are satisfied by $x = 1$, $y = 3$, and by no other pair of values.

112. Solution by Elimination. The solution of a pair of simultaneous and independent equations of the first degree in two unknowns may be obtained by a process called *elimination.*

This process consists in combining the equations in such a way as to derive a single equation containing only one of the unknowns.

113. Elimination by Addition or Subtraction. One of the unknowns may be eliminated by *addition or subtraction* when the coefficients of that unknown are equal numerically, or are made equal, in the two equations.

Example. Solve
$$\begin{cases} 2x + 3y = 7, & (1) \\ 5x - 2y = 8. & (2) \end{cases}$$

Solution. In this case the coefficients of neither x nor y are numerically equal in the two equations, but if we multiply equation (1) by 2, and equation (2) by 3, the coefficients of y will be made so.

Then we have
$$\begin{cases} 4x + 6y = 14, & (3) \\ 15x - 6y = 24. & (4) \end{cases}$$

If now we add equations (3) and (4), $6y$ and $-6y$ will cancel, and we have the equation
$$19x = 38, \qquad (5)$$
which contains only the variable x.

Dividing equation (5) by 19, we get $x = 2$.

Substituting $x = 2$ in equation (1), we find
$$2 \cdot 2 + 3y = 7,$$
from which $3y = 3$, and $y = 1$.

Hence, $\begin{cases} x = 2 \\ y = 1 \end{cases}$ is the solution of the pair of equations (1) and (2).

114. Elimination by Substitution. An equation of the first degree in two unknowns may always be solved for one of these unknowns in terms of the other. If the value of the unknown thus found in one equation be substituted in the other, then an equation is derived containing only one unknown. This is called *elimination by substitution.*

Example. Solve
$$\begin{cases} 2x + 3y = 7, & (1) \\ 5x - 2y = 8. & (2) \end{cases}$$

Solution. Solving equation (1) for y in terms of x, we get
$$y = \frac{7 - 2x}{3}. \qquad (3)$$

Substituting this value of y in equation (2),
$$5x - 2 \cdot \frac{7 - 2x}{3} = 8. \qquad (4)$$

Solving, $x = 2.$

Substituting $x = 2$ in equation (1), $y = 1.$

115. Elimination by Comparison. A third method of elimination consists in expressing the same unknown in terms of the other in each equation and equating these two expressions to each other. This is called *elimination by comparison.*

Example. Solve $\begin{cases} 3y + x = 14, & (1) \\ 2y - 5x = -19. & (2) \end{cases}$

Solving (1) for x, $x = 14 - 3y$. (3)

Solving (2) for x, $x = \frac{19 + 2y}{5}$. (4)

From (3) and (4), by axiom 5, § 96,

$$14 - 3y = \frac{19 + 2y}{5}. \quad (5)$$

Solving (5), $y = 3$.

Substituting in (1), $x = 5$.

Check by substituting $x = 5$, $y = 3$ in (2).

In applying any method of elimination it is desirable first to reduce each equation to the standard form:

$$ax + by = c.$$

ORAL EXERCISES

Solve the following pairs of equations by addition or subtraction:

1. $\begin{cases} x + y = 12, \\ x - y = 4. \end{cases}$

2. $\begin{cases} 2x + y = 16, \\ x - y = 5. \end{cases}$

3. $\begin{cases} 2x + 3y = 13, \\ -2x + 5y = 11. \end{cases}$

4. $\begin{cases} x - 2y = 1, \\ 3x + 2y = 19. \end{cases}$

5. $\begin{cases} 5x + 3y = 11, \\ 2x - 3y = -1 \end{cases}$

6. $\begin{cases} 3x - 2y = 25, \\ 5x - 2y = 39. \end{cases}$

7. $\begin{cases} -2x + 5y = -3, \\ 2x - y = -1. \end{cases}$

8. $\begin{cases} 3x + 5y = 11, \\ 7x + 5y = 19. \end{cases}$

9. $\begin{cases} 3x - 4y = 1, \\ 5x + 4y = 23. \end{cases}$

10. $\begin{cases} 4x - 3y = 8, \\ 2x - 3y = -2. \end{cases}$

WRITTEN EXERCISES

Solve the following pairs of equations by any one of the processes of elimination:

1. $\begin{cases} 3x + 2y = 118, \\ x + 5y = 191. \end{cases}$

2. $\begin{cases} 5x - 8\frac{1}{2} = 7y - 44, \\ 2x = y + \frac{5}{7}. \end{cases}$

3. $\begin{cases} 6x - 3y = 7, \\ 2x - 2y = 3. \end{cases}$

4. $\begin{cases} 3x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x, \\ 2\frac{1}{2}x + \frac{1}{2}y = 1. \end{cases}$

5. $\begin{cases} 5x - 11y - 2 = 4x, \\ 5x - 2y = 63. \end{cases}$

6. $\begin{cases} 3y + 40 = 2x + 14, \\ 9y - 347 = 5x - 420. \end{cases}$

7. $\begin{cases} 5y - 3x + 8 = 4y + 2x + 7, \\ 4x - 2y = 3y + 2. \end{cases}$

8. $\begin{cases} 6y - 5x = 5x + 14, \\ 3y - 2x - 6 = 5 + x. \end{cases}$

9. $\begin{cases} (x+5)(y+7) = (x+1)(y-9) + 112, \\ 2x + 10 = 3y + 1. \end{cases}$

10. $\begin{cases} 73 - 7y = 5x, \\ 2y - 3x = 12. \end{cases}$

11. $\begin{cases} ax = by, \\ x + y = c. \end{cases}$

12. $\begin{cases} \frac{x}{a} = \frac{y}{b}, \\ x + y = s. \end{cases}$

13. $\begin{cases} x + y = a, \\ x - y = b. \end{cases}$

14. $\begin{cases} ax + by = c, \\ fx + gy = h. \end{cases}$

15. $\begin{cases} \frac{3}{x} - \frac{5}{y} = 6, \\ \frac{2}{x} + \frac{3}{y} = 2. \end{cases}$

16. $\begin{cases} \frac{a}{x} + \frac{b}{y} = c, \\ \frac{f}{x} + \frac{g}{y} = h. \end{cases}$

Note.—In examples 15 and 16, consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns.

17. $\begin{cases} a(x + y) - b(x - y) = 2a, \\ a(x - y) - b(x + y) = 2b. \end{cases}$

18. $\begin{cases} (k + 1)x + (k - 2)y = 3a, \\ (k + 3)x + (k - 4)y = 7a. \end{cases}$

19. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

20. $\begin{cases} (a + b)x - (a - b)y = 4ab, \\ (a - b)x + (a + b)y = 2a^2 - 2b^2. \end{cases}$

21. $\begin{cases} \frac{1}{2}(a-b)-\frac{1}{5}(a-3\,b)=b-3, \\ \frac{3}{4}(a-b)+\frac{5}{6}(a+b)=18. \end{cases}$

22. $\begin{cases} a(x+y)+b(x-y)=2, \\ a^2(x+y)-b^2(x-y)=a-b. \end{cases}$

23. $\begin{cases} 7(x-5)=3-\dfrac{y}{2}-x, \\ \frac{1}{4}(x-y)+\frac{1}{2}\,y-\frac{5}{3}(x-1)=-1. \end{cases}$

24. $\begin{cases} mx+ny=m^3+2\,m^2n+n^3, \\ nx+my=m^3+2\,mn^2+n^3. \end{cases}$

25. $\begin{cases} (m+n)x-(m-n)y=2\,lm, \\ (m+l)x-(m-l)y=2\,mn. \end{cases}$

26. $\begin{cases} (a-b)x+(a+b)y=2\,a, \\ (a+b)x+(a-b)y=2\,b. \end{cases}$

27. $\begin{cases} (h+k)x+(h-k)y=2(h^2+k^2), \\ (h-k)x+(h+k)y=2(h^2-k^2). \end{cases}$

28. $\begin{cases} (a-b)x+y(a^2+b^2)=(a+b)^2+a+b-2\,ab, \\ (b-a)x+y(a^2+b^2)=a+b-a^2-b^2. \end{cases}$

In examples 29 to 34 solve without clearing of fractions.

29. $\begin{cases} \dfrac{5}{t}-\dfrac{6}{v}=2, \\ \dfrac{17}{v}+\dfrac{4}{t}=67. \end{cases}$

30. $\begin{cases} \dfrac{3}{x}+\dfrac{1}{y}=21, \\ \dfrac{7}{x}-\dfrac{9}{y}=-19. \end{cases}$

31. $\begin{cases} \dfrac{7}{a}-\dfrac{1}{b}=12\tfrac{1}{2}, \\ \dfrac{3}{a}+\dfrac{12}{b}=24. \end{cases}$

32. $\begin{cases} \dfrac{3}{x}-\dfrac{2}{y}=-4, \\ \dfrac{6}{x}+\dfrac{11}{y}=52. \end{cases}$

33. $\begin{cases} \dfrac{12}{x}-\dfrac{10}{y}=7, \\ \dfrac{9}{x}+\dfrac{2}{y}=10. \end{cases}$

34. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=1, \\ \dfrac{c}{x}+\dfrac{d}{y}=1. \end{cases}$

SYSTEMS OF EQUATIONS IN MORE THAN TWO UNKNOWNS

116. Systems of Equations. Two or more equations involving the same unknowns form, when taken together, a *system of equations.* A system containing as many unknowns as equations may be *simultaneous* or *contradictory, independent* or *dependent,* in the same sense as explained in §§ 108–111 for two equations in two unknowns.

117. Indeterminate Equations. If a *single* equation of the first degree in three or more variables is given, there is no limit to the number of sets of values which satisfy it.

E.g. $3x + 2y + 4z = 24$ is satisfied by $x = 1$, $y = 3$, $z = 3\frac{3}{4}$; $x = 2$, $y = 2$, $z = 3\frac{1}{2}$; $x = 0$, $y = 0$, $z = 6$; etc.

If *two* equations of the first degree in three or more variables are given, they have in general an unlimited number of solutions.

E.g. $3x + 2y + 4z = 24$ and $x + y + z = 6$ are both satisfied by $x = 2$, $y = -1$, $z = 5$; $x = 3$, $y = -1\frac{1}{2}$, $z = 4\frac{1}{2}$; etc.

But if a system of equations of the first degree contains *as many equations as variables,* it has in general one and only one set of values which together satisfy all the equations.

E.g. The system
$$\begin{cases} x + y + z = 6, & (1) \\ 3x - y + 2z = 7, & (2) \\ 2x + 3y - z = 5 & (3) \end{cases}$$
is satisfied by $x = 1$, $y = 2$, $z = 3$, and by no other set of values.

This is the case when the system is *independent* and *simultaneous.*

ORAL EXERCISES

1. If equations (1) and (2) in the above example are added, will the resulting equation contain y?

2. If equation (1) is multiplied by 2 and equation (2) subtracted from it, will the resulting equation contain y? Will it contain z?

3. How would you eliminate z from equations (2) and (3)?

4. How would you eliminate y from equations (2) and (3)?

SOLUTION OF EQUATIONS IN THREE UNKNOWNS

118. Solution of a System of three Equations. An independent and simultaneous system of equations of the first degree in three unknowns may be solved as follows:

From two of the equations, say the 1st and 2d, eliminate one of the unknowns, obtaining one equation in the remaining two unknowns.

From the 1st and 3d equations, or from the 2d and 3d, eliminate the same unknown, obtaining a second equation in the remaining two unknowns.

Solve as usual the two equations in two unknowns thus found. Substitute the values of these two unknowns in one of the given equations, and thus find the value of the third unknown.

The process of elimination by addition or subtraction is usually most convenient.

Example. Solve the system:

$$\begin{cases} x + y + z = 9, & (1) \\ 2x + 3y + z = 17, & (2) \\ x + 2y + 2z = 16. & (3) \end{cases}$$

Solution. If we subtract equation (1) from equation (3), we get an equation containing only y and z,

$$y + z = 7. \qquad (4)$$

If we multiply equation (3) by 2, and then subtract equation (2) from it, we get a second equation containing only y and z,

$$y + 3z = 15. \qquad (5)$$

Hence, we now have the system

$$\begin{cases} y + z = 7, \\ y + 3z = 15, \end{cases}$$

from which we solve, as in § 113, obtaining the values of y and z,

$$y = 3, \ z = 4.$$

Substituting these values of y and z in equation (1), we find

$$x = 2.$$

Hence, the solution of the system is

$$\begin{cases} x = 2, \\ y = 3, \\ z = 4. \end{cases}$$

We may also find the solution by first eliminating y, using (1) and (2), and then using (2) and (3), getting two equations in x and z, from which the values of x and z can be found.

Evidently z could have been eliminated first, using equations (1) and (2) and then (1) and (3), giving a new set of two equations in y and x. Let the student find the solution in this manner.

EXERCISES

Solve each of the following systems of equations:

1. $\left\{\begin{array}{l} 2x + 5y - 7z = 9, \\ 5x - y + 3z = 16, \\ 7x + 6y + z = 34. \end{array}\right.$

2. $\left\{\begin{array}{l} a + b + c = 9, \\ 8a + 4b + 2c = 36, \\ 27a + 9b + 3c = 93. \end{array}\right.$

3. $\left\{\begin{array}{l} 18l - 7m - 5n = 161, \\ 4\frac{2}{5}m - \frac{2}{3}l + n = 18, \\ 3\frac{1}{2}n + 2m + \frac{3}{4}l = 33. \end{array}\right.$

4. $\left\{\begin{array}{l} \frac{a}{3} + \frac{b}{6} + \frac{c}{9} = -2, \\ \frac{a}{6} + \frac{b}{9} + \frac{c}{12} = -4, \\ \frac{a}{9} + \frac{b}{12} + \frac{c}{15} = -4. \end{array}\right.$

5. $\left\{\begin{array}{l} 8z - 3y + x = -2, \\ 3x - 5y - 6z = -46, \\ y + 5x - 4z = -18. \end{array}\right.$

6.* $\left\{\begin{array}{l} \frac{3}{a} = \frac{2}{b}, \\ \frac{2}{a} + \frac{5}{b} - \frac{4}{c} = 17, \\ \frac{7}{a} - \frac{3}{b} + \frac{6}{c} = 8. \end{array}\right.$

7. $\left\{\begin{array}{l} x + 2y - 3z = -3, \\ 2x - 3y + z = 8, \\ 5x - 4y - 7z = -5. \end{array}\right.$

8. $\left\{\begin{array}{l} 2x + 3y - 7z = 19, \\ 5x + 8y + 11z = 24, \\ 7x + 11y + 4z = 43. \end{array}\right.$

Show that this system is not independent.

9. $\left\{\begin{array}{l} x + y = 16, \\ z + x = 22, \\ y + z = 28. \end{array}\right.$

10. $\left\{\begin{array}{l} x + 2y = 26, \\ 3x + 4z = 56, \\ 5y + 6z = 65. \end{array}\right.$

11. $\left\{\begin{array}{l} a + b + c = 5, \\ 3a - 5b + 7c = 79, \\ 9a - 11b = 91. \end{array}\right.$

12. $\left\{\begin{array}{l} l + m + n = 29\frac{1}{4}, \\ l + m - n = 18\frac{1}{4}, \\ l - m + n = 13\frac{3}{4}. \end{array}\right.$

13. $\left\{\begin{array}{l} l + m + n = a, \\ l + m - n = b, \\ l - m + n = c. \end{array}\right.$

14. $\left\{\begin{array}{l} ax + by = p, \\ cy + dz = q, \\ ex + fz = r. \end{array}\right.$

* Use $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ as the unknowns.

15. $$\begin{cases} \frac{1}{a}+\frac{1}{b}=4, \\ \frac{1}{a}+\frac{1}{c}=3, \\ \frac{1}{b}+\frac{1}{c}=2. \end{cases}$$

16. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c. \end{cases}$$

17. $$\begin{cases} u+v+x+y=10, \\ 2u-3v+4x-5y=-12, \\ 3u+4v-5x+6y=20, \\ 4u+5v+6x-7y=4. \end{cases}$$

18. Make a rule for solving a system of four or more linear equations in as many unknowns as equations.

PROBLEMS INVOLVING TWO OR MORE UNKNOWNS

1. A man invests a certain amount of money at 4 % interest and another amount at 5 %, thereby obtaining an annual income of $3100. If the first amount had been invested at 5 % and the second at 4 %, the income would have been $3200. Find each investment.

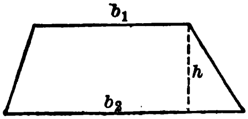
2. The relation between the readings of the Centigrade and the Fahrenheit thermometers is given on page 68 by the equation $F = 32 + \frac{9}{5}C$. The Fahrenheit reading at the melting temperature of tungsten is 2432 degrees higher than the Centigrade. Find the melting temperature in each scale.

3. Going with a current a steamer makes 19 miles per hour, while going against a current $\frac{3}{4}$ as strong it makes 5 miles per hour. Find the speed of each current and the boat.

4. There is a number consisting of 3 digits whose sum is 11. If the digits are written in reverse order, the resulting number is 594 less than the original number. Three times the tens' digit is one more than the sum of the hundreds' and the units' digits.

5. A certain kind of wine contains 20 % alcohol and another kind contains 28 %. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6 % alcohol?

6. The area of a certain trapezoid of altitude 8 is 68. If 4 is added to the lower base and the upper base is doubled, the area is 108. Find both bases.



A trapezoid is a four-sided figure whose upper base, b_1, and lower base, b_2, are parallel, but the other two sides are not. If h is the perpendicular distance between the bases, then the area is $a = \frac{h}{2}(b_1 + b_2)$.

7. The Centigrade reading at the boiling point of alcohol is 96° lower than the Fahrenheit reading. Find both the Centigrade and the Fahrenheit readings at this temperature.

Use C and F as the unknowns. Then one of the equations is the formula connecting Fahrenheit and Centigrade readings obtained on page 68 and the other is $C + 96 = F$.

8. The Fahrenheit reading at the temperature of liquid air is 128 degrees lower than the Centigrade reading. Find both the Centigrade and the Fahrenheit readings at this temperature.

9. The Centigrade reading at the melting point of silver is 796° lower than the Fahrenheit reading. Find both Centigrade and Fahrenheit readings at this temperature.

10. The Fahrenheit reading at the melting point of gold is 992° higher than the Centigrade reading. Find both Centigrade and Fahrenheit readings at this temperature.

11. The upper base of a trapezoid is 6 and its area is 168. If $\frac{1}{8}$ the lower base is added to the upper, the area is 210. Find the altitude and the lower base.

12. A and B can do a piece of work in 18 days, B and C in 24 days, and C and A in 36 days. How long will it require each man to do it working alone, and how long will it require all working together?

13. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. How long will it require each to do it working alone?

14. A beam resting on a fulcrum balances when it carries weights of 100 and 130 pounds at its respective ends. The beam will also balance if it carries weights of 80 and 110 pounds respectively, 2 feet from the ends, provided the fulcrum remains in the same place. Find the distance from the fulcrum to the ends of the beam.

15. A beam carries three weights, A, B, C. A balance is obtained when A is 12 feet from the fulcrum, B is 8 feet from the fulcrum on the same side as A, and C is 20 feet from the fulcrum on the side opposite A. It also balances when the distance of A is 8 feet, B is 10 feet, and C is 18 feet. Find the weights B and C if A is 50 pounds.

16. At 0° Centigrade sound travels 1115 feet per second with the wind on a certain day, and 1065 feet per second against the wind. Find the velocity of sound in calm weather, and the velocity of the wind on this occasion.

17. If the velocities of sound in air, brass, and iron at 0° Centigrade are x, y, z meters per second respectively, then $3x + 2y - z = 2505$, $5x - 2y + z = 151$, and $x + y + z = 8777$. Find the velocity in each.

18. If x, y, z are the Centigrade readings at the temperatures which liquefy hydrogen, nitrogen, and oxygen respectively, then $3x - 8y + 2z = 440$, $-8x + 2y + 4z = 903$, and $x + 4y - 6z = 60$. Find each temperature in both Centigrade and Fahrenheit readings.

19. If x, y, z are the Centigrade readings at the freezing temperatures of hydrogen, nitrogen, and oxygen respectively, then we have $x + y - 3z = 199$, $2x - 5y + z = 328$, and $-4x + 2y + 2z = 156$. Find each temperature.

20. If x, y, z are, respectively, the melting point of carbon, the temperature of the hydrogen flame in air, and the temperature of this flame in pure oxygen, then $10x + 2y + z = 41892$, $15x + y + 2z = 60212$, and $7x + y + z = 29368$. Find each.

21. Two boys carry a 120-pound weight by means of a pole, at a certain point on which the weight is hung. One boy holds the pole 5 ft. from the weight and the other 3 ft. from it. What part of the weight does each boy lift?

Suggestion. Let x and y be the required amounts, then $5x$ is the leverage of the first boy and $3y$ that of the second, and these must be equal as in the case of the teeter. Hence, we have

$$5x = 3y, \text{ and } x + y = 120.$$

22. If, in problem 21, the boys lift P_1 and P_2 pounds respectively, at distances d_1 and d_2, and w is the weight lifted, then

$$P_1d_1 = P_2d_2. \qquad (1)$$

$$P_1 + P_2 = w. \qquad (2)$$

Solve (1) and (2), (*a*) when P_1 and P_2 are unknown, (*b*) when P_1 and w are unknown, (*c*) when P_1 and d_2 are unknown.

23. A weight of 540 pounds is carried on a pole by two men at distances of 4 and 5 feet respectively, from the weight. How much does each lift?

24. There is a number consisting of three digits such that the sum of the hundreds' and units' digits is 7. If the order of the digits is reversed the number is increased by 297 and if the tens' and hundreds' digits are interchanged the number is increased by 450. Find the number.

CHAPTER VIII

SYSTEMS OF EQUATIONS SOLVED BY DETERMINANTS

SYSTEMS OF TWO EQUATIONS

119. Solution of Equations with Numerical Coefficients. We now proceed to a more general study of the solution of a pair of first-degree equations in two unknowns.

Example 1. Solve

$$\begin{cases} 2x + 3y = 4, & (1) \\ 5x + 6y = 7. & (2) \end{cases}$$

Solution. Multiplying (1) by 5 and (2) by 2,

$$\begin{cases} 5 \cdot 2x + 5 \cdot 3y = 5 \cdot 4, & (3) \\ 2 \cdot 5x + 2 \cdot 6y = 2 \cdot 7. & (4) \end{cases}$$

Subtracting (3) from (4), $(2 \cdot 6 - 5 \cdot 3)y = 2 \cdot 7 - 5 \cdot 4.$ (5)

Solving for y, $y = \dfrac{2 \cdot 7 - 5 \cdot 4}{2 \cdot 6 - 5 \cdot 3} = \dfrac{-6}{-3} = 2.$ (6)

In like manner, solve for x by multiplying (1) by 6 and (2) by 3 and subtracting. We then have

$$= \frac{4 \cdot 6 - 7 \cdot 3}{2 \cdot 6 - 5 \cdot 3} = \frac{3}{-3} = -1. \qquad (7)$$

The expressions, $x = \dfrac{4 \cdot 6 - 7 \cdot 3}{2 \cdot 6 - 5 \cdot 3}$ and $y = \dfrac{2 \cdot 7 - 5 \cdot 4}{2 \cdot 6 - 5 \cdot 3}$ show exactly how each coefficient in the original equations enters into the solution, while $x = -1$, $y = 2$ give no information on this point.

Example 2. In this manner, solving,

$$\begin{cases} 7x + 9y = 71, \\ 2x + 3y = 48, \end{cases}$$

we find $x = \dfrac{71 \cdot 3 - 48 \cdot 9}{7 \cdot 3 - 2 \cdot 9}$ and $y = \dfrac{7 \cdot 48 - 2 \cdot 71}{7 \cdot 3 - 2 \cdot 9}.$

In Example 2, the various coefficients are found to occupy the same relative positions in the expressions for x and y as do the corresponding coefficients in Example 1.

120. Solution of General Equations. A convenient rule for reading directly the values of the unknowns in such pairs of equations as the above may be made from the solution of the following *general equations:*

Example 3. Solve

$$\begin{cases} a_1x + b_1y = c_1, & (1) \\ a_2x + b_2y = c_2. & (2) \end{cases}$$

Solution. To eliminate y, multiply equation (1) by b_2 and equation (2) by b_1, getting

$$\begin{cases} a_1b_2x + b_1b_2y = c_1b_2, & (3) \\ a_2b_1x + b_1b_2y = c_2b_1. & (4) \end{cases}$$

Subtracting equation (4) from (3),

$$(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1.$$

Dividing by the coefficient of x,

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}. \qquad (5)$$

Again, to eliminate x multiply equation (1) by a_2, and equation (2) by a_1, and subtract (2) from (1), getting

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \qquad (6)$$

121. Rule for Reading the Results. To remember these results, notice that the coefficients of x and y in the given equation stand in the form of a square, thus

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

and that the denominator in the expressions for both x and y is the diagonal a_1b_2 of this square, minus the diagonal a_2b_1.

The numerator in the expression for x is formed by replacing the a's in this square by the c's, thus,

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

and then reading the diagonals as before obtaining $c_1b_2 - c_2b_1$.

The numerator for y is formed by replacing the b's by the c's, thus

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

and then reading the diagonals, obtaining $a_1c_2 - a_2c_1$.

122. Determinants of the Second Order. A square array like $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ with two numbers on each side, is called a ***determinant of the second order*** and its value is given by the equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \qquad (A)$$

Thus, $\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$ means $4 \cdot 1 - 3 \cdot 2 = -2$.

123. Determinant Formulas. Since any pair of equations of the first degree in two unknowns may be reduced to the *standard form* as given in Example 3, page 83, it follows that the values of x and y there obtained constitute a formula for the solution of any pair of such equations.

Example 4. Solve by determinants:

$$\begin{cases} 5x = 8 + 3y, & (1) \\ 7y = 19 - 2x. & (2) \end{cases}$$

Solution. Putting these equations in the standard form,

$$\begin{cases} 5x - 3y = 8, & (3) \\ 2x + 7y = 19. & (4) \end{cases}$$

The determinant of the coefficients is

$$\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix} = 5 \cdot 7 - (-3) \cdot 2 = 35 + 6 = 41.$$

This determinant with the coefficients of x replaced by the known terms is

$$\begin{vmatrix} 8 & -3 \\ 19 & 7 \end{vmatrix} = 8 \cdot 7 - (-3) \cdot 19 = 56 + 57 = 113.$$

This determinant with the coefficients of y replaced by the known terms is

$$\begin{vmatrix} 5 & 8 \\ 2 & 19 \end{vmatrix} = 5 \cdot 19 - 2 \cdot 8 = 95 - 16 = 79.$$

Hence, the solution is

$$x = \frac{\begin{vmatrix} 8 & -3 \\ 19 & 7 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix}} = \frac{113}{41}, \text{ and } y = \frac{\begin{vmatrix} 5 & 8 \\ 2 & 19 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 2 & 7 \end{vmatrix}} = \frac{79}{41}.$$

124. Rule for Solving two Equations of the first Degree by Determinants. These examples lead to the following rule:

(1) *Reduce the equations to the standard form*

$$a_1x + b_1y = c_1,$$

and

$$a_2x + b_2y = c_2$$

(2) *Form the determinant of the coefficients*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(3) *Find the determinant obtained by replacing the a's by the c's, and also the one obtained by replacing the b's by c's,* i.e.,

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(4) *Write the values of x and y in the form*

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

(5) *Evaluate these determinants thus:*

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

In practice the values of x and y are written down at once as in (4), omitting step (3).

The solution by means of determinants is especially advantageous in case of systems of literal equations.

ORAL EXERCISES

In each of the following give the determinant of the coefficients, the x-numerator, and the y-numerator:

1. $\begin{cases} 3x + 2y = 4, \\ 5x + 3y = 9. \end{cases}$

2. $\begin{cases} 7x + y = 7, \\ 3x + 2y = 14. \end{cases}$

3. $\begin{cases} 4x - y = 12, \\ x - 5y = 3. \end{cases}$

4. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 7. \end{cases}$

WRITTEN EXERCISES

Solve the following system by means of determinants.

1. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

2. $\begin{cases} 4x - 5y = -26, \\ 2x - 3y = -14. \end{cases}$

3. $\begin{cases} 6y - 17 = -5x, \\ 6x - 16 = -5y. \end{cases}$

4. $\begin{cases} \frac{1}{4}(x-3) = -\frac{1}{3}(y-2) + \frac{1}{2}x, \\ \frac{1}{2}(y-1) = x - \frac{1}{3}(x-2). \end{cases}$

5. $\begin{cases} 2x - y = 53, \\ 19x + 17y = 0. \end{cases}$

6. $\begin{cases} ax - by = 0, \\ x - y = c. \end{cases}$

7. $\begin{cases} mx + ny = p, \\ rx + sy = t. \end{cases}$

8. $\begin{cases} x(a-b) + y(a+b) = 2a, \\ x(a+b) - y(a-b) = 2b. \end{cases}$

9. $\begin{cases} (k+1)x + (k-2)y = 3a, \\ (k+3)x + (k-4)y = 7a. \end{cases}$

10. $\begin{cases} 2ax + 2by = 4a^2 + b^2, \\ x - 2y = 2a - b. \end{cases}$

11. $\begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a-b)x + (a+b)y = 2a^2 - 2b^2. \end{cases}$

12. $\begin{cases} mx + ny = m^3 + 2m^2n + n^3, \\ nx + my = m^3 + 2mn^2 + n^3. \end{cases}$

13. $\begin{cases} (m+n)x - (m-n)y = 2lm, \\ (m+l)x - (m-l)y = 2mn. \end{cases}$

14. $\begin{cases} (a-b)x + (a+b)y = 2a, \\ (a-b)x - (a+b)y = 2b. \end{cases}$

15. $\begin{cases} (h+k)x + (h-k)y = 2(h^2 + k^2), \\ (h-k)x + (h+k)y = 2(h^2 - k^2). \end{cases}$

16. $\begin{cases} (a-b)x + y(a^2 + b^2) = a, \\ (b-a)x + y(a^2 + b^2) = b. \end{cases}$

17. $\begin{cases} a(x+y) + b(x-y) = 2, \\ a^2(x+y) - b^2(x-y) = a - b. \end{cases}$

18. $\begin{cases} 7(x-5) = 3 - \dfrac{y}{2} - x, \\ \frac{1}{4}(x-y) + \frac{1}{2}y - \frac{5}{3}(x-1) = -1. \end{cases}$

19. $\begin{cases} \frac{1}{2}(a-b) - \frac{1}{5}(a-3b) = b - 3, \\ \frac{3}{4}(a-b) + \frac{5}{6}(a+b) = 18. \end{cases}$

SYSTEMS OF THREE EQUATIONS

125. Solution by Determinants. Systems of three equations of the first degree in three unknowns may be solved by determinants in a manner similar to the foregoing solution of systems of two equations.

Example. Solve the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3. & (3) \end{cases}$$

Solution. Multiply equation (1) by c_2 and equation (2) by c_1, and subtract (2) from (1), thus eliminating z from these equations. Then multiply equation (1) by c_3, and equation (3) by c_1, and subtract (3) from (1), thus eliminating z from these equations.

We then have the system of two equations in x and y.

$$(a_1c_2 - a_2c_1)x + (b_1c_2 - b_2c_1)y = d_1c_2 - d_2c_1, \qquad (4)$$
$$(a_1c_3 - a_3c_1)x + (b_1c_3 - b_3c_1)y = d_1c_3 - d_3c_1. \qquad (5)$$

Solving this system of two equations by determinants, according to the rule (§ 124) we have for the value of x,

$$x = \frac{\begin{vmatrix} d_1c_2 - d_2c_1 & b_1c_2 - b_2c_1 \\ d_1c_3 - d_3c_1 & b_1c_3 - b_3c_1 \end{vmatrix}}{\begin{vmatrix} a_1c_2 - a_2c_1 & b_1c_2 - b_2c_1 \\ a_1c_3 - a_3c_1 & b_1c_3 - b_3c_1 \end{vmatrix}},$$

$$\text{or } x = \frac{(d_1c_2 - d_2c_1)(b_1c_3 - b_3c_1) - (d_1c_3 - d_3c_1)(b_1c_2 - b_2c_1)}{(a_1c_2 - a_2c_1)(b_1c_3 - b_3c_1) - (a_1c_3 - a_3c_1)(b_1c_2 - b_2c_1)}.$$

Performing the multiplications, we have for the value of x

$$\frac{\cancel{d_1b_1c_2c_3} - d_1b_3c_1c_2 - d_2b_1c_1c_3 + d_2b_3c_1^2 \cancel{- d_1b_1c_2c_3} + d_3b_1c_1c_2 + d_1b_2c_1c_3 - d_3b_2c_1^2}{\cancel{a_1b_1c_2c_3} - a_1b_3c_1c_2 - a_2b_1c_1c_3 + a_2b_3c_1^2 \cancel{- a_1b_1c_2c_3} + a_3b_1c_1c_2 + a_1b_2c_1c_3 - a_3b_2c_1^2}.$$

After cancelling $d_1b_1c_2c_3$ and $-d_1b_1c_2c_3$ in the numerator and $a_1b_1c_2c_3$ and $-a_1b_1c_2c_3$ in the denominator, we find that c_1 is a factor of every term in both numerator and denominator. Dividing this out, and writing the positive terms first, we have

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

126. Expressing the Denominator as a Determinant. The denominator of the fraction in the above value of x contains the nine coefficients of x, y, and z in the given system of equations. If these be written in the form of a square array, thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then the terms of the denominator may be read from this array as follows:

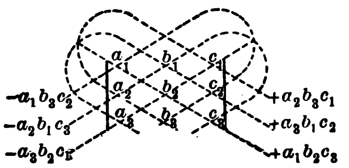
The Positive Terms. (1) *Take the diagonal* $a_1b_2c_3$.

(2) *Take the line* b_1c_2 *parallel to this diagonal, together with* a_3 *in the opposite corner, getting* $a_3b_1c_2$.

(3) *Take the line* a_2b_3 *also parallel to this diagonal, together with* c_1 *in the opposite corner, getting* $a_2b_3c_1$.

These are the positive terms: $a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1$.

The Negative Terms. (1) *Take the diagonal* $a_3b_2c_1$.



(2) *Take the line* a_2b_1 *parallel to this diagonal, together with* c_3 *in the opposite corner, getting* $a_2b_1c_3$.

(3) *Take the line* b_3c_2 *also parallel to this diagonal, together with* a_1 *in the opposite corner, getting* $a_1b_3c_2$.

These are the negative terms: $-a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$.

127. Determinant of the Third Order. A square array like the above is called a determinant of the third order. Its value is given by the equation:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This is called the *determinant of the system* of equations in the example on page 87.

The scheme for reading it is shown in the above diagram.

128. Expressing the Numerators as Determinants. The numerator in the value of x on page 87 is exactly the same polynomial as the denominator with the a's replaced by the d's. Hence, this may also be written as a determinant of the third order by replacing the a's by the d's. Thus,

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2.$$

This is called the **x-determinant.** Similarly the **y-determinant** and **z-determinant** are formed by replacing the b's and c's in turn by the d's.

Hence we have,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_2b_1c_3 - d_1b_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

If we solve the given system for y and z, in a manner similar to the above solution for x, we should find:

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_2d_1c_3 - a_1d_3c_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2},$$

and

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_2b_1d_3 - a_1b_3d_2}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2}.$$

129. Determinant Formulas. The determinant values of x, y, and z may be used as formulas for solving any set of first degree equations in three unknowns. The formulas are directly applicable when the equations are reduced to the standard form (see § 125).

Example. By means of determinants solve the system:

$$\begin{aligned} y + z &= 9 - x, \\ 2x + 3y &= 17 - z, \\ 2z + x &= 16 - 2y. \end{aligned}$$

Solution. Putting the equations in the standard form, we have

$$\begin{aligned} x + y + z &= 9, \\ 2x + 3y + z &= 17, \\ x + 2y + 2z &= 16. \end{aligned}$$

(1) The *determinant of the coefficients* is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 6 + 1 + 4 - 3 - 4 - 2 = 2.$$

(2) The x-, y-, and *z-determinants* are formed by replacing the first, second, and third columns respectively, by the known terms, obtaining

$$\begin{vmatrix} 9 & 1 & 1 \\ 17 & 3 & 1 \\ 16 & 2 & 2 \end{vmatrix} = 54 + 16 + 34 - 48 - 34 - 18 = 4.$$

$$\begin{vmatrix} 1 & 9 & 1 \\ 2 & 17 & 1 \\ 1 & 16 & 2 \end{vmatrix} = 34 + 9 + 32 - 17 - 16 - 36 = 6.$$

$$\begin{vmatrix} 1 & 1 & 9 \\ 2 & 3 & 17 \\ 1 & 2 & 16 \end{vmatrix} = 48 + 17 + 36 - 27 - 34 - 32 = 8.$$

Hence,

$$x = \frac{\begin{vmatrix} 9 & 1 & 1 \\ 17 & 3 & 1 \\ 16 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 2, \quad y = \frac{\begin{vmatrix} 1 & 9 & 1 \\ 2 & 17 & 1 \\ 1 & 16 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 3, \quad z = \frac{\begin{vmatrix} 1 & 1 & 9 \\ 2 & 3 & 17 \\ 1 & 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix}} = 4.$$

The complete solution is, then, $x = 2$, $y = 3$, $z = 4$.

130. Rule for Solving by Determinants. To solve a system of three equations of the first degree in three unknowns by means of determinants, we have the following rule:

(1) *Write the equations in the standard form,*

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

(2) *Form the determinant of the coefficients,* $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(3) *Form the* **x**-, **y**-, *and* **z**- *determinants by replacing in the determinant of the coefficients the* **a**'s, **b**'s, **c**'s, *respectively, by the* **d**'s,

$$\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

(4) *The solution is then*

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

(5) *Evaluate the determinants as shown on page 88.*

In practice, the values of x, y, z are written down at once as in (4), thus omitting steps (2) and (3).

It may seem to the student at first that this method of solution is longer than by ordinary elimination; but when facility is gained in evaluating determinants, it will be found to be a much more speedy and satisfactory method.

Determinants are used in solving first degree equations in four, five, or any number of unknowns, and are among the most useful and far-reaching devices of algebra.

ORAL EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 9 & -1 \\ 3 & -8 \end{vmatrix}$.

4. $\begin{vmatrix} 8 & 6 \\ -4 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} -3 & -4 \\ 6 & -7 \end{vmatrix}$.

6. $\begin{vmatrix} a & -b \\ a & b \end{vmatrix}$.

7. $\begin{vmatrix} x & 7 \\ 7^2 & x^2 \end{vmatrix}$.

8. $\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$.

9. $\begin{vmatrix} 3a & -2b \\ 4b & -5a \end{vmatrix}$.

10. $\begin{vmatrix} 6x & -2x^2 \\ -5x^3 & 6x^4 \end{vmatrix}$.

11. $\begin{vmatrix} 7a^2 & 2a \\ 4a & a^3 \end{vmatrix}$.

12. $\begin{vmatrix} 2xy & -y \\ x & 3 \end{vmatrix}$.

WRITTEN EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 0 & 7 & 0 \\ 4 & 1 & 8 \\ 1 & 4 & 4 \end{vmatrix}$.

4. $\begin{vmatrix} 3 & 0 & 2 \\ 2 & 4 & 1 \\ 1 & 6 & 2 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} -2 & 1 & 2 \\ -3 & 4 & -1 \\ 5 & 3 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} 3 & -1 & 4 \\ 5 & -2 & 5 \\ 7 & -3 & 6 \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 5 & 7 \end{vmatrix}$.

9. $\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$.

10. $\begin{vmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{vmatrix}$.

11. $\begin{vmatrix} 4 & 2 & 0 \\ 7 & 9 & 6 \\ 0 & 4 & 2 \end{vmatrix}$.

12. $\begin{vmatrix} a & 3 & 2 \\ b & 5 & 6 \\ c & 7 & 0 \end{vmatrix}$.

13. $\begin{vmatrix} a-b & a & a+b \\ b-c & b & b+c \\ c-a & c & c+a \end{vmatrix}$.

Solve the following sets of equations by means of determinants:

14. $\begin{cases} 2x - y + z = 18, \\ x - 2y + 3z = 10, \\ 3x + y - 4z = 20. \end{cases}$

15. $\begin{cases} 5x - 3y + z = 15, \\ x - 3y - z = -3, \\ 2x - y + z = 8. \end{cases}$

16. $\begin{cases} 4x + 2y + z = 13, \\ x - y + z = 4, \\ x + 2y - z = 1. \end{cases}$

17. $\begin{cases} 6x + 4y - 4z = 4, \\ 4x - 2y + 8z = 0, \\ x + y + z = 4. \end{cases}$

18. $\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 5, \\ x + y + z = 2. \end{cases}$

19. $\begin{cases} 2x - 8y + 3z = 2, \\ x - 4y + 5z = 1, \\ 3x - 10y - z = 5. \end{cases}$

20. $\begin{cases} x + y + z = 1, \\ x + 3y + 2z = 8, \\ 2x + 8y - 3z = 15. \end{cases}$

21. $\begin{cases} 2x - 3y + z = 5, \\ 3x + 2y - z = 5, \\ x + y + z = 3. \end{cases}$

22. $\begin{cases} x + y + z = 6, \\ 3x - 2y - z = 13, \\ 2x - y + 3z = 26. \end{cases}$

23. $\begin{cases} x + y + z = 6, \\ 4x - y - z = -1, \\ 2x + y - 3z = -6. \end{cases}$

24. $\begin{cases} 2x - 3y - 4z = -4, \\ 4x - 4y - 2z = 2, \\ x + y + z = 6. \end{cases}$

25. $\begin{cases} x + y + z = 0, \\ 5x + 3y + 4z = 2, \\ 2x - 2y + 3z = -8. \end{cases}$

26. $\begin{cases} x + 2y - z = 2, \\ 2x - y + z = 3, \\ x + 2y + z = 8. \end{cases}$

27. $\begin{cases} 2x - y - z = 6, \\ 2x - 2y + z = 10, \\ x + y - 3z = -2. \end{cases}$

28. $\begin{cases} x - y - z = 1, \\ 2x + 3y + z = 20, \\ x - 2y + z = 0. \end{cases}$

29. $\begin{cases} 5x - 2y + 3z = 5, \\ -2x + y - z = -1, \\ -x - y + 2z = 4. \end{cases}$

30. $\begin{cases} 4a - 3b + 2c = 1, \\ a + b - 4c = -7, \\ 7a - 4b + c = -2. \end{cases}$

31. $\begin{cases} 5m - 4n + r = -2, \\ 3m + n + 3r = 5, \\ 2m + 4n + 3r = 4. \end{cases}$

32. $\begin{cases} 9x - 4y - z = -4, \\ 2x + 5y - 6z = -12, \\ -x + 2y + 4z = 30. \end{cases}$

33. $\begin{cases} 4x + 7y + 3z = -4, \\ x - 3y + 2z = 4, \\ 5x + 2y + 4z = -4. \end{cases}$

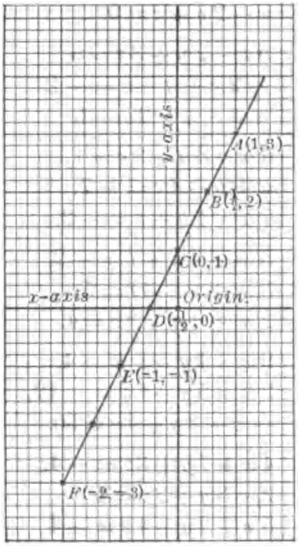
CHAPTER IX

GRAPHIC REPRESENTATION OF EQUATIONS

131. Indeterminate Equations. We have seen (§ 106) that a single equation in two unknowns has indefinitely many pairs of values of the unknowns which satisfy it.

Thus, in the equation $y = 2x + 1$, if we assign any value whatever to x, we find a corresponding value of y. *E.g.* if $x = 1$, $y = 2 \cdot 1 + 1 = 3$.

A picture or map of the solutions of an indeterminate equation in two unknowns may be made by means of the graph.



On a piece of square-ruled paper draw two lines at right angles to each other, called the **axes of coördinates.** Call their intersection the **origin.** Call the **horizontal** line the ***x*-axis,** and the vertical line the ***y*-axis.**

Now measure positive values of x to the *right* from the y-axis, and negative values to the *left;* and measure positive values of y *upward* from the x-axis, and negative values *downward.*

In this manner a point will be determined by each pair of values of x and y, as shown in the accompanying figure.

In the equation $y = 2x + 1$, we have for $x = 1$, $y = 2 \cdot 1 + 1 = 3$. Measuring one unit to the right of the y-axis, and three units up from the x-axis, we locate the point A.

Again, for $x = \frac{1}{2}$, we have $y = 2 \cdot \frac{1}{2} + 1 = 2$. Measuring $\frac{1}{2}$ unit to the right, and 2 units up, we locate the point B.

For $x = 0$, $y = 2 \cdot 0 + 1 = 1$. Measuring 1 unit up from the origin, we locate the point C.

For $x = -2$, $y = 2(-2) + 1 = -4 + 1 = -3$. Measuring 2 units to the left, and 3 units down, we locate the point F.

In this manner we may use as many pairs of values of x and y as we please, each satisfying the equation $y = 2x + 1$, and for each pair we locate a point.

132. The Graph of the First Degree Equation. If this work is done with care, it will be found that a straight line can be drawn passing through all the points thus located by the pairs of values of x and y which satisfy the equation $y = 2x + 1$.

This straight line is called the **graph** of the equation.

133. Coördinates. The values of x and y which locate a point are called the **coördinates of the point.** The x value is called the **abscissa** of the point, and the y value the **ordinate** of the point.

The coördinates of a point are usually written thus (x, y).

E.g. The coördinates of the point A in the graph are $(1, 3)$; those of B are $(\frac{1}{2}, 2)$; of C, $(0, 1)$; of D, $(-1, 0)$; of E, $(-1, -1)$; etc.

134. Linear Equations. An equation of the first degree in two variables is called a *linear equation*, since it can be shown that the graph of every such equation is a straight line.

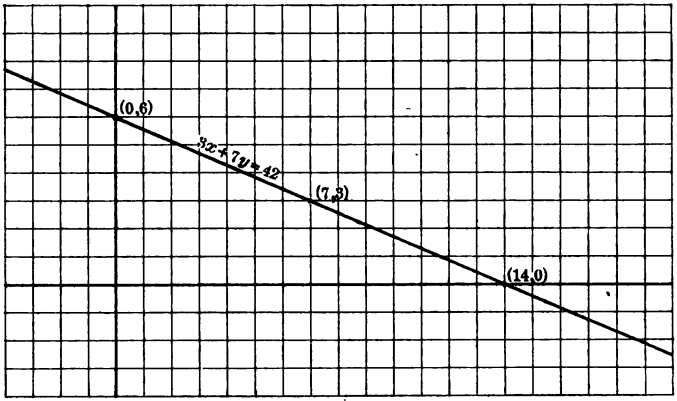
Thus, the equation $y = 2x + 1$, above, has the straight line AF extended indefinitely as its graph.

Since the graph of an equation of the first degree is a straight line, it is only necessary to locate *two* points in order to construct the graph.

E.g. In the equation $y = 2x + 1$, we need to find only the two points where the line cuts the axes. These are the easiest to find, since $y = 0$ where the line cuts the x-axis, and $x = 0$ where it cuts the y-axis.

Thus, we have $x = 0$, $y = 1$; and $y = 0$, $x = -\frac{1}{2}$.

135. Integral Solutions. It is often important to determine those solutions of an indeterminate equation which are *positive integers,* and for this purpose the graph is especially useful.



Example. Find the positive integral solutions of the equation

$$3x + 7y = 42.$$

Solution. Graph the equation carefully on cross-ruled paper, finding it to cut the x-axis at $x = 14$, $y = 0$, and the y-axis at $x = 0$, $y = 6$.

Look now for the *corner points* of the unit squares through which this straight line passes. The coördinates of these points, if there are any such, are the solutions required. In this case the line passes through only one such point whose coördinates are both positive, namely the point (7, 3). Hence the solution sought is $x = 7$, $y = 3$.

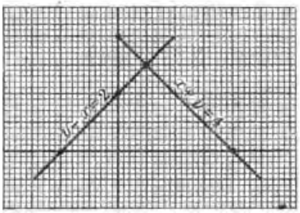
EXERCISES

Solve in positive integers by means of graphs, and check:

1. $x + y = 7$.
2. $x + y = 3$.
3. $x - 27 = -9y$.
4. $7y - 112 = -4x$.
5. $90 - 5x = 9y$.
6. $5x = 29 - 3y$.
7. $140 - 7x - 10y = 0$.
8. $8 - 2x - y = 0$.

136. Two Indeterminate Equations. In the case of two indeterminate equations, the coördinates of any point of intersection of their graphs form a solution of *both equations.*

If the equations are of the first degree the graphs are straight lines. Since these have *only one point* in common, there is only *one solution* of the given pair of equations.



E.g. On graphing $x+y=4$ and $y-x=2$, the lines are found to intersect in the point (1, 3). Hence the solution of this pair of equations is

$$x=1,\ y=3.$$

EXERCISES

Graph the following and thus find the solution of each pair of equations. Construct the graph of each equation by locating two points on it and drawing a straight line through them. Check by substituting in the equations.

1. $\begin{cases} 3x-2y=-2, \\ x+7y=30. \end{cases}$

2. $\begin{cases} x+y=2, \\ 3x+2y=3. \end{cases}$

3. $\begin{cases} x-4y=1, \\ 2x-y=-5. \end{cases}$

4. $\begin{cases} x=-1, \\ 2x-3y=1. \end{cases}$

5. $\begin{cases} 4x=2y+6, \\ x-5=y-1. \end{cases}$

6. $\begin{cases} x=y-5, \\ 5y=x+9. \end{cases}$

7. $\begin{cases} 8x=7y, \\ x+3=5y+3. \end{cases}$

8. $\begin{cases} y=1, \\ 3y+4x=y. \end{cases}$

9. $\begin{cases} 2x-4y=4, \\ x-y=6y-3. \end{cases}$

10. $\begin{cases} x=4, \\ y+x=8. \end{cases}$

11. $\begin{cases} y=-3, \\ 3x+2y=3. \end{cases}$

12. $\begin{cases} x=-2, \\ y=5. \end{cases}$

INCONSISTENT AND DEPENDENT SYSTEMS

137. A pair of linear equations in two unknowns may be such that they either have *no solution* or have an *unlimited number of solutions.*

Example. Solve
$$\begin{cases} x-2y=-2, & (1)\\ 3x-6y=-12. & (2)\end{cases}$$

On graphing these equations they are found to represent two parallel lines. See Fig. 1. Since the lines have no point in common, it follows that the equations have no solution.

Attempting to solve them by means of determinants, § 124, we find:

$$x=\frac{\begin{vmatrix}-2 & -2\\ -12 & -6\end{vmatrix}}{\begin{vmatrix}1 & -2\\ 3 & -6\end{vmatrix}}=\frac{-12}{0},\quad y=\frac{\begin{vmatrix}1 & -2\\ 3 & -12\end{vmatrix}}{\begin{vmatrix}1 & -2\\ 3 & -6\end{vmatrix}}=\frac{-6}{0}.$$

But by § 31, $\frac{-12}{0}$ and $\frac{-6}{0}$ are *impossible operations*, and hence there are no values of x and y which satisfy these equations.

In this case, then, no solution is possible, and the equations are said to be **contradictory**.

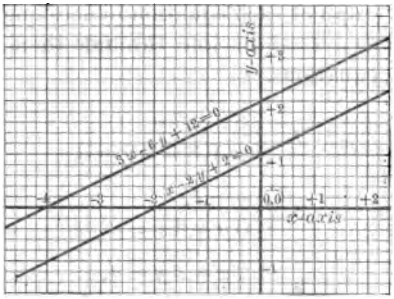


FIG. 1.

Example 2. Solve
$$\begin{cases} 3x-6y=-6, & (1)\\ x-2y=-2. & (2)\end{cases}$$

On graphing these equations, they are found to represent the *same line.* Hence every pair of numbers satisfying one equation must satisfy the other also. See Fig. 2.

Solving these equations by determinants we find:

$$x=\frac{\begin{vmatrix}-6 & -6\\ -2 & -2\end{vmatrix}}{\begin{vmatrix}3 & -6\\ 1 & -2\end{vmatrix}}=\frac{0}{0},\quad y=\frac{\begin{vmatrix}3 & -6\\ 1 & -2\end{vmatrix}}{\begin{vmatrix}3 & -6\\ 1 & -2\end{vmatrix}}=\frac{0}{0}.$$

But by § 31, $\frac{0}{0}$ may represent *any number whatever*. In this case the solution is **indeterminate** and the equations are **dependent**; that is, one may be derived from the other.

Thus, (2) is derived from (1) by dividing both members by 3.

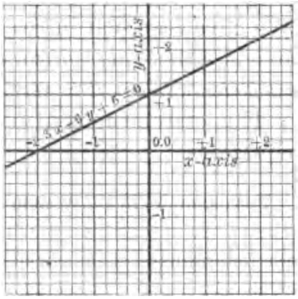


FIG. 2.

138. Test of Dependence by Determinants. The cases in which pairs of linear equations are *dependent* or *contradictory* are those in which the denominators of the expressions for x and y become *zero.* Hence, in order that such a pair of equations may have a *unique* solution, the determinant of the coefficients *must not reduce to zero.* This may be used as a test to determine whether a given pair of equations is independent and consistent.

Note. There is another sense in which the symbols $\frac{a}{0}$ and $\frac{0}{0}$ are used in higher mathematical works, but the above interpretation is the only one which may properly be given to them in elementary algebra.

WRITTEN EXERCISES

In the following, show both by determinants and by the graph, which pairs of equations are independent and consistent, which dependent, and which contradictory.

1. $\begin{cases} 5x-3y=5, \\ 5x-3y=9. \end{cases}$

2. $\begin{cases} x-7+5y=y-x-2, \\ 5x+3y-4=3x-y+3. \end{cases}$

3. $\begin{cases} 7x-3y-4=2x-2, \\ x+y-3=2x-7. \end{cases}$

4. $\begin{cases} x-3y=6, \\ 5x-15y=18. \end{cases}$

5. $\begin{cases} 3y-4x-1=2x-5y+8, \\ 2y-5x+8=3x+y. \end{cases}$

6. $\begin{cases} 3x-6y+5=2x-5y+7, \\ 5x+3y-1=3x+5y+3. \end{cases}$

7. $\begin{cases} 2y+7x=2+6x, \\ 4x-3y=4+3x-5y. \end{cases}$

8. $\begin{cases} 5x-.3=7y+8, \\ 2x+7=4y-9. \end{cases}$

9. $\begin{cases} 5x+2y=6+3x+5y, \\ 3x+y=18-3x+10y \end{cases}$

10. $\begin{cases} 3x+4y=7+5y, \\ x-y=6-2x. \end{cases}$

139. Contradictory and Dependent Systems of Three Equations. We have seen that in the case of two linear equations, the system is either *contradictory or dependent if the determinant of the coefficients is equal to zero.*

This statement is also true in the case of a system of three equations.

For example, in the system:
$$\begin{cases} x + 2y + 3z = 5, \\ 4x - 3y - z = 6, \\ 2x + 4y + 6z = 14. \end{cases}$$
the determinant of the coefficients is
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & -3 & -1 \\ 2 & 4 & 6 \end{vmatrix} = -18 - 4 + 48 + 18 + 4 - 48 = 0.$$

Hence, no definite solution of this system exists, and no further attempt to solve the system need be made.

The solution by means of determinants may be extended to systems of four, five, six, etc., linear equations, and in all cases a definite solution can be found only when the *determinant of the coefficients is different from zero.*

HISTORICAL NOTE

Determinants. The theory of determinants forms one of the most remarkable phases of modern algebra. Like all other developments in algebraic methods, it grew slowly from small beginnings.

Gottfried Wilhelm Leibnitz (1646–1716) in Germany introduced the first notion of determinants in his effort to simplify the expressions arising in the solution of a set of linear equations; but little or no attention was given to the matter for another century.

Lagrange (1736–1813) in France used determinants of the third order in 1773 and proved that the square of a determinant is a determinant.

Vandermonde (1735–1796), also a Frenchman, was the first to give a connected and logical exposition of the theory of determinants.

Gauss (1777–1855) in Germany made systematic use of determinants, and it was he who first introduced the name *determinant*, though **Cauchy** (1789–1857) in France first brought the name into common use.

During the nineteenth century, a long list of mathematicians wrote on this fascinating theme. Possibly no topic in mathematics has attracted greater attention from so large a number of prominent writers.

Gottfried Wilhelm Leibnitz (1646–1716) was born in Leipsic and spent most of his life in Germany, though he lived several years in Paris and made an extended visit to England. Politics and diplomacy occupied much of his attention, but he found time to write extensively on philosophy, law, and mathematics.

Leibnitz was the first to introduce into mathematics the notion of a *determinant.*

CHAPTER X

RATIO VARIATION AND PROPORTION

RATIO AND VARIATION

140. Ratio. In many important applications fractions are called *ratios.*

E.g. $\frac{3}{5}$ is called the ratio of 3 to 5 and is sometimes written $3:5$.

A Ratio is a Number. It is to be understood that a ratio is the *quotient of two numbers* and hence is itself a *number.* We sometimes speak of the ratio of two magnitudes of the same kind, meaning thereby that these magnitudes are expressed in terms of a common unit and a ratio formed from the resulting *numbers.*

E.g. If the heights of two trees are 25 feet and 35 feet respectively, we say the *ratio of their heights* is $\frac{25}{35}$ or $\frac{5}{7}$.

141. Variables, Constants. In many problems, especially in physics, magnitudes are considered which are constantly changing. Number expressions representing such magnitudes are called *variables,* while those which represent fixed magnitudes are *constants.*

E.g. Consider a moving body. If t is the *number* of seconds from the time of starting and s the *number* of feet passed over in that number of seconds, then both the distance and the time are continually changing, and s and t are *variables.*

142. Constant Ratio. Two variables, such as s and t, may be such as to have a *fixed or constant ratio.*

E.g. If a body moves a distance of 5 feet in each second during its period of motion, then for any time t, the distance is $5t$. That is, $s = 5t$, or $s:t = 5$.

143. Direct Variation. When two variables are so related that for all pairs of corresponding values their ratio remains constant, then each one is said to *vary directly as the other.*

For example, if $s : t = k$ (a constant), then s varies directly as t. Also $t : s = \frac{1}{k}$ (likewise a constant). Hence t varies directly as s.

Variation is sometimes indicated by the symbol $\propto$. Thus $s \propto t$ means s varies as t, *i.e.* $\frac{s}{t} = k$, or $s = kt$.

As another illustration of direct variation, consider a rectangle with a fixed base b. Suppose the altitude is continually increased. Then the area A and the altitude h are *variables,* such that $A = b \cdot h$ or $A \div h = b$.

Hence, the area of a rectangle *varies directly as the altitude when the base is constant.*

Likewise, the area *varies directly as the base when the altitude is constant.*

144. Inverse Variation. When two variables are so related that for all pairs of corresponding values their product remains constant, then each one is said to vary *inversely as the other.*

E.g. Consider a rectangle whose area is A, and whose base and altitude are b and h respectively. Then, $A = b \cdot h$.

If, now, the base is multiplied by 2, 3, 4, etc., while the altitude is divided by 2, 3, 4, etc., then the area will remain constant. Hence, b and h may both vary while A remains constant.

The relation $bh = A$ may be written $b = A \times \frac{1}{h}$ or $h = A \times \frac{1}{b}$. Hence, either one is a constant times the reciprocal of the other. For this reason, one is said to vary *inversely as the other.*

If the temperature of a gas remains constant, then the pressure varies inversely as the volume. This is expressed by the equation $pv = k$ (where k is a constant.) This may also be written $p = \frac{k}{v}$ or $v = \frac{k}{p}$.

145. Other Forms of Variation. If $y = kx^2$, k being constant and x and y variables, then y varies *directly as* x^2.

The area of a circle is given by the equation $A = \pi r^2$. Hence, the area varies as the square of the radius.

If $y = kx^3$ then y varies directly as the cube of x, and x varies directly as the cube root of y.

The volume of a sphere is expressed by the equation $v = \frac{4}{3}\pi r^3$. Hence the volume of a sphere varies as the cube of the radius.

If $y = \dfrac{k}{x^2}$, then y *varies inversely as* x^2. If $y = k \cdot wx$, then y *varies jointly as* w *and* x.

If $y = \dfrac{k \cdot w}{x}$, then y varies *directly* as w and *inversely* as x.

The gravitational attraction A between two masses m_1 and m_2 at a distance d from each other is given by the equation $A = k\dfrac{m_1 m_2}{d^2}$ where k is a constant. Hence, their attraction varies jointly as their masses and inversely as the square of their distance apart.

Example. It is known that the resistance offered by a wire to an electric current passing through it varies directly as its length and inversely as the area of its cross section.

If a wire $\frac{1}{8}$ inch in diameter has a resistance of r units per mile, find the resistance of a wire $\frac{1}{4}$ inch in diameter and 3 miles long.

Solution. Let R represent the resistance of a wire of length l and cross-section area $s = \pi \cdot (\text{radius})^2$. Then $R = k \cdot \dfrac{l}{s}$ where k is some constant. Since $R = r$ when $l = 1$ mile and $s = \pi(\frac{1}{16})^2$, we have

$$r = k \cdot \frac{1}{\dfrac{\pi}{256}} \text{ or } k = \frac{\pi r}{256}.$$

Hence, when $l = 3$ and $s = \pi(\frac{1}{8})^2$, we have,

$$R = \frac{\pi r}{256} \cdot \frac{3}{\dfrac{\pi}{64}} = \frac{3}{4}r.$$

That is, the resistance of three miles of the second wire is $\frac{3}{4}$ the resistance *per mile* of the first wire.

PROBLEMS

1. If $z \propto w$, and if $z = 27$ when $w = 3$, find the value of z when $w = 4\frac{1}{3}$.

2. If z varies jointly as w and x, and if $z = 24$ when $w = 2$ and $x = 3$, find z when $w = 3\frac{1}{2}$ and $x = 7$.

3. If z varies inversely as w, and if $z = 11$ when $w = 3$, find z when $w = 66$.

4. If z varies directly as w and inversely as x, and if $z = 28$ when $w = 14$ and $x = 2$, find z when $w = 42$ and $x = 3$.

5. If z varies inversely as the square of w, and if $z = 3$ when $w = 2$, find z when $w = 6$.

6. If q varies directly as m and inversely as the square of d, and $q = 30$ when $m = 1$ and $d = \frac{6}{100}$, find q when $m = 3$ and $d = 600$.

7. If $y^2 \propto x^3$, and if $y = 16$ when $x = 4$, find y when $x = 9$.

8. The weight of a triangular plate of steel of uniform thickness varies jointly as its base and altitude. Find the base when the altitude is 4 and the weight 72, if it is known that the weight is 60 when the altitude is 5 and base 6.

9. The weight of a circular piece of steel cut from a sheet of uniform thickness varies as the square of its radius. Find the weight of a piece whose radius is 13 feet, if a piece of radius 7 feet weighs 196 pounds.

10. If a body starts falling from rest, its velocity varies directly as the number of seconds during which it has fallen. If the velocity at the end of 3 seconds is 96.6 feet per second, find its velocity at the end of 7 seconds; of ten seconds.

11. If a body starts falling from rest, the total distance fallen varies directly as the square of the time during which it has fallen. If in 2 seconds it falls 64.4 feet, how far will it fall in 5 seconds? In 9 seconds?

12. The number of vibrations per second of a pendulum varies inversely as the square root of the length. If a pendulum 39.1 inches long vibrates once in each second, how long is a pendulum which vibrates 3 times in each second?

13. Illuminating gas in cities is forced through the pipes by subjecting it to pressure in the storage tanks. It is found that the volume of a fixed amount of gas varies inversely as the pressure. A certain body of gas occupies 49,000 cu. ft. when under a pressure of 2 pounds per square inch. What space would it occupy under a pressure of $2\frac{4}{5}$ pounds per square inch?

14. The amount of heat received from a stove varies inversely as the square of the distance from it. A person sitting 15 feet from the stove moves up to 5 feet from it. By how much will this multiply the amount of heat received?

15. The weights of bodies of the same shape and of the same material vary as the cubes of corresponding dimensions. If a ball $3\frac{1}{4}$ inches in diameter weighs 14 oz., how much will a ball of the same material weigh whose diameter is $3\frac{1}{2}$ inches?

16. On the principle of problem 15, if a man 5 feet 9 inches tall weighs 165 pounds, what should be the weight of a man of similar build 6 feet tall?

17. The area of a circle varies as the square of its radius. If the area of a certain circle is 254.47 square inches and its radius is 9 inches, find the area of a circle whose radius is 11 inches. Also find the radius of a circle whose area is 212 square inches.

18. The time required for the vibration of a pendulum varies directly as the square root of the length. If a pendulum 39.1 inches long vibrates once in a second, find the time of vibration of a pendulum 54 inches long. Also find the length of a pendulum that vibrates once in 3 seconds.

PROPORTION

146. Definitions. The four numbers a, b, c, d, are said to be *proportional* or *to form a proportion*, if $\frac{a}{b} = \frac{c}{d}$. This is also sometimes written $a : b :: c : d$, and is read *a is to b as c is to d*.

The four numbers are called the **terms of the proportion**; the first and fourth are the **extremes**, the second and third the **means** of the proportion. The first and third are the **antecedents** of the ratios, the second and fourth the **consequents**.

If a, b, c, x are proportional, x is the **fourth proportional** to a, b, c. If a, x, x, b are proportional, x is a **mean proportional** between a and b, and b is a **third proportional** to a and x.

147. Transforming a Proportion. A proportion may be transformed as shown in the following examples.

Example 1. If $a : b :: c : d$ show that $a : c :: b : d$.

Solution. From $\frac{a}{b} = \frac{c}{d}$, we are to show that $\frac{a}{c} = \frac{b}{d}$. Multiplying both members of $\frac{a}{b} = \frac{c}{d}$ by $\frac{b}{c}$, we have $\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = \frac{\cancel{c}}{d} \cdot \frac{b}{\cancel{c}}$ or $\frac{a}{c} = \frac{b}{d}$.

Example 2. If $a : b :: c : d$, show that $a + b : b :: c + d : d$.

Solution. Adding 1 to both members of $\frac{a}{b} = \frac{c}{d}$, we have $\frac{a+b}{b} = \frac{c+d}{d}$.

3. From $a : b :: c : d$ show how to derive $b : a :: d : c$.
4. From $a : b :: c : d$ show how to derive $a : c :: b : d$.
5. From $a : b :: c : d$ show how to derive $d : b :: c : a$.
6. From $a : b :: c : d$ derive $a - b : b :: c - d : d$.
7. From $a : b :: c : d$ derive $a + b : a - b :: c + d : c - d$.

148. Names of Derived Proportions. The proportion $a : b :: c : d$ is said to be taken by **alternation** when it is transformed into $a : c :: b : d$; by **inversion** when it is transformed into $b : a :: d : c$; by **composition** when it is transformed into $a + b : b :: c + d : d$; by **division** when it is transformed into $a - b : b :: c - d : d$; and by **composition and division** when it is transformed into $a + b : a - b :: c + d : c - d$.

WRITTEN EXERCISES

1. What principles in the transformation of equations are involved (*a*) in taking a proportion by inversion, (*b*) by alternation.

2. From $a:b::c:d$ derive $a+b:a::c+d:c$.

3. From $a:b::c:d$ derive $a-b:a::c-d:c$.

4. From $a:b::c:d$ derive $a+c:b+d::a:b$.

5. From $a:b::c:d$ derive $a+b:c+d::b:d$.

6. From $a:b::c:d$ show that $a^2:b^2::c^2:d^2$.

7. From $a:b::c:d$ and $m:n::r:s$, show that $am:bn::cr:ds$.

8. From $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}$ show that

$$\frac{a_1+a_2+\cdots+a_n}{b_1+b_2+\cdots+b_n}=\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}.$$

Suggestion. Let $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\cdots=\frac{a_n}{b_n}=k$.

9. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{ma-b}{mc-d}=\frac{a}{c}$.

10. If $\frac{x}{y}=\frac{z}{w}$, show that $\frac{x+ky}{x-ky}=\frac{z+kw}{z-kw}$.

11. If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{ma+nb}{ma-nb}=\frac{mc+nd}{mc-nd}$.

12. (*a*) Find a fourth proportional to 17, 19, and 187.
    (*b*) Find a mean proportional between 6 and 54.
    (*c*) Find a third proportional to 27 and 189.

13. If $ma=pq$, prove that the following proportions hold:
    (1) $m:p::q:a$.
    (2) $a:q::p:m$.

14. Find the unknown term in each of the following proportions:

(1) $x:42::27:126$; (2) $78:x::13:3$;
(3) $99:117::x:39$; (4) $171:27::57:x$.

ORAL EXERCISES

Solve the following equations:

1. $\frac{x}{4}=\frac{7}{2}$.
2. $\frac{x}{3}=\frac{5}{6}$.
3. $\frac{2x}{7}=\frac{5}{2}$.
4. $\frac{5}{x}=\frac{10}{7}$.
5. $\frac{4}{x}=\frac{7}{9}$.
6. $\frac{3}{7}=\frac{x}{16}$.
7. $\frac{4}{5}=\frac{2x}{15}$.
8. $\frac{3}{5}=\frac{6}{x}$.
9. $\frac{4}{7}=\frac{3}{2x}$.

WRITTEN EXERCISES

1. The area of a circle is πr^2, where r is the radius. Show that $\frac{a_1}{a_2}=\frac{r_1^2}{r_2^2}$, where a_1 and a_2 are the areas of any two circles whose radii are r_1 and r_2 respectively. Also show that $\frac{a_1}{a_2}=\frac{d_1^2}{d_2^2}$, where d_1 and d_2 are the diameters.

2. Compare the areas of two circular fields whose radii are 200 and 300 feet, respectively.

3. The volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius. Show that $\frac{v_1}{v_2}=\frac{r_1^3}{r_2^3}$, where v_1 and v_2 are the volumes of any two spheres whose radii are r_1 and r_2, respectively. Also show that $\frac{v_1}{v_2}=\frac{d_1^3}{d_2^3}$, where d_1 and d_2 are the diameters.

4. Compare the volumes of two spheres whose radii are 2 and 5 inches, respectively.

5. The volumes of two spheres are in the ratio 8 to 27. Find the ratio of their radii.

6. The surface of a sphere is $4\pi r^2$, where r is the radius. Show that $\frac{s_1}{s_2}=\frac{r_1^2}{r_2^2}$, where s_1 and s_2 are the surfaces of two spheres whose radii are r_1 and r_2, respectively.

7. Compare the surfaces of two spheres whose radii are 2 and 5, respectively.

149. Variation Related to Proportion. A problem in variation may usually be stated in the form of a proportion:

For example, in the case of uniform motion, where the distance varies directly as the time, we have $s = kt$ or $\frac{s}{t} = k$. This simply means that if the body moves a distance s_1 in a time t_1 and a distance s_2 in a time t_2, that $\frac{s_1}{t_1} = \frac{s_2}{t_2}$, since both ratios are equal to the constant k. From this, by alternation we have $\frac{s_1}{s_2} = \frac{t_1}{t_2}$.

Again, it is known that if a body starts falling from rest, the distance fallen varies *directly as the square of the time* through which it has fallen. That is, $s = kt^2$, or $\frac{s}{t^2} = k$.

Hence, if the body falls a distance s_1 in a time t_1, and a distance s_2 in time t_2, we have the proportion $\frac{s_1}{t_1^2} = \frac{s_2}{t_2^2}$, since each ratio $= k$. From this proportion by alternation we have $\frac{s_1}{s_2} = \frac{t_1^2}{t_2^2}$.

This proportion may be obtained more directly from the equation $s = kt^2$ as fellows: Since $s_1 = kt_1^2$ and $s_2 = kt_2^2$ then $\frac{s_1}{s_2} = \frac{kt_1^2}{kt_2^2} = \frac{t_1^2}{t_2^2}$.

Example. It is known that a body starting from rest falls a distance of 144.9 feet in 3 seconds. Find how far it will fall in 8 seconds.

Solution. Since the distance through which a body falls is proportional to the square of the time, we have

$$\frac{s_1}{s_2} = \frac{t_1^2}{t_2^2}.$$

In this example, we have $s_1 = 144.9$, $t_1 = 3$, and $t_2 = 8$. Required to find the distance s_2.

Hence,
$$\frac{144.9}{s_2} = \frac{3^2}{8^2},$$

or
$$s_2 = \frac{64 \times 144.9}{9} = 1030.4 \text{ feet.}$$

EXERCISES

1. The volume of a sphere varies directly as the cube of the radius. If a sphere whose radius is 4 inches contains 268.07 cubic inches, find the volume of a sphere whose radius is 6 inches.

2. When a weight is attached to a spring balance the index is displaced a distance proportional to the weight. Thus, if d_1 and d_2 are displacements and w_1, w_2 the corresponding weights, then $\frac{d_1}{d_2}=\frac{w_1}{w_2}$. If a 2-pound weight displaces the index $\frac{1}{4}$ inch, how much will a 50-pound weight displace it?

3. The intensity of light is *inversely* proportional to the square of the distance from the source of the light. That is, if i_1 and i_2 are the measures of intensities at the distances d_1 and d_2, then $\frac{i_1}{i_2}=\frac{d_2^2}{d_1^2}$. If the intensity of a given light at a distance of 2 feet is 20 candle power, find the intensity at 5 feet.

4. If w_1 and w_2 are weights resting on the two ends of a beam, and if the distances from the fulcrum are d_1 and d_2 respectively, then the beam will balance when $\frac{w_1}{w_2}=\frac{d_2}{d_1}$. That is, the weights are *inversely* proportional to the distances.

If a stone weighing 850 pounds at a distance of 1 foot from the fulcrum is to be balanced by a 50-pound weight, where should the weight be applied?

5. Find where the fulcrum should be in order that two boys weighing 110 and 80 pounds, respectively, may balance on the ends of a 16-foot plank.

6. The weight of a body above the earth's surface varies inversely as the square of its distance from the earth's center. If an object weighs 2000 pounds at the earth's surface, what would be its weight if it were 12,000 miles above the center of the earth, the radius of the earth being 4000 miles?

CHAPTER XI

POWERS AND ROOTS

150. Each of the operations thus far studied leads to a **single result.** See § 31.

E.g. Two numbers have one and only one *sum*, and one and only one *product*.

When a number is subtracted from a given number, there is one and only one *remainder*.

When a number is divided by a given number, there is one and only one *quotient*, provided the divisor is not zero.

We are now to study an operation which leads to **more than one result**; namely, the operation of finding roots.

Thus, both 3 and -3 are square roots of 9, since $3 \cdot 3 = 9$, and also $(-3)(-3) = 9$. The two square roots of 9 are written $\pm\sqrt{9} = \pm 3$.

Similarly a and $-a$ are both square roots of a^2, $x + y$ and $-x - y$ are both square roots of $x^2 + 2xy + y^2$.

151. Imaginary Numbers. There is no positive or negative number such that its square is negative.

Thus, $1^2 = +1$ and $(-1)^2 = +1$.

Hence, such an expression as $\sqrt{-1}$ has no meaning in terms of the numbers thus far used.

We now define the symbol $\sqrt{-1}$ by the equation

$$(\sqrt{-1})^2 = -1,$$

and we call $\sqrt{-1}$ the **imaginary unit.**

An *even* root of a negative number is called an **imaginary number.**

Thus, $\sqrt{-4}$, $\sqrt[4]{-2}$, are imaginary numbers.

All numbers which do not contain an imaginary number are called **real numbers.**

152. When irrational and imaginary numbers are admitted to the number system, it can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc.

E.g. The square roots of 9 are $+3$ and -3. The square roots of -9 are $\pm\sqrt{-9} = \pm 3\sqrt{-1}$. The cube roots of 8 are 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$. The fourth roots of 16 are $+2$, -2, $+2\sqrt{-1}$ and $-2\sqrt{-1}$.

Any positive real number has two real roots of *even* degree, one positive and one negative.

E.g. ± 2 are fourth roots of 16. The square roots of 3 are $\pm\sqrt{3}$.

Any real number, positive or negative, has one real root of *odd* degree, whose sign is the same as that of the number itself.

E.g. $\sqrt[3]{27} = 3$, and $\sqrt[5]{-32} = -2$.

153. Principal Root. The positive *even* root of a positive real number, or the real *odd* root of any real number, is called the *principal root.*

The positive square root of a negative real number is also sometimes called the *principal* imaginary root.

E.g. 2 is the principal square root of 4; 3 is the principal 4th root of 81; -4 is the principal cube root of -64; and $+\sqrt{-3}$ is the principal square root of -3.

Unless otherwise stated the radical sign alone is understood to indicate the *principal root.*

Since a number has more than one root, it is necessary to limit certain theorems so as to make them apply to principal roots only.

Thus, the square root of 16 is not necessarily 4 unless the *principal* square root of 16 is understood.

Again, the cube root of 8 is not necessarily 2 unless the *principal* cube root is understood.

Unless this restriction is understood it may be easily shown that the conclusions in §§ 157, 159 are not true.

PRINCIPLES INVOLVING POWERS AND ROOTS

154. From § 44 $(2^3)^2 = (2^2)^3 = 2^6 = 64$.

In general, if n and k are any positive integers,

$$(b^k)^n = (b^n)^k = b^{nk}.$$

For
$$\begin{aligned}(b^k)^n &= b^k \cdot b^k \cdot b^k \cdots \text{ to } n \text{ factors}\\ &= b^{k+k+k \,\dots\, \text{to } n \text{ terms}} = b^{nk}.\end{aligned}$$

Likewise,
$$\begin{aligned}(b^n)^k &= b^n \cdot b^n \cdot b^n \cdots \text{ to } k \text{ factors}\\ &= b^{n+n+n+\cdots\, \text{to } k \text{ terms}} = b^{kn}.\end{aligned}$$

Hence: *The nth power of the kth power of any base is the nkth power of that base.*

155. Again, we have $(2^3 \cdot 3^2)^2 = 2^6 \cdot 3^4$.

In general, if k, r, and n are any positive integers,

$$(a^k b^r)^n = a^{nk} b^{nr}.$$

For
$$\begin{aligned}(a^k b^r)^n &= (a^k b^r) \cdot (a^k b^r) \cdots \text{ to } n \text{ factors.}\\ &= (a^k \cdot a^k \cdots \text{ to } n \text{ factors})(b^r \cdot b^r \cdots \text{ to } n \text{ factors})\\ &= (a^k)^n \cdot (b^r)^n = a^{nk} b^{nr}.\end{aligned}$$

Hence: *The nth power of the product of several factors is the product of the nth powers of those factors.*

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $(3^2)^3$.
2. $(2^3)^2$.
3. $(2 \cdot 5^2)^2$.
4. $(2\,a^2)^3$.
5. $(3\,a^4)^2$.
6. $3(2\,x^2)^2$.
7. $(3 \cdot 2\,x^2)^2$.
8. $a^2(b^3)^2$.
9. $(x^2y)^2$.
10. $(ab^2c)^2$.
11. $(a^3bc)^2$.
12. $(a^2b^3)^2$.
13. $(2\,ab^2)^3$.
14. $6(a^2b)^3$.
15. $a(bc^2)^4$.
16. $c(x^2y^3)^3$.
17. $c(a^2b)^2$.
18. $(cx^2y^3)^3$.
19. $(ca^2b)^2$.
20. $(c^3a^2b)^3$.
21. $(x^2y^3z^4)^2$.
22. $4(mn^3p^2)^3$.
23. $2(a^n)^k$.
24. $(2\,a^n)^k$.
25. $a^n(b^n)^k$.
26. $(a^nb^n)^k$.
27. $2(a^2b^4)^k$.
28. $(2\,a^2b^4)^k$.
29. $a^l(x^my^n)^k$.
30. $(a^lx^my^n)^k$.
31. $4(a^2b^3c^4)^{2n}$.
32. $2(a^{2a}y^{3b}z^{4b})^2$.
33. $(a^{3n}b^{2n}c^p)^q$.

156. From § 44, we have $\left(\frac{2^3}{3^2}\right)^2 = \frac{2^3}{3^2} \cdot \frac{2^3}{3^2} = \frac{2^6}{3^4}$.

In general, $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^{nk}}{b^{nr}}.$$

For we have $$\left(\frac{a^k}{b^r}\right)^n = \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdot \frac{a^k}{b^r} \cdots \text{ to } n \text{ factors}$$
$$= \frac{a^{nk}}{b^{nr}}.$$

Hence: *The **n**th power of the quotient of two numbers equals the quotient of the **n**th powers of those numbers.*

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $\left(\frac{1}{2}\right)^2$.
2. $\left(\frac{1}{a}\right)^3$.
3. $\left(\frac{1}{a^2}\right)^3$.
4. $\left(\frac{1}{a^3}\right)^2$.
5. $\left(\frac{a}{b^2}\right)^2$.
6. $\left(\frac{a^3}{b^4}\right)^2$.
7. $a^2\left(\frac{b}{a}\right)^2$.
8. $\left(\frac{a^2b}{a}\right)^2$.
9. $a^3\left(\frac{x^3}{a}\right)^2$.
10. $\left(\frac{a^2x^3}{a}\right)^2$.
11. $4\left(\frac{xy}{2a}\right)^2$.
12. $\left(\frac{xy}{a}\right)^2$.
13. $\left(\frac{x^ny^m}{a^l}\right)^k$.
14. $x^k\left(\frac{ab^2}{x}\right)^k$.
15. $\left(\frac{x^kab^2}{x}\right)^k$.

157. We may easily verify that $\sqrt{3^4} = 3^{4 \div 2} = 3^2 = 9$.

In general, if k and r are positive integers and b any positive real number, we have:
$$\sqrt[r]{b^{kr}} = b^{kr \div r} = b^k.$$

For, from § 154, $(b^k)^r = b^{kr}$.

Hence, by definition b^k is an rth root of b^{kr}.

That is, $\sqrt[r]{b^{kr}} = b^k = b^{kr \div r}$.

Hence: *The **r**th root of the **kr**th power of any positive real number is the **k**th power of that number.*

E.g. $\sqrt[4]{2^{12}} = 2^{12 \div 4} = 2^3 = 8$.

158. Another general principle is that, if a and b are any real numbers and r any positive integer, then

$$\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$$

For by § 155, $(\sqrt[r]{a} \cdot \sqrt[r]{b})^r = (\sqrt[r]{a})^r \cdot (\sqrt[r]{b})^r = a \cdot b,$
since the rth power of the rth root of a number is the number itself.

Hence, we have $ab = (\sqrt[r]{a} \cdot \sqrt[r]{b})^r.$

Taking the principal rth root of both members,
we have $\sqrt[r]{ab} = \sqrt[r]{a} \cdot \sqrt[r]{b}.$

Hence: *The rth root of the product of two positive real numbers equals the product of the rth roots of the numbers.*

ORAL EXERCISES

Remove the radical signs from each of the following:

1. $\sqrt{a^4}$.
2. $\sqrt{2^6}$.
3. $\sqrt[3]{2^6}$.
4. $\sqrt[3]{a^6}$.
5. $\sqrt[4]{a^8}$.
6. $\sqrt{a^8}$.
7. $\sqrt[3]{a^{12}}$.
8. $\sqrt[4]{a^{12}}$.
9. $\sqrt{a^4b^2}$.
10. $\sqrt[3]{a^3b^6c^9}$.
11. $\sqrt{a^{2n}}$.
12. $\sqrt{a^{2n}b^{4n}}$.
13. $\sqrt[3]{a^{6n}x^3}$.
14. $\sqrt[3]{x^{3k}y^{6n}}$.
15. $\sqrt[4]{x^8y^{12n}}$.
16. $\sqrt[3]{x^{12n}y^9}$.
17. $\sqrt[n]{a^{2n}b^{3n}}$.
18. $\sqrt[n]{a^{4n}b^nx^{kn}}$.
19. $\sqrt[2n]{x^{4n}y^{10n}}$.
20. $\sqrt[2n]{(a-b)^{4n}(a+b)^{2n}}$.
21. $\sqrt[3]{(x+y)^6(x-y)^9}$.

159. Again,

$$\sqrt[r]{\frac{a}{b}} = \frac{\sqrt[r]{a}}{\sqrt[r]{b}}.$$

For we have by § 156, $\left(\dfrac{\sqrt[r]{a}}{\sqrt[r]{b}}\right)^r = \dfrac{(\sqrt[r]{a})^r}{(\sqrt[r]{b})^r} = \dfrac{a}{b}.$

Hence, taking the principal rth root of the first and last members, we have

$$\frac{\sqrt[r]{a}}{\sqrt[r]{b}} = \sqrt[r]{\frac{a}{b}}.$$

That is: *The rth root of the quotient of two positive real numbers equals the quotient of the rth roots of the numbers.*

ORAL EXERCISES

Remove the radical sign from each of the following:

1. $\sqrt{\frac{1}{2^2}}$.
2. $\sqrt{\frac{1}{a^2}}$.
3. $\sqrt{\frac{a^4}{b^6}}$.
4. $\sqrt{\frac{a^2}{b^6c^2}}$.
5. $a\sqrt{\frac{a^2}{b^4}}$.
6. $\sqrt{\frac{a^4}{b^4}}$.
7. $\sqrt[3]{\frac{1}{a^6}}$.
8. $\sqrt[3]{\frac{a^3}{b^6}}$.
9. $a\sqrt[3]{\frac{a^3}{b^9}}$.
10. $\sqrt{\frac{a^6}{b^9}}$.
11. $\sqrt[n]{\frac{a^{2n}}{b^{4n}}}$.
12. $\sqrt[r]{\frac{(x+y)^r}{(x-y)^{2r}}}$.

160. It follows from §§ 154–159 that

(1) *Any positive integral power of a monomial is found by multiplying the exponents of the factors of the monomial by the exponent of the power.*

(2) *If a monomial is a perfect power of the* **k**th *degree, its* **k**th *root may be found by dividing the exponent of each factor by the index of the root.*

These principles may be expressed by the following formulas in which a and b are positive numbers, integral or fractional.

$$(a^kb^r)^n = a^{nk}b^{nr}.$$
$$\sqrt[r]{a^{nr}b^{mr}} = a^nb^m.$$

EXERCISES

Find the following indicated powers and roots, and reduce each expression to its simplest form:

1. $(a^3b^4c^5)^7$.
2. $(2^{a+b}\cdot 3^c\cdot 5^b)^{a-b}$.
3. $\left(\frac{a^3b^4c^5}{2^3\cdot 3^2\cdot 4^3}\right)^2$.
4. $(a^{x-y})^{x^2+xy+y^2}$.
5. $(x^4y^5z^{x+y})^{x-y}$.
6. $\left(\frac{5^2b^2mn}{3^7bc^4}\right)^3$.
7. $(3^5\cdot 4^5\cdot 2^5)^{a-b}$.
8. $\sqrt[a]{3^{2a}\cdot 2^a\cdot 5^{3a}}$.
9. $\sqrt[3]{\frac{-27\cdot 8a^6}{64c^9d^{6a}}}$.
10. $(a^{m+n}b^{m-n}c^{mn})^{m+n}$.
11. $(3^a\cdot 4^b\cdot 5^c)^{abx}$.
12. $\sqrt[2a]{3^{6a}\cdot 4^{2a}\cdot 5^{8a}}$.
13. $\sqrt[a-b]{3^{a^3-b^3}\cdot 4^{a-b}\cdot 5^{a^2-b^2}}$.
14. $\sqrt{64\cdot 25\cdot 256\cdot 625}$.
15. $\sqrt[3]{27\cdot 125\cdot 64\cdot 3^{6a}}$.
16. $\sqrt{\frac{(a-b)^2(a^2+2ab+b^2)}{(a-b)^4(a+b)^2}}$.

SQUARE ROOT OF A POLYNOMIAL

161. The rule for finding the **square root of a polynomial** is derived from a study of the following squares.

From the squares

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$
$$(a+b+c)^2 = [(a+b)+c]^2 = (a+b)^2 + 2(a+b)c + c^2,$$
$$(a+b+c+d)^2 = [(a+b+c)+d]^2$$
$$= (a+b+c)^2 + 2(a+b+c)d + d^2, \text{ etc.}$$

we see that

(1) If to a^2 we add $2\,ab + b^2 = (2\,a+b)b$, we get $(a+b)^2$;

(2) If to $(a+b)^2$ we add $2(a+b)c + c^2 = [2(a+b)+c]c$, we get $(a+b+c)^2$;

(3) If to $(a+b+c)^2$ we add $2(a+b+c)d + d^2 = [2(a+b+c)+d]d$, we get $(a+b+c+d)^2$, etc.

Hence, in squaring a polynomial:

*For every new **term** added to the root there is a new **part** added to the power. This new part consists of twice the sum of the **preceding** terms of the root plus the **last** term of the root, all multiplied by the last term of the root.*

This is expressed by the formula:

$$\boldsymbol{(a+b+c+d)^2 = a^2 + (2\,a+b)b + [2(a+b)+c]c + [2(a+b+c)+d]d.}$$

WRITTEN EXERCISES

Write the following squares in the above form:

1. $(x+y+z)^2$.

2. $(x+y+z+v)^2$.

3. $(x+2\,y+3\,z+4\,v)^2$.

4. $(2\,a+b+3\,c+d)^2$.

5. $(b+4\,c+2\,d+e)^2$.

6. $(x+2\,a+3\,b+c)^2$.

Example 1. Find the square root of

$$9x^4 - 12x^3 + 28x^2 - 16x + 16.$$

$$\begin{array}{lr}
\textit{Solution.} & a \;+\; b \;+ c \qquad\qquad\qquad\qquad\quad \\
\text{Square root (to be found).} & \underline{3x^2 - 2x + 4 \qquad\qquad\qquad\qquad\quad} \\
\text{Given square} = & 9x^4 - 12x^3 + 28x^2 - 16x + 16 \\
a^2 = (3x^2)^2 = & \underline{9x^4 \qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
2a = 2\cdot 3x^2 = 6x^2 & -12x^3 + 28x^2 - 16x + 16 \\
(2a + b)b = (6x^2 - 2x)(-2x) = & \underline{-12x^3 + \;\;4x^2 \qquad\qquad\quad} \\
2(a + b) = 6x^2 - 4x & 24x^2 - 16x + 16 \\
[2(a + b) + c]c = (6x^2 - 4x + 4)\cdot 4 = & \underline{24x^2 - 16x + 16} \\
 & 0
\end{array}$$

Explanation. The first term of the root is $\sqrt{9x^4} = 3x^2$. The $9x^4$ is subtracted from the square.

The second term of the root is $-12x^3 \div (2\cdot 3x^2) = -2x$.

The second part of the square is $(6x^2 - 2x)(-2x) = -12x^3 + 4x^2$, corresponding to $(2a + b)b$. This is now subtracted.

The third term of the root is $24x^2 \div (2\cdot 3x^2) = 4$.

The third part of the square is $(6x^2 - 4x + 4)\cdot 4 = 24x^3 - 16x + 16$, corresponding to $[2(a + b) + c]c$. This is now subtracted.

Thus, at each step a new term of the root is found by dividing the first term of the remainder by twice the first term of the root; and then a new part of the power is built up and subtracted.

Since the final remainder is zero, the square root is exact.

Example 2. Find the square root of

$$16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25.$$

$$\begin{array}{rr}
\textit{Solution.} & \\
 & a \qquad +b \quad\; +c+d \qquad\qquad\qquad\qquad\qquad\qquad\quad \\
\text{Square root} & \underline{4x^3 - 3x^2 + 2x - 5 \qquad\qquad\qquad\qquad\qquad\quad} \\
\text{Given square} & 16x^6 - 24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
a^2 = (4x^3)^2 = & \underline{16x^6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad} \\
 & -24x^5 + 25x^4 - 52x^3 + 34x^2 - 20x + 25 \\
(2a + b)b = & \underline{-24x^5 + \;\;9x^4 \qquad\qquad\qquad\qquad\qquad\quad} \\
 & 16x^4 - 52x^3 + 34x^2 - 20x + 25 \\
[2(a + b) + c]c = & \underline{16x^4 - 12x^3 + \;\;4x^2 \qquad\qquad\quad} \\
 & -40x^3 + 30x^2 - 20x + 25 \\
[2(a + b + c) + d]d = & \underline{-40x^3 + 30x^2 - 20x + 25}
\end{array}$$

162. From the preceding examples we have the following:

Rule. (1) *Arrange the polynomial according to ascending or descending powers of some letter.*

(2) *Find the square root of the first term, and write it as the first term of the root.*

(3) *Subtract the square of this first term of the root.*

(4) *Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.*

(5) *Add this second term of the root to twice the first term, and multiply the sum by the second term. This product is the second part of the square and is to be subtracted.*

(6) *Now use the sum of the first two terms of the root to find the third term, just as the first term was used to find the second; and continue in this manner till all the terms of the root are found.*

EXERCISES

Find the square roots of the following:

1. $m^2 + 4\,mn + 6\,ml + 4\,n^2 + 12\,ln + 9\,l^2$.
2. $4\,x^4 + 8\,ax^3 + 4\,a^2x^2 + 16\,b^2x^2 + 16\,ab^2x + 16\,b^4$.
3. $4\,x^6 - 12\,x^5 + 13\,x^4 - 14\,x^3 + 13\,x^2 - 4\,x + 4$.
4. $16\,a^6 + 24\,a^5 + 25\,a^4 + 20\,a^3 + 10\,a^2 + 4\,a + 1$.
5. $1 + 2\,x + 3\,x^2 + 4\,x^3 + 5\,x^4 + 4\,x^5 + 3\,x^6 + 2\,x^7 + x^8$.
6. $9\,a^2 - 6\,ab + 30\,ac + 6\,ad + b^2 - 10\,bc - 2\,bd + 25\,c^2 + 10\,cd + d^2$.
7. $9\,a^2 - 30\,ab - 3\,ab^2 + 25\,b^2 + 5\,b^3 + \frac{b^4}{4}$.
8. $\frac{4}{9}\,a^2x^4 - \frac{4}{3}\,abx^3z + \frac{8}{3}\,a^2bx^2z^2 + b^2x^2z^2 - 4\,ab^2xz^3 + 4\,a^2b^2z^4$.
9. $a^2 - 6\,ab + 10\,ac - 14\,ad + 9\,b^2 - 30\,bc + 42\,bd + 25\,c^2 - 70\,cd + 49\,d^2$.

10. $9a^6 - 24a^3b^4 - 18a^3c^5 + 6a^3d^2 + 16b^8 + 24b^4c^5 - 8b^4d^2$
$+ 9c^{10} - 6c^5d^2 + d^4.$

11. $x^{10} - 8x^5w^5 + 16w^{10} - 4x^5y^3 + 16y^3w^5 + 4y^6 + 6x^5z^4$
$- 24z^4w^5 - 12y^3z^4 + 9z^8.$

12. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729.$

13. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

14. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64.$

THE SQUARE ROOT OF AN ARITHMETIC NUMBER

163. Rule for Finding the First Term of the Square Root of an Integral Number. Since $1^2 = 1$ and $9^2 = 81$, the square of a number of one figure contains either *one* or *two* figures.

Since $10^2 = 100$ and $99^2 = 9801$, the square of a number of two figures contains either *three* or *four* figures.

Similarly, the square of a number of three figures contains either *five* or *six* figures, and so on.

Rule. Hence, to find the first figure in the root:

(1) *Separate the number into groups of two figures each, counting from units' place toward the left. The last group may contain only one figure.*

(2) *Take the square root of the largest square in the left-hand group; this is the first figure of the root. There are as many figures in the root as there are groups in the number.*

Examples. 1. To find the first figure and the number of figures in the square root of 450,769 we write it thus, 45 07 69.

Since there are three groups of two figures each, the square root contains three figures, and hence it starts with a digit in hundreds' place.

Since 36 is the largest square in 45, the first figure in the root is $\sqrt{36} = 6$.

2. Similarly, the square root of 6,762,436, written 6 76 24 36, contains four figures, of which the first one is in thousands' place.

Since 4 is the largest square in 6, the first figure of the root is $\sqrt{4} = 2$.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and state whether it stands in units', tens', or hundreds' place:

1. 8947.
2. 6205.
3. 19,140.
4. 72,048.
5. 90,401.
6. 63,401.
7. 1428.
8. 194,670.
9. 7347.
10. 73,470.
11. 14,051.
12. 140,051.
13. 107.
14. 4091.
15. 10,007.
16. 100,007.

164. The square root of an arithmetic number may be found by the process used for polynomials in § 162.

Illustrative Example 1. Find the square root of 405769.

Solution.

$$\begin{array}{rr|l}
 & \text{SQUARE} & \text{SQUARE ROOT} \\
 & & a+\ b+c \\
 & 40\ 57\ 69 & \underline{600+30+7=637} \\
a^2=\underline{600^2=} & \underline{36\ 00\ 00} & \\
2a=1200 & 4\ 57\ 69 & \\
b=\underline{30} & & \\
2a+b=\underline{1230} & \underline{3\ 69\ 00} & =(2a+b)b \\
2(a+b)=1260 & 88\ 69 & \\
c=\underline{7} & & \\
2(a+b)+c=\underline{1267} & \underline{88\ 69} & =[2(a+b)+c]c \\
 & 0 &
\end{array}$$

Explanation. The first figure in the root is the square root of the largest square in the left-hand group, and since there are three groups, the root starts with 600, which corresponds to a of the formula (§ 161).

Subtracting the square of 600, we have a remainder 4 57 69.

The trial divisor is $2a = 1200$ and when 4 57 69 is divided by 1200, the largest number of tens in the quotient is 3. Hence 30 corresponds to b of the formula.

The complete divisor is $2a + b = 1230$.

The next trial divisor is $2(a + b) = 1260$ and $8869 \div 1260$ gives 7 as the largest number of units. This is c of the formula.

165. In case a square consists of **a whole number and a decimal part**, the figures in the *integral part* of the square root are found exactly as in Example 1, page 121. To find the *decimal part* of the root, we proceed as in the next illustrative example.

ORAL EXERCISES

Give the first figure in the square root of each of the following, and tell in which place it stands:

1. 12.645.
2. 1.2645.
3. 126.45.
4. 941.61.
5. 94.16.
6. 9.416.
7. 49.29.
8. 4.929.
9. 492.9.

Illustrative Example 2. Find the square root of 67.7329.

Solution.

	SQUARE	SQUARE ROOT
		$a + b + c$
	67.73 29	$8 + .2 + .03 = 8.23$
$a^2 = 8^2 =$	64	
$2a = 16$	3.73 29	
$b = .2$		
$2a + b = 16.2$	3.24	$= (2a + b)b$
$2(a + b) = 16.4$	.49 29	
$c = .03$		
$2(a + b) + c = 16.43$	.49 29	$= [2(a + b) + c]c$
	0	

Explanation. There is only one group of two figures to the left of the decimal point. Hence the first figure of the root is units' place.

In getting the second figure of the root, the trial divisor is $2a = 16$. The quotient is $b = .2$ since $.2 \times 16 = 3.2$. The quotient could not be .3 since $.3 \times 16 = 4.8$.

Similarly, in getting the third figure, we divide .4929 by 16.4 and the quotient is .03 since $.03 \times 16.4 = .492$.

Since the square of a decimal contains twice as many decimal places as the number itself, there will be one decimal figure in the root for every two in the square.

Illustrative Example 3. Find the square root of 9.1204.

Solution.

$$\begin{array}{rrr}
 & \text{SQUARE} & \text{SQUARE ROOT} \\
 & & a+b \\
 & 9.12\ 04 & 3+.02=3.02 \\
a^2=3^2= & \underline{9\qquad\quad} & \\
2a=6 & .12\ 04 & \\
\underline{b=\ .02} & & \\
2a+b=\underline{6.02} & \underline{.12\ 04} & =(2a+b)b \\
 & 0 &
\end{array}$$

Explanation. Since there is only one group to the left of the decimal point, the first figure of the root is in units' place.

In this case, in dividing .1204 by 6, the quotient is .02 since .02 × 6 = .12; that is, there is a zero in tenths' place, and there are only two terms in the root.

166. The Trial Divisor. From these examples we see that

(1) *Any trial divisor is twice that part of the root which has already been found.*

(2) *Any term of the root is found by dividing the last remainder by the trial divisor.*

(3) *In taking a quotient as a term of the root, allowance must be made because the last term is to be added to the trial divisor to form the complete divisor.*

EXERCISES

Find the square root of each of the following:

1. 294,849.
2. 37,636.
3. 872,356.
4. 599,076.
5. 3481.
6. 7569.
7. 1849.
8. 73,441.
9. 100,489.
10. 265.69.
11. 87.4225.
12. 170,569.
13. 357.21.
14. 16,641.
15. 32,761.
16. 2332.89.
17. 1197.16.
18. 6272.64.

167. The First Digit in the Square Root of a Decimal Number. In case a number has no integral part, the first term of its square root is found as in the following examples:

1. In .17 42 the first digit in the root is .4 since the square of .4 is .16, the largest square in .17.

2. In .05 42 the first digit in the root is .2 since the square of .2 is .04, the largest square in .05.

3. In .00 70 the first digit in the root is .08 since the square of .08 is .0064, the largest square in .0070.

4. In .00 07 the first digit in the root is .02 since the square of .02 is .00 04, the largest square in .0007.

5. In .00 00 69 the first digit in the root is .008 since the square of .008 is .000064, the largest square in .000069.

168. From these examples we have the following

Rule. To find the first digit in the square root of a decimal number:

(1) *Divide the number into groups of two figures each, counting from the decimal point toward the right, adding a zero if necessary to complete the last group.*

(2) *Take the square root of the largest square contained in the first group which is not all zeros, and prefix to it as many zeros as there are complete groups of zeros.*

Illustrative Example. Find the square root of .06783.

Solution.

		SQUARE	SQUARE ROOT
			$a +\ b +\ c$
		.06 78 30	.2 + .06 + .0004
$a^2 = .2^2$		= .04	
$2a = 2 \times .2 =$	.4	.02 78	
$b =$	.06		
$2a + b =$	.46	.02 76	$=(2a + b)b$
$2(a + b) =$	.52	.00 02 30 00	
$c =$	.0004		
$2(a + b) + c =$	.5204	.00 02 08 16	$=[2(a = b) + c]c$
		.00 00 21 84	

Explanation. According to the rule, .2 is the first term of the root because .04 is the largest square in .06 and there is no group preceding .06. The process is the same as in the case of an integral square, but special care is now needed in handling the decimal points, which is done exactly as in operations upon decimals in the process of division in arithmetic.

For instance, in finding the third term in this example, we divide .00023 by $2(.26) = .52$ and the quotient lies between .0004 and .0005. Hence $c = .0004$. Zeros are annexed to .00023 to correspond to the number of decimal places in the product $.5204 \times .0004$.

The three terms of the root thus found are $.2 + .06 + .0004 = .2604$.

To find the next term of the root we would divide .00002184 by $2(.2604) = .5208$, finding the quotient .00004. We would then add .00004 to .5208 and multiply the sum by .00004, annexing zeros to the dividend as before.

169. Approximate Square Roots. Evidently the process in this example may be carried on indefinitely. .2604 is an **approximation** to the square root of .06783; in fact, the square of .2604 differs from .06783 by only .00002184. The nearest approximation using three decimal places is .260. If the fourth figure were 5, or any digit greater than 5, then .261 would be the nearest approximation using three decimal places. Hence, four places must be found in order to be sure of the nearest approximation to three places; and five places must be found in order to be sure of the nearest approximation to four places, and so on.

Find the square root of each of the following, correct to two significant figures:

1. 387.
2. 5267.
3. 2.92.
4. 27.29.
5. 51.
6. 3.824.
7. 2.
8. 3.
9. 5.
10. 7.
11. 8.
12. 11.
13. .02.
14. .003.
15. .5.
16. .005.
17. .2.
18. .3.
19. .45.
20. .05.
21. .6.
22. .7.
23. .8.
24. .9.
25. .0067.
26. .0091.
27. .00087.
28. .000191.
29. .000471.
30. .00186.
31. .7004.
32. .0981.

170. Cube and Square Roots Compared. We have seen that the process for finding the **square root** of a polynomial is obtained by studying the relation of the expressions $a+b$, $a+b+c$, $a+b+c+d$, etc., to their respective squares.

The process for finding the **cube root** of a polynomial is obtained by studying the relation of the cubes,

$$\begin{aligned}&\boldsymbol{a^3+3a^2b+3ab^2+b^3} \text{ or } \boldsymbol{a^3+(3a^2+3ab+b^2)b},\\ &\boldsymbol{a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c},\\ \text{and }&\boldsymbol{a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c}\\ &\boldsymbol{+[3(a+b+c)^2+3(a+b+c)d+d^2]d},\end{aligned}$$

to their cube roots, $a+b$, $a+b+c$, and $a+b+c+d$.

An example will illustrate the process.

Example 1. Find the cube root of

$$27m^3+108m^2n+144mn^2+64n^3.$$

$$\begin{array}{rl|l}
\textbf{Given cube,} & 27m^3+108m^2n+144mn^2+64n^3 & a+b \\
a^3=(3m)^3= & \underline{27m^3\qquad\qquad\qquad\qquad\qquad} & \underline{3m+4n\textbf{, cube root}} \\
3a^2=3(3m)^2=27m^2 & 108m^2n+144mn^2+64n^3 = & \textbf{1st remainder} \\
3ab=3(3m)(4n)=36mn & & \\
b^2=(4n)^2=16n^2 & & \\
3a^2+3ab+b^2=27m^2+36mn+16n^2 & \underline{108m^2n+144mn^2+64n^3} = & (3a^2+3ab+b^2)b \\
 & \qquad\qquad\qquad\qquad 0 &
\end{array}$$

Explanation. The cube root of the first term, namely $3m$, is the first term of the root and corresponds to a of the formula. Cubing $3m$ gives $27m^3$, which is the a^3 of the formula.

Subtracting $27m^3$ leaves $108m^2n+144mn^2+64n^3$, which is the $(3a^2+3ab+b^2)b$ of the formula.

Since b is not yet known, we cannot find completely either factor of $(3a^2+3ab+b^2)b$, but since a has been found, we can get the first term of the factor $3a^2+3ab+b^2$; viz., $3a^2$ or $3(3m)^2=27m^2$, which is the partial divisor. Dividing $108m^2n$ by $27m^2$ we have $4n$, which is the b of the formula.

Then $3a^2+3ab+b^2=3(3m)^2+3(3m)(4n)+(4n)^2=27m^2+36mn+16n^2$ is the complete divisor. This expression is then multiplied by $b=4n$, giving $108m^2n+144mn^2+64n^3$, which corresponds to $(3a^2+3ab+b^2)b$ of the formula. On subtracting, the remainder is zero and the process ends. Hence, $3m+4n$ is the required root.

Example 2. Find the cube root of

$$33x^4 - 9x^5 + x^6 - 63x^3 + 66x^2 - 36x + 8.$$

We first arrange the terms with respect to the exponents of x.

$$\begin{array}{rl}
 & \qquad\qquad\quad a+\quad b+c \\
 & \qquad\qquad\quad \underline{x^2-3x+2,\ \text{cube root}} \\
\text{Given cube,} & x^6-9x^5+33x^4-63x^3+66x^2-36x+8 \\
a^3=(x^2)^3= & \underline{x^6\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
3a^2=3(x^2)^2=3x^4 & \big|\,-9x^5+33x^4-63x^3+66x^2-36x+8 \\
3a^2+3ab+b^2=3x^4-9x^3+9x^2 & \big|\,\underline{-9x^5+27x^4-27x^3\qquad\qquad\qquad\qquad} \\
3(a+b)^2=3(x^2-3x)^2=3x^4-18x^3+27x^2 & \big|\,6x^4-36x^3+66x^2-36x+8 \\
3(a+b)^2+3(a+b)c+c^2=3x^4-18x^3+33x^2-18x+4 & \big|\,\underline{6x^4-36x^3+66x^2-36x+8} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

The cube root of x^6, or x^2, is the first term of the root. The first partial divisor, which corresponds to $3a^2$ of the formula, is $3(x^2)^2 = 3x^4$. Dividing $-9x^5$ by $3x^4$ we have $-3x$, which is the second term of the root, corresponding to b of the formula.

After these two terms of the root have been found, we use $3(a+b)^2 = 3(x^2-3x)^2 = 3x^4 - 18x^3 + 27x^2$ as the new partial divisor and find that the next term of the root is 2.

The complete divisor is $3(a+b)^2 + 3(a+b)c + c^2 = 3x^4 - 18x^3 + 33x^2 - 18x + 4$. On multiplying this expression by 2 and subtracting, the remainder is zero. Hence the root is $x^2 - 3x + 2$.

In case there are four terms in the root, the next partial divisor is $3(a+b+c)^2$ and $3(a+b+c)^2 + 3(a+b+c)d + d^2$ is the complete divisor. The process is then precisely the same as in the preceding step.

In practice, the next term of the root is always found by dividing the remainder by three times the square of the first term of the root.

ORAL EXERCISES

1. How are the terms of a polynomial arranged for the extraction of its cube root?

2. How is the first term of the root found?

3. How is the first partial divisor found? the second?

4. How is the second term of the root found? the third?

WRITTEN EXERCISES

Find the cube roots of each of the following:

1. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.
2. $8a^3 - 12a^2b + 6ab^2 - b^3$.
3. $1728x^6 + 1728x^4y^3 + 576x^2y^6 + 64y^9$.
4. $a^3 + 3a^2b + 3a^2 + 3b^2 + 3ab^2 + 6ab + 3a + 3b + b^3 + 1$.
5. $x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3x^2y + 3x^2z + 3y^2z + 3yz^2 + 6xyz$.
6. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$.
7. $a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$.
8. $8x^6 - 36x^5 + 114x^4 - 207x^3 + 285x^2 - 225x + 125$.
9. $27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6$.
10. $1 - 9y^2 + 39y^4 - 99y^6 + 156y^8 - 144y^{10} + 64y^{12}$.
11. $125x^6 - 525x^5y + 60x^4y^2 + 1547x^3y^3 - 108x^2y^4 - 1701xy^5 - 729y^6$.
12. $64l^{12} - 576l^{10} + 2160l^8 - 4320l^6 + 4860l^4 - 2916l^2 + 729$.
13. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
14. $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9$.
15. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.
16. $343a^6 - 441a^5b + 777a^4b^2 - 531a^3b^3 + 444a^2b^4 - 144ab^5 + 64b^6$.
17. $a^{18} + 12a^{15} + 60a^{12} + 160a^9 + 240a^6 + 192a^3 + 64$.
18. $27l^{12} + 189l^{11} + 198l^{10} - 791l^9 - 594l^8 + 1701l^7 - 729l^6$.

ROOTS OF NUMBERS EXPRESSED IN ARABIC FIGURES

171. The cube root of a number expressed in Arabic figures may be found by the process used for polynomials as in the case of square root, pp. 120–125. An example will illustrate the process.

Example 1. Find the cube root of 389,017.

In order to decide how many digits there are in the root, we observe that $10^3 = 1000$, $100^3 = 1,000,000$. Hence the cube root of 389,017 lies between 10 and 100, that is, it contains two digits. Since $70^3 = 343,000$ and $80^3 = 512,000$, it follows that 7 is the largest number possible in tens' place. The work is arranged as follows:

$$
\begin{array}{r|l}
 & \qquad\quad a + b \\
\text{The given cube,} & 389\,017\;\underline{|\,70 + 3}, \text{ cube root.} \\
a^3 = 70^3 = & 343\,000 \\
\cline{2-2}
3a^2 = 3 \cdot 70^2 = 14700 & 46\,017 \quad \text{1st remainder.} \\
3ab = 3 \cdot 70 \cdot 3 = 630 & \\
b^2 = 3^2 \qquad = \underline{9} & \\
3a^2 + 3ab + b^2 = \overline{15339} & \underline{46\,017} = (3a^3 + 3ab + b^2)b. \\
 & \qquad 0
\end{array}
$$

Having decided as above that the a of the formula is 7 tens, we cube this and subtract, obtaining 46,017 as the remaining part of the power.

The first partial divisor, $3a^2 = 14,700$, is divided into 46,017, giving a quotient 3, which is the b of the formula. Hence, the first complete divisor, $3a^2 + 3ab + 3b^2$, is 15,339 and the product, $(3a^2 + 3ab + b^2)b$, is 46,017. Since the remainder is zero, the process ends and 73 is the cube root sought.

172. The cube of any number from 1 to 9 contains one, two, or three digits; the cube of any number between 10 and 99 contains four, five, or six digits; the cube of any number between 100 and 999 contains seven, eight, or nine digits, etc. Hence it is evident that if the digits of a number are separated into groups of three figures each, counting from units' place toward the left, the number of groups thus formed is the same as the number of digits in the root.

The left-hand group may contain one, two, or three digits, as the case may be.

Example 2. Find the cube root of 13,997,521.

$$\begin{array}{rrl}
 & & \quad a+b\ \ +c \\
\text{The given cube,} & 13\,997\,521 & |\underline{200+40+1=241}, \text{ cube root.}\\
a^3=200^3= & \underline{8\,000\,000} & \\
3\,a^2=120000 & 5\,997\,521 & \\
3\,ab=\ \ 24000 & & \\
b^2=\underline{\ \ \ 1600} & & \\
145600 & \underline{5\,824\,000} & =(3\,a^2+3\,ab+b^2)b \\
3(a+b)^2=172800 & 173\,521 & \\
3(a+b)c=\ \ \ \ 720 & & \\
c^2=\underline{\ \ \ \ \ \ \ 1} & & \\
173521 & \underline{173\,521} & =[3(a+b)^2+3(a+b)c+c^2]c. \\
 & 0 &
\end{array}$$

Since the root contains three digits, the first one is the cube root of 8, the largest integral cube in 13.

The first partial divisor, $3 \cdot 200^2 = 120{,}000$, is completed by adding $3\,ab = 3 \cdot 200 \cdot 40 = 24{,}000$, and $b^2 = 1600$.

The second partial divisor, $3(a + b)^2$, which stands for $3(200 + 40)^2 = 172{,}800$, is completed by adding $3(a + b)c$ which stands for $3 \cdot 240 \cdot 1 = 720$, and c^2 which stands for 1^2. At this step the remainder is zero and the root sought is 241.

EXERCISES

Find the square root of each of the following:

1. 58,081.
2. 795,664.
3. 11,641,744.

Find the cube root of each of the following:

4. 110,592.
5. 571,787.
6. 7,301,384.
7. 205,379.
8. 31,855,013.
9. 5,929,741.
10. 2,146,689.
11. 19,902,511.
12. 817,400,375.

173. Since the cube of a decimal fraction has three times as many places as the given decimal, it is evident that the cube root of a decimal fraction contains one decimal place for every three digits in the cube. Hence for the purpose of determining the places in the root, the decimal part of a cube should be separated into groups of three digits each, counting from the decimal point toward the right.

Example. Approximate the cube root of 34.567 to two places of decimals.

$$\begin{array}{rrl}
 & & a+b+\ c+d \\
 & 34.567 & \underline{|3+.2+.05+.007}=3.257 \\
a^3=3^3= & \underline{27.000} & \\
3a^2=3\cdot 3^2=27 & 7.567 & \\
3ab=3\cdot 3(.2)=\ \ 1.8 & & \\
b^2=(.2)^2=\underline{\ \ .04} & & \\
28.84 & \underline{5.768} & =(3a^2+3ab+b^2)b \\
3a'^2=3(3.2)^2=30.72 & 1.799000 & \\
3a'c=3(3.2)(.05)=\ \ .48 & & \\
c^2=(.05)^2=\underline{\ \ .0025} & & \\
31.2025 & \underline{1.560125} & =(3a'^2+3a'c'+c^2)c \\
3a''^2=3(3.25)^2=31.6875 & .238875000 & \\
3a''d=3(3.25)(.007)=\ \ .06825 & & \\
d^2=(.007)^2=\underline{\ \ .000049} & & \\
31.755799 & \underline{.222290593} & =(3a''^2+3a''d+d^2)d \\
 & .016584407 &
\end{array}$$

For the sake of brevity a' is used for $a+b$ and a'' for $a+b+c$. The decimal points are handled exactly as in arithmetic work.

174. Evidently the above process can be carried on indefinitely. 3.257 is an **approximation** to the cube root of 34.567. In fact the cube of 3.257 differs from 34.567 by less than the small fraction .017. The nearest approximation using two decimal places is 3.26. If the third decimal place were any digit less than 5, then 3.25 would be the nearest approximation using two decimal places. Hence, three places must be found in order to be sure of the nearest approximation to two places.

EXERCISES

Approximate the cube root of each of the following to two places of decimals:

1. 21.4736.
2. 6.5428.
3. 58.
4. 2.
5. 3.
6. .003.
7. .3917.
8. .5.
9. .05.
10. 6410.37.
11. .004178.
12. 200.002.
13. 572.274.
14. 31.7246.
15. 54913.416.

CHAPTER XII

EXPONENTS AND RADICALS

FRACTIONAL AND NEGATIVE EXPONENTS

175. The meaning heretofore attached to the word *exponent* cannot apply to a fractional or negative number.

E.g. Such an exponent as $\frac{2}{3}$ or -5 cannot indicate the *number of times* a base is used as a factor.

It is possible, however, to interpret fractional and negative exponents in such a way *that the laws of operations which govern positive integral exponents shall apply to these also.*

176. The laws for positive integral exponents are:

I. $a^m \cdot a^n = a^{m+n}$. § 44

II. $a^m \div a^n = a^{m-n}$. § 46

III. $(a^m)^n = a^{mn}$. § 154

IV. $(a^m \cdot b^n)^p = a^{mp}b^{np}$. § 155

V. $(a^m \div s^n)^p = a^{mp} \div s^{np}$. § 156

177. Fractional Exponents. Assuming Law I to hold for positive fractional exponents and letting r and s be positive integers, we determine as follows the meaning of $b^{\frac{r}{s}}$ (read b *exponent r divided by s*).

By definition, $\left(b^{\frac{r}{s}}\right)^s = b^{\frac{r}{s}} \cdot b^{\frac{r}{s}} \cdots$ to s factors,

which by Law I $= b^{\frac{r}{s}+\frac{r}{s}+\cdots \text{ to } s \text{ terms}} = b^{s \times \frac{r}{s}} = b^r$.

Hence, $b^{\frac{r}{s}}$ is one of the s equal factors of b^r.

That is, $b^{\frac{r}{s}} = \sqrt[s]{b^r}$, and in particular $b^{\frac{1}{s}} = \sqrt[s]{b}$.

Hence, $b^{\frac{1}{n}}$ is an nth root of b.

Similarly, from $\left(b^{\frac{1}{s}}\right)^r = b^{\frac{1}{s}} \cdot b^{\frac{1}{s}} \cdots$ to r factors, $= b^{\frac{r}{s}}$,

we show that $\qquad b^{\frac{r}{s}} = \left(b^{\frac{1}{s}}\right)^r = (\sqrt[s]{b})^r.$

Hence, $\qquad b^{\frac{r}{s}} = \sqrt[s]{b^r} = (\sqrt[s]{b})^r.$

Thus, a positive fractional exponent means *a root of a power or a power of a root, the numerator indicating the power and the denominator indicating the root.*

E.g. $a^{\frac{2}{3}} = \sqrt[3]{a^2} = (\sqrt[3]{a})^2$; $8^{\frac{2}{3}} = \sqrt[3]{64} = 4$, or $(\sqrt[3]{8})^2 = 2^2 = 4.$

178. Zero Exponents. Assuming that Law I holds for zero exponents, and letting t be any positive number, we have

$$b^t \cdot b^0 = b^{t+0} = b^t.$$

Hence, $\qquad b^0 = \frac{b^t}{b^t} = 1.$

It follows that any number with the exponent zero is equal to 1.

179. Negative Exponents. Assuming Law I to hold also for negative exponents, and letting t be a positive number, integral or fractional, we determine as follows the meaning of b^{-t} (*read b exponent negative t*).

By Law I, $\qquad b^t \cdot b^{-t} = b^{t-t} = b^0 = 1. \qquad$ § 44

Therefore, $\qquad b^{-t} = \frac{1}{b^t}.$

Hence a number with a negative exponent means *the same as the reciprocal of the number with a positive exponent of the same absolute value.*

E.g. $a^{-2} = \frac{1}{a^2}$; $4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{2^3} = \frac{1}{8}.$

ORAL EXERCISES

In the following replace the radicals by fractional exponents:

1. $3\sqrt{2}$.
2. $3\sqrt{x^3y}$.
3. $2\sqrt[3]{x^4}$.
4. $d\sqrt[3]{a^2b^5}$.
5. $\sqrt{a^3}\sqrt{b^{10}}$.
6. $r\sqrt[4]{b^3}$.
7. $\sqrt[4]{b^9} \cdot \sqrt[5]{a^6}$.
8. $c\sqrt[3]{a^5b^2}$.
9. $\sqrt[3]{(a-b)^5}$.

180. New Form of Notation. It thus appears that fractional and negative exponents simply provide *new notations for indicating operations already well known.* Sometimes one notation is more convenient and sometimes the other.

Fractional and negative exponents are also called *powers.*

E.g. $x^{\frac{2}{3}}$ may be read *x to the* $\frac{2}{3}$ *power*, and x^{-4} may be read *x to the negative 4th power.*

Radical signs may now be replaced by fractional exponents, or fractional exponents by radical signs.

E.g. $\sqrt[3]{x^2} + 3\sqrt[5]{x^3} \cdot \sqrt{y} + 5\sqrt[7]{x^4}\sqrt[3]{y^2} \equiv x^{\frac{2}{3}} + 3\,x^{\frac{3}{5}}y^{\frac{1}{2}} + 5\,x^{\frac{4}{7}}y^{\frac{2}{3}}.$

In a fraction, *any factor may be changed from numerator to denominator, or from denominator to numerator by changing the sign of its exponent,* as shown in the following examples:

1. $\dfrac{a^2b^{-3}}{c} = \dfrac{a^2 \cdot \dfrac{1}{b^3}}{c} = \dfrac{\dfrac{a^2}{b^3}}{c} = \dfrac{a^2}{b^3c}.$

2. $\dfrac{ab}{x^2} = abx^{-2}$, since $abx^{-2} = ab \cdot \dfrac{1}{x^2} = \dfrac{ab}{x^2}.$

3. $ab^{-3}c^2 = ac^2 \cdot \dfrac{1}{b^3} = \dfrac{ac^2}{b^3}.$

4. $32^{-\frac{4}{5}} = \dfrac{1}{32^{\frac{4}{5}}} = \dfrac{1}{(\sqrt[5]{32})^4} = \dfrac{1}{2^4} = \dfrac{1}{16}.$

WRITTEN EXERCISES

In the following replace fractional exponents by radicals:

1. $a^{\frac{2}{3}}$.
2. $x^{\frac{2}{5}}$.
3. $18^{\frac{1}{2}}$.
4. $a^{\frac{5}{8}}$.
5. $x^{\frac{3}{7}}$.
6. $a^{\frac{7}{3}}$.
7. $m^{\frac{4}{5}}n^{\frac{3}{4}}$.
8. $a^{\frac{3}{4}} \cdot b^{\frac{5}{3}}$.
9. $ax^{\frac{7}{2}}$.
10. $bx^{\frac{3}{4}}$.
11. $a^{\frac{2}{3}}b^{\frac{7}{5}}$.
12. $m^{\frac{3}{4}}n^{\frac{4}{5}}l^{\frac{7}{8}}$.
13. $m^{\frac{3}{2}}n^{\frac{2}{3}}l^{\frac{4}{5}}$.
14. $\dfrac{a^{\frac{2}{3}}b^{\frac{3}{2}}}{a^{\frac{1}{2}}b^{\frac{2}{3}}}$.
15. $\dfrac{4\,a^{\frac{5}{6}} \cdot b^{\frac{3}{2}}}{5\,a^{\frac{1}{2}}b^{\frac{1}{6}}}$.
16. $\dfrac{c^{\frac{1}{2}}d^{\frac{1}{4}}}{d^{\frac{1}{3}}c^{\frac{1}{6}}}$.

In the following change all expressions having negative exponents to equivalent expressions having positive exponents:

17. $\frac{1}{b^{-\frac{1}{2}}}$.

18. $a^{-2}b^{3}$.

19. $\frac{3}{m^{-7}}$.

20. $\frac{3a^{-\frac{2}{3}}}{c^{-2}b^{3}}$.

21. $\frac{n^{-\frac{1}{3}}m^{-1}}{4}$.

22. $\frac{3a^{-\frac{2}{3}}b^{-2}}{4a^{-\frac{1}{2}}b^{-\frac{1}{2}}}$.

23. $3(a+b)^{-\frac{2}{3}}$.

24. $\left(\frac{1}{a}\right)^{-\frac{3}{4}}$.

25. $\left(\frac{1}{27}\right)^{-\frac{1}{3}}$.

181. Extended Laws of Exponents. Fractional and negative exponents were defined so as to conform to Law I, §§ 177, 179. It is now possible to show that, when so defined, they also conform to Laws II, III, IV, and V there given. Thus:

Law II. $a^m \div a^n = a^{m-n}$ *for all values of* m *and* n, *positive or negative, integral or fractional.*

E.g.

$$a^2 \div a^5 = a^{2-5} = a^{-3}.$$

$$a^{\frac{1}{2}} \div a^{-\frac{1}{3}} = a^{\frac{1}{2}-\left(-\frac{1}{3}\right)} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}.$$

Law III. $(a^m)^n = a^{mn}$ *for all values of* m *and* n.

E.g.

$$(a^{-3})^2 = a^{2(-3)} = a^{-6}.$$

$$\left(a^{\frac{1}{3}}\right)^{-\frac{2}{5}} = a^{\frac{1}{3}\left(-\frac{2}{5}\right)} = a^{-\frac{2}{15}}.$$

Law IV. $(a^m \cdot b^n)^p = a^{mp} \cdot b^{np}$ *for all values of* m, n, *and* p.

E.g.

$$\left(a^{\frac{1}{2}}b^{-3}\right)^{\frac{2}{3}} = a^{\frac{1}{2}\cdot\frac{2}{3}}b^{-3\cdot\frac{2}{3}} = a^{\frac{1}{3}}b^{-2}.$$

$$(a^{-3}b^2)^{-\frac{1}{6}} = a^{-3\left(-\frac{1}{6}\right)}b^{2\left(-\frac{1}{6}\right)} = a^{\frac{1}{2}}b^{-\frac{1}{3}}.$$

Law V. $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$ *for all values of* m, n, *and* p.

E.g.

$$\left(\frac{a^{-3}}{b^{\frac{1}{2}}}\right)^{\frac{2}{3}} = \frac{a^{-3\cdot\frac{2}{3}}}{b^{\frac{1}{2}\cdot\frac{2}{3}}} = \frac{a^{-2}}{b^{\frac{1}{3}}}.$$

$$\left(\frac{a^3}{b^{-2}}\right)^{-\frac{1}{6}} = \frac{a^{3\left(-\frac{1}{6}\right)}}{b^{-2\left(-\frac{1}{6}\right)}} = \frac{a^{-\frac{1}{2}}}{b^{\frac{1}{3}}}.$$

182. Affecting a Monomial with an Exponent. From Laws III, IV, and V, it follows that *any monomial is affected with a given exponent by multiplying the exponent of each factor of the monomial by the given exponent.*

Since exponents may be integral or fractional, positive or negative, the statement just made is equivalent to the two principles given in § 160.

Example 1. $(a^{\frac{1}{2}}b^{-2}c^{3})^{-\frac{2}{3}} = a^{-\frac{2}{3}\cdot\frac{1}{2}}b^{-\frac{2}{3}\cdot(-2)}c^{-\frac{2}{3}\cdot 3} = a^{-\frac{1}{3}}b^{\frac{4}{3}}c^{-2}.$

Example 2. $\left(\dfrac{3a^2x^6}{by^4}\right)^{-\frac{1}{2}} = \dfrac{3^{-\frac{1}{2}}a^{-1}x^{-3}}{b^{-\frac{1}{2}}y^{-2}} = \dfrac{b^{\frac{1}{2}}y^2}{3^{\frac{1}{2}}ax^3}.$

Example 3. $\left(\dfrac{8x^9}{27y^6}\right)^{-\frac{1}{3}} = \left(\dfrac{27y^6}{8x^9}\right)^{\frac{1}{3}} = \dfrac{27^{\frac{1}{3}}y^2}{8^{\frac{1}{3}}x^3} = \dfrac{3y^2}{2x^3}.$

ORAL EXERCISES

Remove the parenthesis in each of the following:

1. $(a^{\frac{1}{3}}b^{\frac{1}{2}}c)^{\frac{1}{2}}.$
2. $(2xy^{\frac{2}{3}}z^{\frac{1}{4}})^{2}.$
3. $(3a^{\frac{2}{3}}b^{\frac{1}{5}}c^{2})^{3}.$
4. $(8x^2y^3)^{-\frac{1}{3}}.$
5. $(16a^{-2}b^{-4}c^{6})^{\frac{1}{4}}.$
6. $(32a^{-5}b^{10}c)^{\frac{1}{5}}.$
7. $(27a^{3}b^{-6}x^{2})^{\frac{1}{3}}.$
8. $(m^{-\frac{3}{2}}n^{\frac{2}{3}}p^{\frac{4}{5}})^{\frac{1}{4}}.$
9. $(r^{\frac{2}{5}}s^{\frac{3}{4}})^{\frac{1}{6}}.$
10. $(x^{-2}y^{3}z^{4})^{\frac{1}{12}}.$
11. $(a^{-\frac{4}{5}}b^{3}c^{-\frac{1}{2}})^{-\frac{1}{2}}.$
12. $(x^{-\frac{1}{3}}y^{-\frac{1}{4}}z^{-\frac{1}{2}})^{-12}.$

WRITTEN EXERCISES

Perform the operations indicated by the exponents in each of the following, writing the results without negative exponents and in as simple form as possible:

1. $(\frac{16}{25})^{-\frac{1}{2}}.$
2. $(\frac{27}{64})^{-\frac{1}{3}}.$
3. $(\frac{25}{36})^{\frac{3}{2}}.$
4. $(27a^{-9})^{\frac{1}{3}}.$
5. $(x^{-\frac{2}{3}}y^{\frac{4}{5}})^{-\frac{3}{4}}.$
6. $25^{\frac{3}{2}}.$
7. $25^{-\frac{3}{2}}.$
8. $25^{0}.$
9. $(\frac{27}{8})^{-\frac{2}{3}}.$
10. $(\frac{8}{27})^{\frac{2}{3}}.$
11. $(0.25)^{\frac{1}{2}}.$
12. $(0.25)^{-\frac{1}{2}}.$
13. $(.0009)^{\frac{3}{2}}.$
14. $(.027)^{\frac{1}{3}}.$
15. $(32a^{-5}b^{10})^{\frac{1}{5}}.$
16. $8^{\frac{1}{3}}\cdot 4^{-\frac{1}{2}}.$

17. $\left(\frac{a^{-8}}{16}\right)^{-\frac{1}{4}}$. 19. $(\frac{1}{32})^{-\frac{1}{5}}(\frac{1}{81})^{-\frac{3}{4}}$. 21. $(-\frac{243}{32})^{\frac{1}{5}}\div(\frac{16}{81})^{-\frac{1}{4}}$.

18. $(27\,x^6y^{-3}z^{-1})^{-\frac{1}{3}}$. 20. $\sqrt[3]{\frac{512}{729}}\cdot(\frac{256}{729})^{-\frac{1}{2}}$. 22. $\left(\frac{x^3y^{-4}}{x^{-2}y}\right)^3\left(\frac{x^{-3}y^2}{xy^{-1}}\right)^5$.

23. $\sqrt[4]{81\,a^{-4}b^8}(-27\,a^3b^{-6})^{-\frac{1}{3}}$. 26. $\sqrt{16\,a^{-4}b^{-6}}\cdot\sqrt[3]{8\,a^3b^{-6}}$.

24. $\left(\frac{m^{-1}n}{m^{\frac{1}{2}}n^{-\frac{2}{3}}}\right)^{-2}\div\left(\frac{m^{-3}}{n^{-1}}\right)^{-\frac{1}{3}}$. 27. $(-2^{-2}a^{-3}b^{-6})^{-\frac{1}{3}}(-2^{-\frac{1}{3}}a^{-\frac{1}{2}}b^{-1})^2$.

25. $\left(\frac{a^3b^{-2}}{a^{-2}b^3}\right)^{\frac{1}{5}}\div\left(\frac{a^3b^{-3}}{a^{-3}b^3}\right)^{-1}$. 28. $\left(\frac{81\,r^{-16}s^4t}{625\,r^4s^8t}\right)^{\frac{1}{4}}\left(\frac{9\,r^2s^{-4}t}{25\,r^{10}s^0t}\right)^{-\frac{1}{2}}$.

Multiply:

29: $x^{-2}+x^{-1}y^{-1}+y^{-2}$ by $x^{-1}-y^{-1}$.

30. $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}$.

31. $x^{\frac{4}{5}}+x^{\frac{3}{5}}y^{\frac{1}{5}}+x^{\frac{2}{5}}y^{\frac{2}{5}}+x^{\frac{1}{5}}y^{\frac{3}{5}}+y^{\frac{4}{5}}$ by $x^{\frac{1}{5}}-y^{\frac{1}{5}}$.

32. $\sqrt[4]{a^3}+\sqrt[5]{b^2}$ by $\sqrt[4]{a^3}-\sqrt[5]{b^2}$.

33. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$ by $x^{\frac{2}{3}}-y^{\frac{2}{3}}$.

34. $x-3\,x^{\frac{2}{3}}y^{-\frac{1}{2}}+3\,x^{\frac{1}{3}}y^{-1}-y^{-\frac{3}{2}}$ by $x^{\frac{2}{3}}-2\,x^{\frac{1}{3}}y^{-\frac{1}{2}}+y^{-1}$.

35. $x^{\frac{3}{2}}+xy^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^{\frac{9}{4}}$ by $x^{\frac{1}{2}}-y^{\frac{3}{4}}$.

Divide:

36. $x^2-x^{\frac{11}{6}}y+x^{\frac{3}{2}}y-x^{\frac{1}{2}}y^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{4}{3}}-y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{3}}y+y$.

37. $3\,a^{\frac{7}{4}}-ab^{\frac{2}{3}}+4\,ab^2-3\,a^{\frac{3}{4}}b+b^{\frac{5}{3}}-4\,b^3$ by $3\,a^{\frac{3}{4}}-b^{\frac{2}{3}}+4\,b^2$.

38. $x^2-3\,x^{\frac{5}{3}}+6\,x^{\frac{4}{3}}-7\,x+6\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}+1$ by $x^{\frac{2}{3}}-x^{\frac{1}{3}}+1$.

39. $4\,x^{\frac{5}{4}}b^{-2}-17\,x^{\frac{1}{4}}b^2+16\,x^{-\frac{3}{4}}b^6$ by $2\,x^{\frac{1}{2}}-b^2-4\,x^{-\frac{1}{2}}b^4$.

Find the square root of:

40. $4\,x^2-4\,xy^{\frac{1}{3}}+4\,xz^{-\frac{1}{2}}+y^{\frac{2}{3}}-2\,y^{\frac{1}{3}}z^{-\frac{1}{2}}+z^{-1}$.

41. $a^{-\frac{2}{3}}-2\,a^{-\frac{1}{3}}b^{\frac{2}{3}}+b^{\frac{4}{3}}+2\,a^{-\frac{1}{3}}c^2+c^4-2\,b^{\frac{2}{3}}c^2$.

42. $b^{-\frac{1}{2}}-2\,b^{-\frac{1}{4}}c^{\frac{1}{3}}+c^{\frac{2}{3}}+2\,b^{-\frac{1}{4}}d^{\frac{1}{5}}+2\,b^{-\frac{1}{4}}e^{-\frac{1}{2}}-2\,c^{\frac{1}{3}}d^{\frac{1}{5}}+d^{\frac{2}{5}}$
$+2\,d^{\frac{1}{5}}e^{-\frac{1}{2}}-2\,c^{\frac{1}{3}}e^{-\frac{1}{2}}+e^{-1}$.

HISTORICAL NOTE

The **full use of exponents**, including fractional and negative, was the culmination of a long historical development. **Bhaskara**, an Indian mathematician of the 11th century, used the initial syllable of the word for *unknown quantity*, repeating this to denote powers. Similar devices prevailed for five hundred years.

For example, in the 16th century, $x^2 + 5x - 4$ would have been written $1\,Z\,p\,5\,R\,m\,4$, where *Z* stands for *zensus*, i.e. x^2, *p* for plus, *R* for *root* (of the *square*), and *m* for *minus*.

Bombelli (1572) wrote this expression $1\overset{2}{\smile}\,p\,5\overset{1}{\smile}\,m\,4$, where $\overset{1}{\smile}$, $\overset{2}{\smile}$ are used to indicate the first and second powers of the unknown quantity, respectively. Clearly this notation would not permit the introduction of the powers of more than one quantity into any one expression.

Stevinus (1586) used ①, ②, ③, for first, second, and third powers, but this has the same limitation as the notation of Bombelli.

Vieta (1591) wrote A*quad*, A*cub*, ⋯ for A^2, A^3, ⋯ . This had the advantage of permitting the introduction of powers of more than one quantity.

Harriot (1631) wrote *aa* for a^2, *aaa* for a^3, etc.

Finally **Descartes** (1637) used the present exponential notation for a^2, a^3, a^4, etc. He did not use fractional or negative exponents.

Wallis (1659) used fractional and negative exponents and explained their meaning in much the same manner as in §§ 177–179 of this book.

Newton (1676) used expressions such as a^n, containing literal exponents, without any restriction as to whether the exponents were fractional or integral, positive or negative.

Euler (1707–1783) showed that logarithms could most naturally be regarded as exponents and that the laws of exponents determine the laws of operations by means of logarithms.

ORAL EXERCISES

1. Read $3x^4 + 2x^3 - 7x + 5$, using Bombelli's notation.
2. In what important respect did Vieta's notation differ from those that preceded him?
3. Read $5x^3 - 7x^2 + 4x + 9$, using Harriot's notation.
4. Who first introduced the notation a^3, etc.?
5. Who first used fractional exponents? When?

REDUCTION OF RADICAL EXPRESSIONS

183. Radical Expression. An expression containing a root indicated by the radical sign or by a fractional exponent is called a *radical expression.* The expression whose root is indicated is the **radicand.**

E.g. $\sqrt[3]{5}$ and $(1+x)^{\frac{2}{3}}$ are radical expressions. In each case the **index** of the radical is 3.

A reduction of a radical expression consists in *changing its form without changing its value.*

Each reduction is based upon one or more of the Laws I to V, as extended in § 176.

184. Case I. To Remove a Factor from the Radicand. This reduction is possible only when the radicand contains a factor which is a perfect power of the degree indicated by the index of the root, as shown in the following examples:

Example 1. $\sqrt{72}=\sqrt{36\cdot 2}=\sqrt{36}\sqrt{2}=6\sqrt{2}$. See § 158.

Example 2. $(a^3x^2y^6)^{\frac{1}{3}}=(a^3y^6\cdot x^2)^{\frac{1}{3}}=(a^3y^6)^{\frac{1}{3}}\cdot(x^2)^{\frac{1}{3}}=ay^2x^{\frac{2}{3}}$.

This reduction involves Law IV, and may be written in symbols thus:

$$\sqrt[r]{x^{kr}y}=\sqrt[r]{x^{kr}}\sqrt[r]{y}=x^k\sqrt[r]{y}.$$

ORAL EXERCISES

In each of the following remove as many factors as possible from the radicand:

1. $\sqrt{45}$.
2. $\sqrt{125}$.
3. $\sqrt{50}$.
4. $\sqrt{72}$.
5. $3\sqrt{x^2y}$.
6. $2\sqrt[3]{x^4}$.
7. $\sqrt{(a+b)^3}$.
8. $\sqrt[3]{(a-b)^5}$.
9. $\sqrt[5]{a^6b^4c^9}$.
10. $\sqrt[5]{a^3b^7c^{13}}$.
11. $\sqrt{a^3\sqrt[4]{b^9}}$.
12. $\sqrt[3]{b^7\sqrt[4]{a^5}}$.
13. $\sqrt[4]{32}$.
14. $\sqrt[4]{48}$.
15. $\sqrt[6]{128}$.
16. $\sqrt[3]{a^3b}$.
17. $\sqrt[2]{7x^2-14xy+7y^2}$.
18. $\sqrt[3]{4a^3-12a^2b+12ab^2-4b^3}$.

185. Case II. To Introduce a Factor into the Radicand. This process simply reverses the steps of the foregoing reduction, and hence also involves Law IV.

Example 1. $6\sqrt{2} = \sqrt{6^2} \cdot \sqrt{2} = \sqrt{36 \cdot 2} = \sqrt{72}$. See § 184.

Example 2. $ay^2x^{\frac{2}{3}} = \sqrt[3]{(ay^2)^3} \cdot \sqrt[3]{x^2} = \sqrt[3]{(ay^2)^3x^2} = \sqrt[3]{a^3y^6x^2}$.

Example 3. $x\sqrt[r]{y} = \sqrt[r]{x^r}\sqrt[r]{y} = \sqrt[r]{x^ry}$.

ORAL EXERCISES

In each of the following introduce into the radicand the factor which appears as the coefficient of the radical:

1. $2\sqrt{5}$.
2. $3(7)^{\frac{1}{2}}$.
3. $3\sqrt{xy}$.
4. $d\sqrt[3]{a^2b}$.
5. $c\sqrt[3]{a^5b^2}$.
6. $a\sqrt{x}$.
7. $n\sqrt{m^5}$.
8. $r\sqrt{b^3}$.
9. $3\sqrt[3]{a+b}$.
10. $2\sqrt[5]{cx}$.
11. $2\sqrt[3]{x^4}$.
12. $a^2\sqrt[4]{c^3}$.
13. $a\sqrt[n]{b}$.
14. $a^2\sqrt[n]{a}$.
15. $b\sqrt[n]{b}$.
16. $c^2\sqrt[4]{c}$.
17. $3\sqrt[3]{3}$.
18. $a^4\sqrt[n]{a}$.
19. $x\sqrt[2m]{2}$.
20. $2a\sqrt[n]{a^2}$.
21. $3ab\sqrt[2p]{a^3b^2}$.

186. Case III. To Reduce a Fractional Radicand to the Integral Form. This reduction involves Law IV or Law V, and may always be accomplished.

Example 1. $\sqrt{\frac{3}{5}} = \sqrt{\frac{15}{25}} = \sqrt{\frac{1}{25} \cdot 15} = \frac{1}{5}\sqrt{15}$. Law IV

Example 2. $\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} = \left(\frac{a^2-b^2}{(a+b)^2}\right)^{\frac{1}{2}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{[(a+b)^2]^{\frac{1}{2}}} = \frac{(a^2-b^2)^{\frac{1}{2}}}{a+b}$.

Law V

Example 3. $\frac{\sqrt[3]{40}}{\sqrt[3]{5}} = \sqrt[3]{\frac{40}{5}} = \sqrt[3]{8} = 2$.

In symbols, we have $\sqrt[r]{\frac{a}{b}} = \sqrt[r]{\frac{ab^{r-1}}{b^r}} = \frac{\sqrt[r]{ab^{r-1}}}{\sqrt[r]{b^r}} = \frac{1}{b}\sqrt[r]{ab^{r-1}}$.

ORAL EXERCISES

In each of the following reduce the radicand to the integral form:

1. $\sqrt{\frac{3}{4}}$.
2. $\sqrt{\frac{5}{9}}$.
3. $\sqrt{\frac{1}{2}}$.
4. $\sqrt{\frac{1}{3}}$.
5. $\sqrt[3]{\frac{1}{2}}$.
6. $\sqrt[3]{\frac{1}{4}}$.
7. $\sqrt{\frac{1}{a^3}}$.
8. $\sqrt{\frac{1}{a}}$.
9. $\sqrt[3]{\frac{1}{a^2}}$.
10. $\sqrt{\frac{1}{8}}$.
11. $\sqrt[3]{\frac{1}{32}}$.
12. $\sqrt[3]{\frac{a}{b}}$.
13. $\sqrt{\frac{5}{12}}$.
14. $\sqrt{\frac{7}{18}}$.
15. $\sqrt[3]{\frac{a}{xy^2}}$.
16. $\sqrt{\frac{b}{ac^3}}$.
17. $\sqrt[3]{\frac{a}{b^2c^3}}$.
18. $\sqrt[3]{\frac{ab}{x^2y^4}}$.
19. $\sqrt[2]{\frac{1}{27}}$.
20. $\sqrt[3]{\frac{1}{9}}$.

187. Case IV. To Reduce a Radical to an Equivalent Radical of Lower Index. This reduction can be made when the radicand is a perfect power corresponding to *some factor of the index.*

Example 1. $\sqrt[6]{8} = 8^{\frac{1}{6}} = 8^{\left(\frac{1}{3 \cdot 2}\right)} = (8^{\frac{1}{3}})^{\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}.$

Example 2. $\sqrt[4]{a^2 + 2\,ab + b^2} = \sqrt{\sqrt{a^2 + 2\,ab + b^2}} = \sqrt{a + b}.$

This reduction involves Law III as follows:

$$(x^{\frac{1}{r}})^{\frac{1}{s}} = (x^{\frac{1}{s}})^{\frac{1}{r}} = x^{\frac{1}{rs}}, \qquad \text{See § 181}$$

from which we have $\sqrt[rs]{x} = \sqrt[r]{\sqrt[s]{x}} = \sqrt[s]{\sqrt[r]{x}}.$

By this reduction the finding of a root whose index is a composite number is made to depend upon roots of lower degree.

E.g. A fourth root may be found by taking the square root twice; a sixth root, by taking a square root and then a cube root, etc.

In some cases the reduction may be made by expressing the power and the root involved by means of a fractional exponent and then reducing this fraction to its lowest terms.

E.g. $\sqrt[6]{2^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2},$

and $\sqrt[4]{a^2 + 2\,ab + b^2} = \sqrt[4]{(a + b)^2} = (a + b)^{\frac{2}{4}} = (a + b)^{\frac{1}{2}}.$

ORAL EXERCISES

Reduce each of the following radicals to an equivalent radical of lower index:

1. $\sqrt[6]{(a-b)^3}$.
2. $\sqrt[4]{a^6}$.
3. $\sqrt[10]{x^5}$.
4. $\sqrt[4]{64}$.
5. $\sqrt[4]{x^2-2xy+y^2}$.
6. $\sqrt[6]{125}$.
7. $\sqrt[6]{(x+y)^3}$.
8. $\sqrt[6]{(x+y)^2}$.
9. $\sqrt[4]{a^2b^6}$.
10. $\sqrt[6]{a^9b^3}$.
11. $\sqrt[4]{225}$.
12. $\sqrt[4]{625a^6}$.
13. $\sqrt[4]{(a-b)^6}$.
14. $\sqrt[6]{25y^2a^4}$.
15. $\sqrt[8]{16x^4y^{12}}$.

188. Case V. To Reduce a Radical to an Equivalent Radical of Higher Index. This reduction is possible whenever the required index is a *multiple* of the given index. Thus:

$$x^{\frac{r}{s}} = (x^{\frac{r}{s}})^{\frac{t}{t}} = x^{\frac{rt}{st}}. \qquad \text{Law III. § 181}$$

Example 1. $\sqrt{a} = a^{\frac{1}{2}} = (a^{\frac{1}{2}})^{\frac{3}{3}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$.

Example 2. $\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$.

189. Radicals of the Same Order. Two radical expressions are said to be of the *same order* when their indicated roots have the *same index*.

By case V two radicals of *different* orders may be changed to equivalent radicals of the *same* order. The index of the new radical will be a common multiple of the given indices.

WRITTEN EXERCISES

Reduce each of the following pairs of radicals to radicals of the same order:

1. $x^{\frac{2}{5}}, x^{\frac{1}{3}}$.
2. $3\sqrt{x^2y},\ 2\sqrt[3]{x^4}$.
3. $\sqrt{(a+b)},\ \sqrt[3]{(a-b)^2}$.
4. $\sqrt[4]{3},\ \sqrt[6]{2}$.
5. $\sqrt[9]{3},\ \sqrt[6]{2}$.
6. $ax^{\frac{7}{2}},\ bx^{\frac{3}{4}}$.
7. $4(ab)^{\frac{3}{4}},\ 3(cx)^{\frac{2}{3}}$.
8. $a(xy)^{\frac{2}{3}},\ b(cd)^{\frac{5}{6}}$.

190. Radicals in the Simplest Form. In general, radical expressions should be reduced at once to the lowest possible order and the radicand made *integral and as small* as possible. A radical is then said to be in its **simplest form.**

ADDITION AND SUBTRACTION OF RADICALS

191. Similar Radicals. Two radical expressions are said to be **similar** when they are of the same order and have the same radicands.

E.g. $3\sqrt{7}$ and $5\sqrt{7}$ are similar radicals as are also $a\sqrt[7]{x^4}$ and $b\sqrt[7]{x^4}$.

If two radicals can be reduced to similar radicals, they may be added or subtracted according to § 28.

Example 1. Find the sum of $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{98}$.

By § 184, $\sqrt{8} = 2\sqrt{2}$, $\sqrt{50} = 5\sqrt{2}$, and $\sqrt{98} = 7\sqrt{2}$.
Hence $\sqrt{8} + \sqrt{50} + \sqrt{98} = 2\sqrt{2} + 5\sqrt{2} + 7\sqrt{2} = 14\sqrt{2}$.

Example 2. Simplify $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}}$.

By § 186, $\sqrt{\frac{1}{5}} = \frac{1}{5}\sqrt{5}$, $\sqrt{20} = 2\sqrt{5}$, $\sqrt{3\frac{1}{5}} = \sqrt{\frac{16}{5}} = 4\sqrt{\frac{1}{5}} = \frac{4}{5}\sqrt{5}$.
Hence $\sqrt{\frac{1}{5}} - \sqrt{20} + \sqrt{3\frac{1}{5}} = \frac{1}{5}\sqrt{5} - 2\sqrt{5} + \frac{4}{5}\sqrt{5} = -\sqrt{5}$.

If two radicals cannot be reduced to equivalent similar radicals, their sum can only be indicated.

E.g. The sum of $\sqrt{2}$ and $\sqrt[3]{5}$ is $\sqrt{2} + \sqrt[3]{5}$.

ORAL EXERCISES

Reduce the following to similar radicals:

1. $\sqrt{\frac{1}{2}}$, $\sqrt{2}$, $\sqrt{8}$.
2. $\sqrt{\frac{1}{3}}$, $\sqrt{3}$, $\sqrt{12}$.
3. $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$.
4. $\sqrt{\frac{3}{4}}$, $\sqrt{12}$, $\sqrt{27}$.
5. $\sqrt{5}$, $\sqrt{20}$, $\sqrt{45}$.
6. $\sqrt{a^3b}$, $\sqrt{a^3b^3}$, $\sqrt{ab^5}$.
7. $\sqrt{\frac{1}{8}}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{1}{32}}$.
8. $\sqrt{50}$, $\sqrt{\frac{1}{2}}$, $\sqrt{72}$.
9. $\sqrt{48}$, $\sqrt{\frac{1}{3}}$, $\sqrt{75}$.
10. $\sqrt[3]{\frac{1}{4}}$, $\sqrt[3]{16}$, $\sqrt[3]{54}$.

WRITTEN EXERCISES

In the following, reduce to similar radicals and add as indicated:

1. $3\sqrt{45} + 2\sqrt{125}$.
2. $d\sqrt[3]{a^2b^5} + c\sqrt[3]{a^5b^2}$.
3. $a^{\frac{4}{3}} + a^{\frac{7}{3}}$.
4. $n\sqrt{m^5} + m^{\frac{3}{2}}$.
5. $18^{\frac{1}{2}} + \sqrt{32}$.
6. $\sqrt{12} + 48^{\frac{1}{2}}$.
7. $3(50)^{\frac{1}{2}} + 4(72)^{\frac{1}{2}}$.
8. $(40)^{\frac{1}{3}} + (625)^{\frac{1}{3}}$.
9. $\sqrt{28} + 3\sqrt{7} - 2\sqrt{63}$.
10. $\sqrt[3]{24} - \sqrt[3]{81} - \sqrt[3]{\frac{3}{125}}$.
11. $\sqrt[5]{a^6} + \sqrt[5]{a^{11}} - \sqrt[5]{32\,a}$.
12. $2\sqrt{48} - 3\sqrt{12} + 3\sqrt{\frac{1}{3}}$.
13. $\sqrt{\frac{1}{7}} + \sqrt{63} + 5\sqrt{7}$.
14. $\sqrt{99} - 11\sqrt{\frac{1}{11}} + \sqrt{44}$.
15. $2\sqrt{\frac{4}{7}} + 3\sqrt{\frac{9}{7}} + \sqrt{175}$.
16. $\sqrt[4]{\frac{9}{625}} + 6\sqrt{\frac{1}{3}} - \sqrt{12}$.
17. $\sqrt[6]{9} + \sqrt[9]{27} + \sqrt[3]{-24}$.
18. $(x^2 + 1)\sqrt{a^3 + a^2b} - \sqrt{(a^2 - b^2)(a - b)}$.

MULTIPLICATION OF RADICALS

192. Radicals of Same Order. Radicals of the *same order* are multiplied by multiplying the radicands. If they are not of the same order, they may be reduced to the same order according to § 189.

E.g. $\sqrt{a} \cdot \sqrt[3]{b} = a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = \sqrt[6]{a^3} \cdot \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$.

In many cases this reduction is not desirable. Thus, $\sqrt[3]{x^2} \cdot \sqrt{y^3}$ is written $x^{\frac{2}{3}}y^{\frac{3}{2}}$ rather than $\sqrt[6]{x^4y^9}$.

193. Radicals with Same Base. Radicals with the *same base* are multiplied by first expressing them by means of fractional exponents and then adding the exponents.

E.g. $\sqrt[5]{x^2} \cdot \sqrt{x^3} = x^{\frac{2}{5}} \cdot x^{\frac{3}{2}} = x^{\frac{2}{5}+\frac{3}{2}} = x^{\frac{19}{10}}$.

Again, $\sqrt{2} \cdot \sqrt[3]{2} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}}$, and $2\sqrt{5} \cdot 3\sqrt{5} = 6 \cdot 5 = 30$.

ORAL EXERCISES

Find the following products:

1. $\sqrt{ab} \cdot \sqrt{a^2b^3}$.
2. $\sqrt[3]{ax} \cdot \sqrt[3]{xy}$.
3. $\sqrt{2} \cdot \sqrt{8}$.
4. $\sqrt{8} \cdot \sqrt{18}$.
5. $\sqrt{3} \cdot \sqrt{27}$.
6. $\sqrt{a} \cdot \sqrt{a^3}$.
7. $\sqrt{a^3b} \cdot \sqrt{ab}$.
8. $\sqrt[3]{a^2b} \cdot \sqrt[3]{ab^3}$.
9. $\sqrt[3]{xy} \cdot \sqrt[3]{x^2y^2}$.
10. $\sqrt[3]{3\,a} \cdot \sqrt[3]{9\,a^2}$.
11. $\sqrt{a-b} \cdot \sqrt{a+b}$.
12. $\sqrt{a+3} \cdot \sqrt{a+4}$.
13. $a^{\frac{2}{3}} \cdot a^{\frac{1}{3}}$.
14. $a^{\frac{2}{5}} \cdot a^{\frac{2}{5}}$.
15. $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$.
16. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$.
17. $x^{\frac{3}{4}} \cdot x^{\frac{1}{2}}$.
18. $x^{\frac{2}{3}} \cdot x^{\frac{1}{6}}$.

194. The principles just enumerated are used in multiplying polynomials containing radicals.

WRITTEN EXERCISES

Find the following products:

1. $(3+\sqrt{11})(3-\sqrt{11})$.
2. $(3\sqrt{2}+4\sqrt{5})(4\sqrt{2}-5\sqrt{5})$.
3. $(2+\sqrt{3}+\sqrt{5})(3+\sqrt{3}-\sqrt{5})$.
4. $(3\sqrt{2}-2\sqrt{18}+2\sqrt{7})(2\sqrt{2}-\sqrt{18}-\sqrt{7})$.
5. $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})(a^2+ab+b^2)$.
6. $(\sqrt{\sqrt{13}+3})(\sqrt{\sqrt{13}-3})$.
7. $(\sqrt{2+3\sqrt{5}})(\sqrt{2+3\sqrt{5}})$.
8. $(3\,a-2\sqrt{x})(4\,a+3\sqrt{x})$.
9. $(3\sqrt{3}+2\sqrt{6}-4\sqrt{8})(3\sqrt{3}-2\sqrt{6}+4\sqrt{8})$.
10. $(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})$.
11. $(a-\sqrt{b}-\sqrt{c})(a+\sqrt{b}+\sqrt{c})$.
12. $(2\sqrt{\tfrac{2}{3}}+3\sqrt{\tfrac{1}{2}}+4\sqrt{\tfrac{3}{2}})(2\sqrt{\tfrac{2}{3}}-5\sqrt{\tfrac{1}{2}})$.
13. $(\sqrt[3]{a^2}+\sqrt[3]{b^2})(\sqrt[6]{a^2}+\sqrt[3]{a^2b^2}+\sqrt[9]{b^3})$.
14. $(\sqrt[4]{x^3}-y^3)^3$.

DIVISION OF RADICALS

195. Radicals are divided in accordance with Laws II and V. That is, the exponents are *subtracted* when the *bases* are the *same*, and the bases are *divided* when the *exponents* are the *same*. See § 181.

Example 1. $\sqrt[5]{x^2} \div \sqrt{x^3} = x^{\frac{2}{5}} \div x^{\frac{3}{2}} = x^{\frac{2}{5}-\frac{3}{2}} = x^{-\frac{11}{10}}.$

Example 2. $x^{\frac{2}{3}} \div y^{\frac{2}{3}} = \left(\frac{x}{y}\right)^{\frac{2}{3}} = (xy^{-1})^{\frac{2}{3}} = \sqrt[3]{x^2y^{-2}}.$

Example 3. $\sqrt{a} \div \sqrt[3]{b} = a^{\frac{3}{6}} \div b^{\frac{2}{6}} = \left(\frac{a^3}{b^2}\right)^{\frac{1}{6}} = \sqrt[6]{a^3b^{-2}}.$

ORAL EXERCISES

Find the following quotients:

1. $\sqrt{12} \div \sqrt{3}.$
2. $\sqrt{8} \div \sqrt{2}.$
3. $\sqrt{18} \div \sqrt{2}.$
4. $\sqrt{27} \div \sqrt{3}.$
5. $\sqrt{15} \div \sqrt{5}.$
6. $\sqrt{a^3} \div \sqrt{a}.$
7. $\sqrt{a^3b} \div \sqrt{ab}.$
8. $\sqrt[3]{a^2x} \div \sqrt[3]{ax}.$
9. $\sqrt[3]{(a+b)^2} \div \sqrt[3]{a+b}.$
10. $\sqrt[3]{x^2y^2} \div \sqrt[3]{xy}.$
11. $\sqrt{32} \div \sqrt{8}.$
12. $\sqrt[3]{32} \div \sqrt[3]{4}.$
13. $a^{\frac{1}{2}} \div a^{\frac{1}{3}}.$
14. $a^{\frac{1}{3}} \div a^{\frac{1}{6}}.$
15. $x^{\frac{1}{2}} \div x^{\frac{1}{4}}.$
16. $x^{\frac{5}{6}} \div x^{\frac{1}{3}}.$
17. $x^{\frac{2}{3}} \div x^{\frac{1}{4}}.$
18. $x^{\frac{4}{5}} \div x^{\frac{7}{10}}.$

WRITTEN EXERCISES

Perform the following divisions:

1. $(\sqrt{a^3} + 2\sqrt{a^5} - 3\sqrt{a}) \div 6\sqrt{a}.$
2. $(\sqrt{a} + \sqrt[3]{b} - c) \div \sqrt{c}.$
3. $(2\sqrt[3]{9} + 3\sqrt[3]{12} - 4\sqrt[3]{15}) \div \sqrt[3]{3}.$
4. $(4\sqrt[5]{7} - 8\sqrt[5]{21} + 6\sqrt[5]{42}) \div 2\sqrt[5]{7}.$
5. $(\sqrt[3]{x^4} - 2\sqrt[3]{x^5} + 3\sqrt[3]{x^8}) \div \sqrt[3]{x}.$
6. $(\sqrt[4]{x^7} + 7\sqrt[4]{x^5} + 2\sqrt[4]{x^3}) \div \sqrt[4]{x^3}.$

196. Rationalizing the Denominator. In case division by a radical expression cannot be carried out conveniently as in the foregoing examples, we indicate the division in the fractional form and then *rationalize* the denominator when possible.

Example 1. $\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}}=\dfrac{\sqrt{10}}{5}.$

Example 2. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}=\dfrac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}=\dfrac{a+\sqrt{ab}}{a-b}.$

Evidently, this reduction is always possible when the divisor is a *monomial* or *binomial* radical expression of the second order.

The expression by which numerator and denominator are multiplied is called the *rationalizing factor.*

For a monomial denominator, $\sqrt{x}$, the rationalizing factor is $\sqrt{x}$ itself. If the denominator is $\sqrt{x}+\sqrt{y}$ the rationalizing factor is $\sqrt{x}-\sqrt{y}$ and if the denominator is $\sqrt{x}-\sqrt{y}$ the rationalizing factor is $\sqrt{x}+\sqrt{y}$.

WRITTEN EXERCISES

Reduce each of the following to equivalent fractions having a rational denominator:

1. $\dfrac{3}{2-\sqrt{5}}.$

2. $\dfrac{7}{\sqrt{5}+\sqrt{3}}.$

3. $\dfrac{\sqrt{27}}{\sqrt{3}+\sqrt{11}}.$

4. $\dfrac{2-\sqrt{7}}{2+\sqrt{7}}.$

5. $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$

6. $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}.$

7. $\dfrac{3\sqrt{3}-2\sqrt{2}}{3\sqrt{3}+2\sqrt{2}}.$

8. $\dfrac{\sqrt{a^2+1}-\sqrt{a^2-1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}.$

9. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}.$

10. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}.$

197. In **approximating the value** of an expression such as $\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, reduce it to the form $\frac{10+2\sqrt{21}}{4}$. Note the resulting economy of work.

WRITTEN EXERCISES

Reduce each of the following to the simplest form for approximating the numerical value:

1. $\frac{3\sqrt{5}+4\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
2. $\frac{\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.
3. $\frac{4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.
4. $\frac{11\sqrt{5}-3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.
5. $\frac{7\sqrt{5}+3\sqrt{8}}{2\sqrt{5}-3\sqrt{2}}$.
6. $\frac{5\sqrt{19}-3\sqrt{7}}{3\sqrt{7}-\sqrt{19}}$.
7. $\frac{3\sqrt{2}-\sqrt{5}}{\sqrt{5}-6\sqrt{2}}$.
8. $\frac{5\sqrt{6}-7\sqrt{13}}{3\sqrt{13}-7\sqrt{6}}$.
9. $\frac{2\sqrt{3}+3\sqrt{2}}{6\sqrt{10}-5\sqrt{15}}$.

QUADRATIC SURDS

198. A surd is an indicated root of a rational number, which is not reducible to a rational number.

E.g. $\sqrt{2}$ is a surd since it cannot be reduced to a rational number. $\sqrt[3]{4}$, $\sqrt[5]{3}$ are surds for the same reason. $\sqrt{9}$ is not a surd since $\sqrt{9}=3$. $\sqrt{2+\sqrt{2}}$ is not a surd, since $2+\sqrt{2}$ is not a rational number.

199. Order of a Surd. The *order of a surd* is indicated by the **index** of the root.

E.g. $\sqrt[3]{4}$ is a surd of the *third order*, or of index three; $\sqrt[5]{3}$ is a surd of the fifth order or of index five.

200. Quadratic Surd. Surd expressions containing no indicated roots except square roots are called **quadratic surds.**

E.g. $\sqrt{7}$, $\sqrt{2}+\sqrt{3}$, $3+\sqrt{5}$, $\frac{3}{\sqrt{7}-\sqrt{5}}$, are quadratic surds.

201. Square Root of a Binomial Quadratic Surd. A binomial quadratic surd is sometimes a perfect square, and its square root may be found by inspection.

In order to do this we square the expression $\sqrt{a}+\sqrt{b}$ and examine the result, thus:

$$(\sqrt{a}+\sqrt{b})^2=(\sqrt{a})^2+2\sqrt{a}\sqrt{b}+(\sqrt{b})^2=a+2\sqrt{ab}+b=a+b+2\sqrt{ab}.$$

Hence, if in a quadratic surd of the form $x+2\sqrt{y}$, x is the *sum of two numbers, a and b, and y is the product of these numbers, then $\sqrt{a}+\sqrt{b}$ is the square root of $x+2\sqrt{y}$.*

Example 1. Find the square root of $5+\sqrt{24}$.

Solution. First put this in the *type form* $x+2\sqrt{y}$, thus,

$$5+\sqrt{24}=5+\sqrt{4\cdot 6}=5+2\sqrt{6}.$$

We now seek two numbers whose sum is 5 and whose product is 6. There are two integers, namely 3 and 2, which fulfill these conditions.

Hence, $\sqrt{5+\sqrt{24}}=\sqrt{3}+\sqrt{2}.$

Check. $(\sqrt{3}+\sqrt{2})^2=(\sqrt{3})^2+2\sqrt{3\cdot 2}+(\sqrt{2})^2=3+2+2\sqrt{3\cdot 2}$
$=5+2\sqrt{6}.$

Example 2. Find the square root of $8-\sqrt{60}$.

Solution. $8-\sqrt{60}=8-2\sqrt{15}$. The two integers whose sum is 8 and whose product is 15 are 5 and 3.

Hence, $\sqrt{8-2\sqrt{15}}=\sqrt{5}-\sqrt{3}.$

Check. $(\sqrt{5}-\sqrt{3})^2=5+3-2\sqrt{15}=8-\sqrt{60}.$

WRITTEN EXERCISES

Find the square root of each of the following:

1. $3-2\sqrt{2}$.
2. $7+\sqrt{40}$.
3. $8-\sqrt{60}$.
4. $7+4\sqrt{3}$.
5. $24-6\sqrt{7}$.
6. $28+3\sqrt{12}$.
7. $9-6\sqrt{2}$.
8. $10-4\sqrt{6}$.
9. $9+4\sqrt{5}$.
10. $9+2\sqrt{14}$.
11. $12-2\sqrt{35}$.
12. $12-8\sqrt{2}$.
13. $11+6\sqrt{2}$.
14. $11-4\sqrt{6}$.
15. $11+4\sqrt{7}$.
16. $11-2\sqrt{30}$.
17. $10+2\sqrt{21}$.
18. $13-2\sqrt{22}$.

PROBLEMS INVOLVING RADICALS

202. By means of **radicals and roots** we are now able to solve equations of the type $x^2 = a$.

Example 1. Solve the equation $x^2 = 32$.

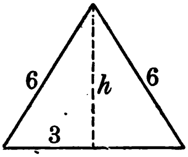
Solution. Taking square roots, $x = \pm\sqrt{32} = \pm 4\sqrt{2}$.
If it is desired to express this root in decimals, we approximate the square root of 2 and multiply the result by 4.

Example 2. Solve $x^2 = 6 + 2\sqrt{5}$.

Solution. By § 201, $x = \sqrt{6 + 2\sqrt{5}} = \pm(\sqrt{5} + 1)$.

In case such an equation arises in a concrete problem, we must decide by direct application to the problem which of the two roots gives the required solution.

Example 3. Find the altitude of an equilateral triangle whose sides are each 6.



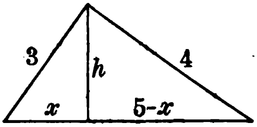
Solution. By the right triangle proposition

$$h^2 = 6^2 - 3^2 = 36 - 9 = 27. \text{ (See the figure.)}$$

Hence, $h = \pm\sqrt{27} = \pm 3\sqrt{3}$.

Clearly the altitude of a triangle cannot be negative. Hence, the altitude is $h = 3\sqrt{3}$.

Example 4. Find the altitude h on the longest side of a triangle whose sides are 3, 4, 5.



Solution. Using the notation of the figure,

$h^2 = 9 - x^2$ and also $h^2 = 16 - (5 - x)^2$.

Hence, equating the two values of h^2,

$$9 - x^2 = 16 - (5 - x)^2,$$

or

$$9 - \cancel{x^2} = 16 - 25 + 10x - \cancel{x^2}.$$

Hence, $x = 1\frac{4}{5}$.

Substituting this value of x in $h^2 = 9 - x^2$,

we have $h^2 = 9 - (\frac{9}{5})^2 = 9 - \frac{81}{25} = \frac{144}{25}$.

Hence, $h = \pm\sqrt{\frac{144}{25}} = \pm\frac{12}{5}$.

The negative result is not applicable, and hence the altitude is $h = \frac{12}{5}$.

PROBLEMS INVOLVING RADICALS

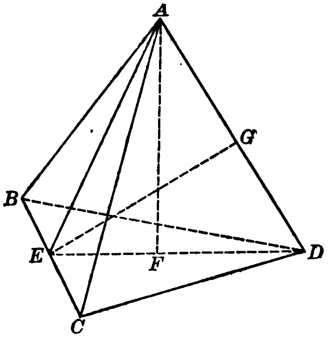
1. Find the area of an equilateral triangle whose sides are 10.

2. Find the area of an equilateral triangle whose sides are a.

3. Find the area of a triangle whose sides are 6, 8, and 10.

4. Find the area of a triangle whose base is a and whose two other sides are a and b.

A three-sided pyramid all of whose edges are equal is called a regular tetrahedron. In the figure AB, AC, AD, BC, BD, CD are all equal.



5. Find the altitude of a regular tetrahedron whose edges are each 6. Also the area of the base.

Hint. **First find the altitudes AE and DE, and then find the altitude of the triangle AED on the side DE, *i.e.* find AF. Use Example 4 above.**

6. Find the volume of a regular tetrahedron whose edges are each 6. Also, find the volume if the edges are 10.

The volume of a tetrahedron is $\frac{1}{3}$ the product of the base and the altitude.

7. Find the volume of a regular tetrahedron whose edges are each a.

8. In the figure above find EG if the edges are each a.

9. If in the figure EG is 12, compute the volume.

Use problem 8 to find the edge, then use problem 7 to find the volume.

10. Express the volume of the tetrahedron in terms of EG. That is, if $EG = b$, find a general expression for the volume in terms of b.

11. If the altitude of a regular tetrahedron is 10, compute the edge accurately to two places of decimals.

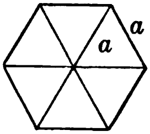
12. Express the edge of a regular tetrahedron in terms of its altitude.

13. Express the volume of a regular tetrahedron in terms of its altitude.

14. Express the edge of a regular tetrahedron in terms of its volume.

15. Express the altitude of a regular tetrahedron in terms of its volume.

16. Express EG of the above figure in terms of the volume of the tetrahedron.



17. Find the area of a regular hexagon whose sides are a.

Suggestion. The regular hexagon may be divided into six equilateral triangles whose sides are a.

18. Find the length of a side of an equilateral triangle whose area is 25. See Example 2 above.

19. Find the length of a side of an equilateral triangle whose area is A.

Solve Example 18 by substituting in the formula thus obtained.

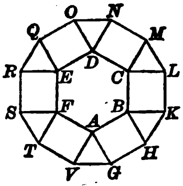
20. Find the length of a side of a regular hexagon whose area is 50.

Suggestion. First find the area of a regular hexagon whose sides are a.

21. Find the length of a side of a regular hexagon whose area is A.

Solve Example 20 by means of the formula thus obtained.

22. Find the area of a regular dodecagon whose sides are a. Express the result in simplest form.



A regular dodecagon (twelve-sided figure) is made up of a regular hexagon, six squares and six equilateral triangles all having equal sides.

23. Find the area of a regular dodecagon whose sides are 10.

24. Find the length of a side of a regular dodecagon whose area is A. Express the result in simplest form.

25. Find the volume of a pyramid whose altitude is 7 and whose base is a regular hexagon whose sides are 7.

The volume of a pyramid or of a cone is $\frac{1}{3}$ the product of its base and its altitude.

26. If the volume of the pyramid in problem 25 were 100 cubic inches, what would be its altitude, a side of the base and the altitude being equal? Approximate the result to two places of decimals.

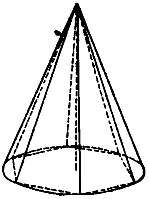
27. If in a right prism the altitude is equal to a side of the base, find the volume, the base being an equilateral triangle whose sides are a.

The volume of a right prism or cylinder equals the product of its base and its altitude.

28. Find the volume of the prism in problem 27, if its base is a regular hexagon whose side is a.

29. Express the side of the base of the prism in problem 27 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.

30. Express the side of the base of the prism in problem 28 in terms of its volume. State and solve a particular problem by means of the formula thus obtained.



31. In the accompanying figure the altitude is supposed to be three times the side a of the regular hexagonal base. Express the difference between the volume of the pyramid and that of its circumscribed cone in terms of a.

The volume of a cone equals $\frac{1}{3}$ the product of its base and altitude.

CHAPTER XIII

THE NUMBER SYSTEM OF ALGEBRA

203. Growth of the Number System. The number system of Algebra is the result of a *gradual growth* starting with the integers of Arithmetic. Each step in this growth was due largely to a desire to make the operations of addition, subtraction, multiplication, and division possible without any exceptions.

204. Operations on the Positive Integers. Starting with the integers of Arithmetic, that is, with the positive integers, the operations of addition and multiplication are always possible in the sense that the result is again a positive integer.

E.g. $2 + 5 = 7$; $3 \cdot 8 = 24$.

However, division of one positive integer by another does not always lead to another integer, and in order to make this operation possible without exception, *we need the positive fractions.*

E.g. $8 \div 4 = 2$; but $2 \div 3 = \frac{2}{3}$.

Again, subtraction of one positive number from another does not always lead to a positive number, and in order to make this operation possible without exception, *we need the negative numbers and zero.*

E.g. $5 - 8 = -3$; and $5 - 5 = 0$.

205. Possible Operations. With the number system enlarged to include *fractions*, *negative numbers*, and *zero*, the operations of addition, subtraction, multiplication, and division are always possible, with one single exception, namely, *it is not possible to divide by zero.* See § 31.

206. Division and Subtraction always possible in Extended Systems. We therefore say that the fractions are brought into the number system to make division possible in all cases (except division by zero); and that the negative numbers and zero are brought in to make subtraction possible without exception.

These statements may be expressed as follows:

We require the number system to be such that the equation $ax + b = 0$ shall have a solution for all values of a and b.

207. Further Extensions of the Number System. The operation of finding roots is *not possible* in all cases, unless other numbers besides positive and negative integers and fractions are admitted to the number system.

E.g. The number $\sqrt{2}$ is not an integer since $1^2 = 1$ and $2^2 = 4$. Suppose $\sqrt{2} = \frac{a}{b}$, a fraction reduced to its lowest terms, so that a and b have no common factor. Then $\frac{a^2}{b^2} = 2$. But this is impossible, for if b^2 is exactly divisible by a^2, then a and b must have factors in common. Hence, $\sqrt{2}$ is not a fraction.

Therefore, if the operation of finding roots is to be possible without exception, a further addition must be made to the number system as explained below.

208. If a positive number is not the square of either an integer or a fraction, a number may be found in terms of integers and fractions whose square differs from the given number by as little as we please.

E.g. 1.41, 1.414, 1.4142, are successive numbers whose squares differ by less and less from 2. In fact $(1.4142)^2$ differs from 2 by less than .0004, and by continuing the process by which these numbers are found, a number may be reached whose square differs from 2 by as little as we please.

1., 1.4, 1.41, 1.414, 1.4142, etc., are successive approximations to the number which we call the square root of 2, and which we represent by the symbol $\sqrt{2}$.

209. The Irrational Number. If a number is neither an integer nor a fraction, but if it can be approximated by means of integers and fractions to any specified degree of accuracy, then such a number is called an *irrational number.*

E.g. $\sqrt{2}$, $3\sqrt{2}$, $5\sqrt[3]{2}$, etc., are irrational numbers, whereas $\sqrt{4}$, $\sqrt[3]{8}$ are rational numbers.

The Number π. It will be found in higher work that there are other irrational numbers besides indicated roots. For instance, the number π, which is the ratio of the circumference of a circle to its diameter, is an irrational number, though it is not an indicated root.

It is shown in higher algebra that irrational numbers correspond to definite points on the line of the number scale, just as do integers and fractions.

210. The Real Number System. We, therefore, enlarge the number system so as to include *irrational numbers* as well as integers and fractions. The set of numbers consisting of all rational and irrational numbers is called the *real number system.*

Certain irrational numbers are brought into the number system by the requirement that *every positive number shall have at least one nth root.* This requirement may also be expressed by saying that the number system shall be such that every equation of the form $x^n = a$ shall have at least one solution. Numbers like π are not brought in by this requirement.

211. Imaginary Numbers. Even with the number system as thus enlarged, it is not possible to find roots in all cases. The exception is the *even root of a negative number,* which is called an *imaginary number.*

E.g. $\sqrt{-4}$ is neither $+2$ nor -2, since $(+2)^2 = +4$, and $(-2)^2 = +4$ and no approximation to this root can be found as in the case of $\sqrt{2}$.

The Imaginary Unit. The expression $\sqrt{-1}$ is defined by the equation $(\sqrt{-1})^2 = -1$, and is usually represented by the letter i. See § 150.

212. Type Form of Complex Numbers. Every expression which can be reduced to the form $a+bi$, in which a and b are real numbers, is called a *complex number*. a is called the *real part*, and bi, the *imaginary part* of the complex number $a+bi$.

The form $a+bi$ is the **type form** of the complex number.

In $a+bi$, if $a=0$, then bi is called a *pure imaginary*. If $b=0$, then a is simply a *real number*.

The complex numbers are brought into the number system by the requirement that every equation of the form $ax^2+bx+c=0$ shall have a solution.

213. The Complete Number System of Algebra. The set of numbers consisting of the *integers* and *fractions* of arithmetic, together with the *negative numbers*, the *irrational numbers*, and the *complex numbers*, is called the *number system of algebra.*

With the number system as thus enlarged, it can be shown that all the algebraic operations, namely, addition, subtraction, multiplication, division, raising to powers, and extracting roots, when applied to numbers in the system lead again to numbers in the system, *the only exception being division by zero.*

In this sense the number system of algebra is *complete in itself.*

Example 1. Reduce $4+\sqrt{-36}$ to the type form $a+bi$.

Solution. $4+\sqrt{-36}=4+\sqrt{36(-1)}=4+6\sqrt{-1}=4+6i$.

Example 2. Reduce $6+\sqrt{-14}$ to type form.

Solution. $6+\sqrt{-14}=6+\sqrt{14\cdot(-1)}=6+\sqrt{14}\sqrt{-1}=6+\sqrt{14}\,i$.

ORAL EXERCISES

Reduce each of the following to type form:

1. $\sqrt{-16}+7$.
2. $\sqrt{-25}+3$.
3. $\sqrt{12}+\sqrt{-12}$.
4. $\sqrt{-6}+\sqrt{8}$.
5. $\sqrt{-2}+13$.
6. $\sqrt{32}+\sqrt{-4}$.
7. $5-\sqrt{-16}$.
8. $6-\sqrt{-10}$.
9. $7-\sqrt{-18}$.

HISTORICAL NOTE

Number. "The conception of 'number' has been much extended in recent times. With the Greeks it included the ordinary positive whole numbers; **Diophantus** (died about 330 A.D.) added rational fractions to the domain of numbers. Later negative numbers and imaginaries came gradually to be recognized.

"**Descartes** fully grasped the notion of the negative; **Gauss**, that of the imaginary. With **Euclid**, a ratio, whether rational or irrational, was not a number. The recognition of ratios and irrationals as numbers took place in the sixteenth century, and found expression with **Newton**.

"By the ratio method, the continuity of the real number system has been based on the continuity of space, but in recent times three theories of irrationals have been advanced by **Weiertrass**, T. W. R. **Dedekind**, G. **Cantor**, and **Heine**, which prove the continuity of numbers without borrowing it from space. They are based on the definition of numbers by regular sequences, the use of series and limits, and some new mathematical conceptions." — CAJORI, *A History of Mathematics*.

That there *are* numbers which are not roots of rational equations became known early in the 19th century. One of the first definite numbers found to be such was the number π (1882).

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

214. Complex numbers are added by combining the real parts and the imaginary parts separately.

Thus,

$$\begin{array}{r} a+bi \\ c+di \\ \hline a+c+(b+d)i \end{array}$$

The real part of the sum is $a + c$, and the imaginary part is $(b + d)i$.

Similarly, $(a + bi)-(c + di)=(a - c)+(b - d)i$.

Example 1. Simplify $(2+\sqrt{-4})+(3+\sqrt{-9})+(\sqrt{-16})$.

Solution. $2+\sqrt{-4}=2+2\sqrt{-1}$; $3+\sqrt{-9}=3+3\sqrt{-1}$; and $\sqrt{-16}=4\sqrt{-1}$.

Hence, adding the real parts separately, and the imaginary parts separately, the sum is $(2+3)+(2+3+4)(\sqrt{-1})=5+9\sqrt{-1}=5+9i$.

Example 2. Simplify $(1+\sqrt{-5})+(5+\sqrt{-3})$
$+(2-\sqrt{-10})$.

Solution: $1+\sqrt{-5}=1+\sqrt{5}\cdot\sqrt{-1}$; $5+\sqrt{-3}=5+\sqrt{3}\cdot\sqrt{-1}$; $2-\sqrt{-10}=2-\sqrt{10}\cdot\sqrt{-1}$.

Adding the real and the imaginary parts separately, the sum is

$$(1+5+2)+(\sqrt{5}+\sqrt{3}-\sqrt{10})\sqrt{-1}=8+(\sqrt{5}+\sqrt{3}-\sqrt{10})i.$$

WRITTEN EXERCISES

Simplify the following:

1. $2\sqrt{-8}$.
2. $\sqrt{-4x^2}+3\sqrt{-9x^2}$.
3. $\sqrt{-8}+\sqrt{-32}$.
4. $3\sqrt{-1}+\sqrt{-9}$.
5. $7\sqrt{-4}-2\sqrt{-9}$.
6. $\sqrt{-4a^2}+\sqrt{-9b^2}$.
7. $3+2\sqrt{-16}+5-\sqrt{-36}$.
8. $\sqrt{-1}+3\sqrt{-4}+\sqrt{-9}$.
9. $\sqrt{-4}-\sqrt{-1}+\sqrt{-16}$.
10. $\sqrt{-9}-\sqrt{-1}+\sqrt{-25}$.
11. $\sqrt{-64}+7\sqrt{-1}-3\sqrt{-4}$.
12. $\sqrt{-81}+\sqrt{-64}-\sqrt{-100}$.
13. $3\sqrt{-4x^4}-\sqrt{-9x^4}+\sqrt{-16x^4}$.
14. $\sqrt{-a^2-2a-1}+\sqrt{-a^2}-\sqrt{-1}$.
15. $\sqrt{-4x^2-4x-1}-\sqrt{-4x^2}+\sqrt{-4}$.
16. $3x\sqrt{-x^2}-3x^2\sqrt{-1}$.
17. $5x\sqrt{-9x^2}+3x^2\sqrt{-4}$.
18. $4a^2\sqrt{-16a^2}+2a^3\sqrt{-25}$.
19. $7\sqrt{-4a^4}+2a\sqrt{-a^2}+4a^2\sqrt{-4}$.
20. $3mn\sqrt{-1}+4m\sqrt{-n^2}-n\sqrt{-m^2}$.
21. $ab\cdot i+2a\sqrt{-b^2}+3b\sqrt{-a^2}$.
22. $x\sqrt{-y^2}+y\sqrt{-x^2}+xy\cdot i$.
23. $4a\sqrt{-b^2c^4}+c^2\sqrt{-a^2b^2}+bc^2\sqrt{-a^2}$.

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

215. Multiplication and Division of Pure Imaginary Numbers. By the definition of the imaginary unit, $(\sqrt{-1})^2=-1$.

This forms the basis for the laws that govern the multiplication and division of pure imaginaries.

Examples:

$$\sqrt{-4}\cdot\sqrt{-9}=2\sqrt{-1}\cdot 3\sqrt{-1}=6(\sqrt{-1})^2=-6.$$

and in general,

$$\sqrt{-a}\cdot\sqrt{-b}=\sqrt{a}\cdot\sqrt{-1}\cdot\sqrt{b}\cdot\sqrt{-1}=\sqrt{ab}(\sqrt{-1})^2=-\sqrt{ab}.$$

Again, $$\frac{\sqrt{-4}}{\sqrt{-9}}=\frac{2\sqrt{-1}}{3\sqrt{-1}}=\frac{2}{3},$$

and in general, $$\frac{\sqrt{-a}}{\sqrt{-b}}=\frac{\sqrt{a}\cdot\sqrt{-1}}{\sqrt{b}\cdot\sqrt{-1}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

Also, $$\frac{\sqrt{-a}}{\sqrt{b}}=\frac{\sqrt{a}\cdot\sqrt{-1}}{\sqrt{b}}=\frac{\sqrt{a}}{\sqrt{b}}\cdot\sqrt{-1}=\frac{\sqrt{a}}{\sqrt{b}}i.$$

216. Multiplication of Complex Numbers. Two complex numbers of the form $a+bi$ and $c+di$ are multiplied like two binomials, remembering that $(i)^2=-1$.

Thus,

$$\begin{array}{l}
a+bi\\
\underline{c+di}\\
ac+bci\\
\underline{\qquad adi+bd(i)^2}\\
ac+(bc+ad)i-bd=ac-bd+(bc+ad)i.
\end{array}$$

217. The Powers of *i*. The powers of the imaginary unit, $\sqrt{-1}=i$, are of special interest. These are based upon the definition, $(\sqrt{-1})^2=-1$, or $i^2=-1$.

Hence we have

$$\begin{cases}
i^1=i.\\
i^2=-1.\\
i^3=i^2\cdot i=(-1)i=-i.\\
i^4=i^2\cdot i^2=(-1)(-1)=1.
\end{cases}
\qquad
\begin{cases}
i^5=i^4\cdot i=1\cdot i=i.\\
i^6=i^4\cdot i^2=1(-1)=-1.\\
i^7=i^4\cdot i^3=1(-i)=-i.\\
i^8=i^4\cdot i^4=1\cdot 1=1.
\end{cases}$$

218. Division by a Complex Number. The division of $a+bi$ by $c+di$ may be indicated in the fractional form:

$$\frac{a+bi}{c+di}.$$

Multiplying both terms of the fraction by $c-di$, we have

$$\frac{(c-di)(a+bi)}{(c-di)(c+di)}=\frac{ac+bci-adi+bd}{c^2-(di)^2}=\frac{ac+bd+(bc-ad)i}{c^2+d^2},$$

in which the denominator is a real number. The fraction may be written

$$\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$$

which is in the form $A+Bi$, where $A=\frac{ac+bd}{c^2+d^2}$ and $B=\frac{bc-ad}{c^2+d^2}$.

Hence the quotient $(a+bi)\div(c+di)$ has been reduced to the standard form of a complex number.

Note that $$\frac{\sqrt{a}}{\sqrt{-b}}=\frac{\sqrt{a}}{\sqrt{b}\cdot\sqrt{-1}}=\frac{\sqrt{a}\sqrt{-1}}{\sqrt{b}(\sqrt{-1})^2}=-\frac{\sqrt{a}}{\sqrt{b}}i.$$

ORAL EXERCISES

Simplify the following:

1. $\sqrt{-4}\cdot\sqrt{-9}$.
2. $\sqrt{-4}\cdot\sqrt{-16}$.
3. $\sqrt{-9}\cdot\sqrt{-25}$.
4. $\sqrt{-16}\cdot\sqrt{-36}$.
5. $\sqrt{-9}\div\sqrt{-4}$.
6. $\sqrt{-16}\div\sqrt{-4}$.
7. $\sqrt{-25}\div\sqrt{-9}$.
8. $\sqrt{-36}\div\sqrt{-36}$.
9. $\sqrt{-100}\div\sqrt{-25}$.
10. $\sqrt{-81}\div\sqrt{-9}$.
11. $i\cdot i=i^2$.
12. $i\cdot i\cdot i=i^3$.
13. $-i^5$.
14. i^7.
15. i^9.
16. i^{10}.
17. i^{11}.
18. i^{12}.
19. i^{15}.
20. i^{17}.
21. $-i^{21}$.
22. i^{14}.
23. $-i^{16}$.
24. i^{18}.
25. i^{20}.
26. i^{22}.
27. i^{26}.

WRITTEN EXERCISES

Find the following products:

1. $\sqrt{-3}\cdot\sqrt{-12}$.
2. $-\sqrt{-2}\cdot\sqrt{-8}$.
3. $-\sqrt{-5}\cdot\sqrt{-125}$.
4. $4\sqrt{-18}\cdot\sqrt{-2}$.
5. $\sqrt{-x^2}\cdot\sqrt{-x^2}$.
6. $\sqrt{-6}\cdot\sqrt{-18}\cdot\sqrt{-3}$.
7. $\sqrt{-2}\cdot\sqrt{-32}\cdot(\sqrt{-1})^2$.
8. $3\sqrt{-3}\cdot(-\sqrt{-27})$.
9. $2\sqrt{-2}\cdot 4\sqrt{-3}$.
10. $2\sqrt{-32}\cdot\sqrt{-8}$.
11. $(2+3i)(1-2i)$.
12. $(1-4i)(4-i)$.
13. $(\sqrt{2}+2i)(\sqrt{8}-4i)$.
14. $(4-3i)(9+2i)$.
15. $(7+2i)(2-7i)$.
16. $(x+yi)(x-yi)$.
17. $(x+yi)(x+yi)$.
18. $(x-yi)(x-yi)$.
19. $(a+bi)^2-(a-bi)^2$.
20. $(1+\sqrt{3}i)(1-\sqrt{3}i)$.
21. $\dfrac{-1+\sqrt{-3}}{2}\cdot\dfrac{-1-\sqrt{-3}}{2}$.
22. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^2$.
23. $\left(\dfrac{-1-\sqrt{-3}}{2}\right)^2$.
24. $(2a-i)(2a+i)$.
25. $(x^2+xi+1)(x^2-xi+1)$.
26. $(-x^3i-x^2+xi+1)(xi-1)$.
27. $(a^3-ab^2-a^2bi-b^3i)(a+bi)$.
28. $(a^3-ab^2+a^2bi-b^3i)(a-bi)$.
29. $(x^6-x^2y^4+x^4y^2i-y^6i)(x^2-y^2i)$.
30. $(x^6-x^2y^4-x^4y^2i+y^6i)(x^2+y^2i)$.
31. $(a^5-a^4bi+a^3b^2-a^2b^3i+ab^4-b^5i)(a+bi)$.
32. $(a^5+a^4bi+a^3b^2+a^2b^3i+ab^4+b^5i)(a-bi)$.
33. $(x^{10}+x^8y^2i+x^6y^4+x^4y^6i+x^2y^8+y^{10}i)(x^2-y^2i)$.
34. $(x^{10}-x^8y^2i-x^6y^4+x^4y^6i+x^2y^8-y^{10}i)(x^2+y^2i)$.
35. $(x^4-x^2y^2i+y^4)(x^4+x^2y^2i+y^4)$.

WRITTEN EXERCISES

Simplify the following indicated quotients:

1. $\dfrac{\sqrt{-3}}{\sqrt{-1}}$.
2. $\dfrac{\sqrt{-9}}{\sqrt{-4}}$.
3. $\dfrac{\sqrt{-16}}{\sqrt{4}}$.
4. $\dfrac{\sqrt{-4}+\sqrt{-9}}{\sqrt{-1}}$.
5. $\dfrac{\sqrt{-20}}{\sqrt{-4}}$.
6. $\dfrac{\sqrt{-8}+\sqrt{-32}}{\sqrt{-2}}$.
7. $\dfrac{1+i}{1-i}$.
8. $\dfrac{1-i}{1+i}$.
9. $\dfrac{2+3i}{1+i}$.
10. $\dfrac{3-2i}{3+2i}$.
11. $\dfrac{3+2i}{2-3i}$.
12. $\dfrac{1-4i}{4-i}$.
13. $\dfrac{1-3i}{3-i}$.
14. $\dfrac{i-2}{3+i}$.
15. $\dfrac{2}{1-\sqrt{-1}}$.
16. $\dfrac{3}{\sqrt{3}+\sqrt{-1}}$.
17. $\dfrac{1-\sqrt{-1}}{1+\sqrt{-1}}$.
18. $\dfrac{\sqrt{2}+\sqrt{-3}}{\sqrt{2}-\sqrt{-3}}$.
19. $\dfrac{5}{2-3\sqrt{-5}}$.
20. $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$.

HISTORICAL NOTE

The Imaginary Number had its origin entirely in the attempt to solve quadratic and higher equations. It was not until the time of **Argand** (1768–1825) and **Gauss** (1777–1855) that the imaginary was fully admitted to be a number. The notation i for $\sqrt{-1}$ was first used by Gauss.

Cardan (1545), who understood the solution of the cubic, and who showed that imaginary roots enter in pairs, that is, if a cubic has one imaginary root it has two, decided "not to commit himself to any explanation as to the meaning of these 'sophistic' quantities which he said were ingenious though useless."

During the eighteenth century **Euler** and others used the imaginary fully in mathematical operations without attaining a full insight into its nature.

In modern mathematics the imaginary plays many important rôles which are far beyond the scope of such a book as this. Even in electrical engineering, formulas containing the imaginary are in constant use.

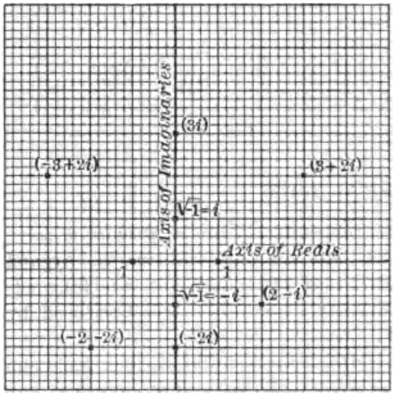
GRAPHIC REPRESENTATION OF COMPLEX NUMBERS

219. Graphs of Real Numbers. We have learned to represent real numbers by *points on a straight line,* — positive numbers on one side of the zero point, and negative numbers on the other side.

We have also learned to represent pairs of real numbers by *points in the plane.* See § 131.

We will now study a device for representing complex numbers in a similar manner.

220. The Axis of Reals and the Axis of Imaginaries. As in ordinary graphing, we construct two axes at right angles to each other. The horizontal line is called the **axis of reals,** and the vertical line, the **axis of imaginaries.**



Any purely real number, such as 4, or − 10, is laid off on the axis of reals. A pure imaginary is laid off on the axis of imaginaries.

Thus, the numbers i and $3i$ correspond to points on the imaginary axis one unit and three units, respectively, above the origin, while $-i$ and $-2i$ correspond to points on the same axis one unit and two units, respectively, below the origin.

A reason for making the vertical axis the axis of pure imaginaries appears in the following:

If the positive half of the axis of reals is rotated about the origin counter-clockwise through an angle of 180°, it is changed into the position of the negative half of this axis, and the point representing any positive number is changed into the position of the corresponding negative number.

But the positive numbers are also changed into corresponding negative numbers by multiplying each by -1. Hence multiplying by -1 may be interpreted graphically as *rotating the positive number line into the negative number line.*

Now by the definition of the imaginary unit, we have

$$(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1.$$

That is, multiplying by $\sqrt{-1}$ *twice in succession* is equivalent to multiplying by -1.

Hence, if $\sqrt{-1} \times \sqrt{-1} \times a$ changes any positive number a into $-a$, that is, rotates the positive real axis through 180°, we may interpret $\sqrt{-1} \times a$ as *rotating the positive real axis one half as far, that is, through* 90°, *or into the vertical position.*

Thus, $a\sqrt{-1}$ is represented graphically by a point on the vertical axis, at a distance a above the origin, and $-a\sqrt{-1}$ is represented by a corresponding point at a distance a below the origin.

221. Representation of Complex Numbers. Since a complex number, like $3+2i$, has a real part 3, and a pure imaginary part $2i$, we measure 3 units to the right along the real axis, and 2 units up parallel to the imaginary axis, and take the point $(3, 2)$ thus located as the graphic representation of the complex number, $3+2i$.

Similarly, to represent $-2-2i$, we measure 2 units to the left along the real axis, and 2 units down parallel to the imaginary axis, and take the point $(-2, -2)$ thus located as the graphic representation of the complex number, $-2-2i$.

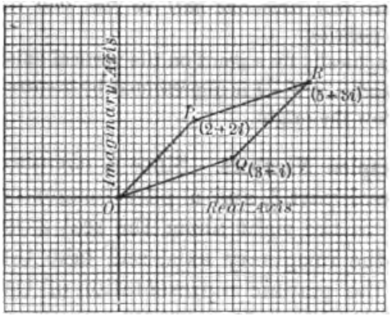
In this manner any complex number $a+bi$ is represented by the point whose abscissa is a, and whose ordinate is b.

Thus, *every point in the plane is the graphic representation of a certain complex number, and every complex number is represented by a certain point in the plane.*

222. The Complex Plane. This plane, in which the vertical axis is the axis of pure imaginaries, is called *the complex plane*, to distinguish it from the real plane in which real numbers are plotted on the vertical axis as well as on the horizontal axis.

223. Graphic Operations. By means of the graphic representation of complex numbers, it is possible to perform *graphically* all the operations of algebra on complex numbers. We will study graphic *addition* of such numbers. Graphic subtraction, multiplication, and division, and the finding of roots, are more complicated and are reserved for a more advanced course.

224. Geometric Representation of Addition. The sum of the numbers, $3 + i$ and $2 + 2i$ is $5 + 3i$, and is represented by the point R shown in the figure.



The point R is obtained by connecting P and Q with the origin, and then drawing PR parallel to OQ and QR parallel to OP. This makes $OPRQ$ a parallelogram.

The construction for locating the point representing the *sum of any two complex numbers* is like the one just described. If P and Q are points representing the numbers $a + bi$, and $c + di$, then in the parallelogram $OPRQ$, the point R represents $(a + c) + (b + d)i$, provided O is the origin.

EXERCISES

1. On squared paper draw axes, and locate the points representing the following numbers: $2i$, $-4i$, 4, -3, $3 + 5i$, $-3 + 2i$, $-3 - 4i$, $2 - 3i$.

2. Construct the parallelogram representing the addition of $3 + 2i$ and $1 + 4i$.

3. Construct parallelograms representing the following indicated additions: $(2 - 2i) + (3 + i)$; $(-1 + 2i) + (-2 - 3i)$.

CHAPTER XIV

QUADRATIC EQUATIONS

225. Quadratic Equations. Equations containing a square of the unknown quantity, and no higher power, are called *quadratic equations*, or equations of the second degree.

Thus, $x^2 = 4$, $x^2 + 7x = 0$ and $x^2 + 5x + 6 = 0$ are quadratic equations.

226. Solution by Factoring. A quadratic equation may frequently be solved by *factoring*, as in the following examples.

Example 1. Solve $x^2 - 5x + 6 = 0$.

Factoring, $(x - 3)(x - 2) = 0$.

Since a product is zero when one of its factors is zero, it follows that the roots of this equation are obtained by putting $x - 3 = 0$ and $x - 2 = 0$, and solving, thus obtaining $x = 2$ and $x = 3$.

Substituting these values in $(x-3)(x-2)=0$, we have $(3-3)(3-2) = 0 \cdot 1 = 0$, and $(2-3)(2-2) = -1 \cdot 0 = 0$.

Example 2. Solve $6x^2 + 5x = 6$.

Transposing and factoring, $(2x + 3)(3x - 2) = 0$.

From $2x + 3 = 0$, we have $x = -\frac{3}{2}$ and from $3x - 2 = 0$ we have $x = \frac{2}{3}$.

Example 3. Solve $x^2 + 7x = 0$.

Factoring, $x(x + 7) = 0$.

From $x + 7 = 0$ we have $x = -7$, and $x = 0$ gives the other root directly.

227. From these examples we get the following **rule**:

To solve a quadratic equation by factoring,

(1) *Transpose all terms to the left member.*

(2) *Factor the left member.*

(3) *Put each factor equal to zero, and solve for x.*

ORAL EXERCISES

Solve the following equations by factoring:

1. $x^2 - 2x + 1 = 0.$
2. $x^2 - 3x + 2 = 0.$
3. $x^2 - 4x + 3 = 0.$
4. $x^2 - 5x + 4 = 0.$
5. $x^2 + x - 2 = 0.$
6. $x^2 - x - 2 = 0.$
7. $x^2 - 2x - 3 = 0.$
8. $x^2 + 2x - 3 = 0.$
9. $x^2 - 3x - 4 = 0.$
10. $x^2 + 3x - 4 = 0.$
11. $x^2 + 3x + 2 = 0.$
12. $x^2 + 5x + 4 = 0.$
13. $x^2 - 5x + 4 = 0.$
14. $x^2 + 4x - 5 = 0.$
15. $x^2 - 4x - 5 = 0.$
16. $x^2 + 6x + 5 = 0.$

228. Many equations of higher degree than the second may be solved by factoring.

Example 1. Solve $2x^3 - x^2 - 5x - 2 = 0.$

We find by the *factor theorem* that $x - 2$ and $x + 1$ are factors of the left member. By division, the remaining factor is found to be $2x + 1$.

Hence, we have $(x - 2)(x + 1)(2x + 1) = 0$.

Hence, $x = 2$, $x = -1$, and $x = -\frac{1}{2}$ are roots of the equation.

Example 2. Solve $x^3 - 9x^2 + 26x - 24 = 0.$

By the *factor theorem* we find that $x - 4$ is a factor. Dividing, we find that $x^2 - 5x + 6$ is the quotient.

Hence the equation reduces to $(x - 4)(x - 2)(x - 3) = 0$, and the roots are $x = 4$, $x = 2$, $x = 3$.

WRITTEN EXERCISES

Solve the following equations by factoring:

1. $x^2 + 5x = -6.$
2. $x^2 - 4x = 21.$
3. $x^2 - 5x = 24.$
4. $x^2 + 9x = 10.$
5. $x^2 - 9x = 10.$
6. $x^2 + 4x = 60.$
7. $x^2 - 5x = -4.$
8. $x^2 - 5x = 36.$
9. $x^2 + 7x = 60.$
10. $x^2 - 10x = 11.$

Solve the following equations by factoring:

11. $2x^2 - x - 3 = 0.$

12. $3x^2 - 13x - 10 = 0.$

13. $3x^2 + 11x = 4.$

14. $2x^2 + 11x = 6.$

15. $6x^2 + x = 2.$

16. $5x^2 + 13x = 6.$

17. $4x^2 + 4x = 3.$

18. $8x^2 - 2x = 15.$

19. $7x^2 + 11x = 6.$

20. $4x^2 - 10x = 24.$

21. $x^3 - x^2 - 17x - 15 = 0.$

22. $x^3 = -2x^2 + 5x + 6.$

23. $x^3 - 12x^2 + 35x - 24 = 0.$

24. $x^3 - 4x^2 - 39x - 54 = 0.$

25. $x^3 + 6x^2 - 31x - 36 = 0.$

26. $x^3 + 3x^2 = 28x.$

27. $5x^3 + 9x^2 = 2x.$

28. $10x^3 - 5x^2 = 15x.$

29. $x^3 + 8x^2 - 3x = 90.$

30. $x^3 - 11x^2 + 34x = 24.$

229. Completing the Square. Any quadratic equation may be solved by *completing the square.*

As a preliminary step consider the trinomial square.

From $(x + a)^2 = x^2 + 2ax + a^2$, we see that the third term is the square of half the coefficient of x in the second term.

Hence, if we have given only the two terms $x^2 + 2ax$, we must add a^2 in order to *complete the square.*

E.g. to complete the square in $x^2 + 6x$ we add the square of half the coefficient of x, or $3^2 = 9$. That is, $x^2 + 6x + 9$ is *a complete square.*

In general, to complete the square in $x^2 + px$, we add $\left(\frac{1}{2}p\right)^2 = \frac{p^2}{4}$, since $x^2 + px + \frac{p^2}{4}$ is a complete square.

ORAL EXERCISES

Complete the trinomial square in each of the following:

1. $x^2 + 2x.$

2. $x^2 + 4x.$

3. $x^2 + 6x.$

4. $x^2 + 8x.$

5. $x^2 + 3x.$

6. $x^2 + 5x.$

7. $(3x)^2 + 2(3x).$

8. $(2x)^2 + 4(2x).$

9. $16x^2 + 2(4x).$

10. Complete the square in each of the above exercises, after replacing the sign + by −.

230. Solution of a Quadratic by Completing the Square.

Example 1. Solve the equation: $x^2 + 6x + 5 = 0.$ (1)

Transposing in (1), $x^2 + 6x = -5.$ (2)

To complete the square add $3^2 = 9$ to both members, obtaining

$$x^2 + 6x + 3^2 = 3^2 - 5 = 4. \tag{3}$$

Taking square roots of both sides, $x + 3 = \pm\sqrt{4} = \pm 2.$

Hence, $x = -3 + 2 = -1,$

and $x = -3 - 2 = -5.$

Example 2. Solve the equation:

$$x^2 - 12x + 42 = 56. \tag{1}$$

Transposing, $x^2 - 12x = 14.$ (2)

Completing the square by adding $(\frac{12}{2})^2 = 6^2 = 36$ to both sides,

$$x^2 - 12x + 36 = 14 + 36 = 50. \tag{3}$$

Taking square roots, $x - 6 = \pm\sqrt{50} = \pm 5\sqrt{2}.$ (4)

Transposing, $x = 6 \pm 7.071.$ (5)

Hence, $x = 6 + 7.071 = 13.071,$

and also, $x = 6 - 7.071 = -1.071.$

The steps involved in the above solutions are:

(1) *Write the equation in the form* $x^2 + px = q.$

(2) *Complete the square by adding* $(\frac{1}{2}p)^2$ *to each member.*

(3) *Take the square root of both members of this equation.*

(4) *Solve each of the first degree equations thus obtained.*

WRITTEN EXERCISES

In solving the following quadratic equations the result may in each case be reduced so that the number remaining under the radical sign shall be 2, 3, or 5. Use $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, and get results to two decimal places.

1. $x^2 - 4x = 8.$
2. $x^2 = 3 - 6x.$
3. $4x = 16 - x^2.$
4. $x^2 + 6x = 9.$
5. $x^2 + 6x = 11.$
6. $x^2 - 12x = 12.$
7. $x^2 - 8x = -14.$
8. $x^2 = 2x + 1.$
9. $x^2 - 4x = 16.$
10. $x^2 = 23 + 4x.$
11. $8 = x^2 + 4x.$
12. $23 - 6x = x^2.$
13. $7 + 2x = x^2.$
14. $25 - x^2 = 5x.$
15. $x^2 + 3x = 9.$

231. The Hindu Method of Completing the Square. In case the coefficient of x^2 is not unity, as in $3x^2+8x=4$, both members may be divided by this coefficient, and the solution is then like that of Examples 1 and 2 on page 170.

However, the following method is sometimes desirable:

$$3x^2+8x=4. \tag{1}$$

Multiplying each member of equation (1) by $4 \cdot 3 = 12$, we get

$$36x^2+96x=48. \tag{2}$$

This can now be written in the form $x^2+px=q$, namely, $(6x)^2+16(6x)=48$, in which $p=16$ and $6x$ is the unknown. Hence, we add $(\frac{16}{2})^2=8^2=64$ to complete the square, and get

$$(6x)^2+16(6x)+64=48+64=112. \tag{3}$$

Taking square roots, $$6x+8=\pm\sqrt{112}=\pm 4\sqrt{7}. \tag{4}$$

Hence, $$6x=-8\pm 4\sqrt{7},$$

and $$x=-\tfrac{4}{3}\pm\tfrac{2}{3}\sqrt{7}. \tag{5}$$

In this solution both sides were multiplied by 4 times the original coefficient of x^2, and then the number added to complete the square was found to be *the square of the original coefficient of x.*

The advantage of this form of solution is that fractions are avoided until the last step, and *the number added to complete the square is equal to the square of the coefficient of x in the original equation.*

This is called the **Hindu method** of completing the square because it was first used by the Hindus.

NOTE. — Fractions would also be avoided in the above solution if equation (1) were multiplied by 3 instead of $4 \cdot 3$. This is the case only when the coefficient of x is an *even* number.

Any quadratic equation may be solved by the following

Rule. 1. *Write the equation in the form* $ax^2+bx=-c$.

2. *Multiply both sides by* $4a$, *and put the equation thus:*

$$(2ax)^2+2b(2ax)=-4ac.$$

3. *Complete the square by adding* b^2 *to both sides.*

4. *Solve as though* $2ax$ *were the unknown.*

EXERCISES

In the solution of the following equations, the roots which contain surds may be left in the simplified radical form.

1. $2x^2 + 3x = 2$.
2. $3x^2 + 5x = 2$.
3. $3x = 9 - 2x^2$.
4. $6x + 1 = -3x^2$.
5. $2x^2 = 5x + 3$.
6. $4x = 2x^2 - 1$.
7. $2x^2 - 3x = 14$.
8. $3x^2 = 9 + 2x$.
9. $4x^2 = 2x + 1$.
10. $6x - 1 = 3x^2$.
11. $2x^2 + 4x = 23$.
12. $3x^2 - 7 = 4x$.
13. $2x^2 - 5 = 3x$.
14. $4x^2 = 6x - 1$.
15. $2x = 1 - 5x^2$.
16. $3x - 20 = -2x^2$.
17. $2x + 3x^2 = 9$.
18. $4x^2 - 1 = 3x$.
19. $4x = 7 - 2x^2$.
20. $2x + 1 = 5x^2$.
21. $3x^2 + 4x = 7$.
22. $3x + 9 = 2x^2$.
23. $2x - 1 = -4x^2$.
24. $5x^2 + 16x = -2$.
25. $4x^2 + 1 = 8x$.
26. $2x^2 - 3x = 20$.
27. $2x^2 - 3 = -5x$.
28. $3x^2 + 4x = 8$.
29. $10 - 4x = 5x^2$.
30. $1 + 4x^2 = -6x$.
31. $5 - 3x = 2x^2$.
32. $7 + 4x = 2x^2$.
33. $6x^2 + 12x = 2$.
34. $6x^2 - 12x = -2$.
35. $6x^2 + 12x = -2$.
36. $6x^2 - 12x = 2$.
37. $3x^2 + 2x = 5$.
38. $2 + 3x = 2x^2$.
39. $8x + 1 = -4x^2$.
40. $8 + 4x = 3x^2$.
41. $10 + 4x = 5x^2$.
42. $2 + 5x = 3x^2$.
43. $3x + 14 = 2x^2$.
44. $3x^2 - 2x = 5$.
45. $2x^2 + 4x = 1$.
46. $3x - 1 = -4x^2$.
47. $23 + 4x = 2x^2$.
48. $3x - 1 = 2x^2$.
49. $5x^2 - 8x = 3$.
50. $7x^2 + 5x - 9 = 0$.
51. $6x^2 - 11x + 3 = 0$.
52. $3x^2 - 14x + 6 = 0$.

232. Solution of Quadratics by Formula. A *formula* for solving *any* quadratic may be obtained as follows:

Example. Solve $ax^2 + bx + c = 0$. (1)

Solution.

Transposing, $ax^2 + bx = -c$.

Multiplying by $4a$, $4a^2x^2 + 4abx = -4ac$. (2)

Equation (2) may be written in the form

$$(2ax)^2 + 2b(2ax) = -4ac.$$

Completing the square as if $2ax$ were the unknown,

$$(2ax)^2 + 2b(2ax) + b^2 = b^2 - 4ac. \qquad (3)$$

Taking square roots, $2ax + b = \pm\sqrt{b^2 - 4ac}$. (4)

Transposing, $2ax = -b \pm \sqrt{b^2 - 4ac}$.

Dividing by $2a$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (5)

Calling the two values of x in the result x_1 and x_2, we have:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a};\quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \qquad (6)$$

Any quadratic equation may be reduced to the form of (1) and may therefore *be solved by substituting in the formulas* 6.

233. Method of Checking Results. A convenient method of checking the roots of a quadratic is obtained by finding the relations between the roots and the coefficients of the equation,

$$ax^2 + bx + c = 0.$$

By addition we obtain, $x_1 + x_2 = \dfrac{-2b}{2a} = \dfrac{-b}{a}$. See (6), § 232.

By multiplication, $x_1x_2 = \dfrac{(-b + \sqrt{b^2 - 4ac})}{2a} \dfrac{(-b - \sqrt{b^2 - 4ac})}{2a}$

$$= \frac{(b - \sqrt{b^2 - 4ac})(b + \sqrt{b^2 - 4ac})}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

Hence in the equation $ax^2 + bx + c = 0$, *the sum of the roots equals* $-\dfrac{b}{a}$, *and the product of the roots equals* $\dfrac{c}{a}$.

Example. Solve $2x^2-4x+1=0$.

Substituting $a=2$, $b=-4$, $c=1$, in the formula, we get

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 2\cdot 1}}{2\cdot 2},$$

from which $x_1=\dfrac{2+\sqrt{2}}{2}$ and $x_2=\dfrac{2-\sqrt{2}}{2}$.

Check. $x_1+x_2=\dfrac{2+\sqrt{2}}{2}+\dfrac{2-\sqrt{2}}{2}=\dfrac{4}{2}=2=-\left(\dfrac{-4}{2}\right)=-\dfrac{b}{a}$

and $x_1x_2=\dfrac{2+\sqrt{2}}{2}\cdot\dfrac{2-\sqrt{2}}{2}=\dfrac{4-2}{4}=\dfrac{1}{2}=\dfrac{c}{a}$.

WRITTEN EXERCISES

By means of the formula (§ 232), find the solutions of each of the following equations and check each by the method of § 233:

1. $2x^2-3x-4=0$.
2. $3x^2+2x-1=0$.
3. $3x^2-2x-1=0$.
4. $4x^2+6x+1=0$.
5. $x^2-7x+12=0$.
6. $5x^2+8x+3=0$.
7. $5x^2-8x+3=0$.
8. $5x^2+8x-3=0$.
9. $5x^2-8x-3=0$.
10. $2x-3x^2+7=0$.
11. $3x-9x^2+1=0$.
12. $7x^2-3x-2=0$.
13. $6x^2+7x+1=0$.
14. $4x^2+5x-3=0$.
15. $4x^2-5x-3=0$.
16. $8x^2+3x-5=0$.
17. $7x^2+x-3=0$.
18. $7x^2-x-4=0$.
19. $x^2-2ax=3b-a^2$.
20. $x^2-6ax=49c^2-9a^2$.
21. $x^2-\dfrac{mx}{2}+2mn=4nx$.
22. $x^2-2ax+4ab=b^2+3a^2$.
23. $x^2-abx+a^2b-ax=ab^2-bx$.
24. $x^2+9-c=6x$.
25. $nx^2+m^2n=mn^2x+mx$.
26. $x^2+2a^2+3a-2=(3a+1)x$.

FRACTIONAL EQUATIONS LEADING TO QUADRATICS

234. When a quadratic is obtained by clearing an equation of fractions it may happen that one of the roots of this quadratic will fail to satisfy the original equation. See § 102. *This will be the case if any denominator in the given equation becomes zero on substituting a root of the quadratic.* Otherwise, both roots will satisfy the given equation.

Example. Solve $\dfrac{x}{x-1}+\dfrac{x-1}{x}=\dfrac{x^2+x-1}{x(x-1)}.$

Solution. Clearing of fractions by multiplying both members by $x(x-1)$,

$$x^2+x^2-2x+1=x^2+x-1.$$

Transposing, $x^2-3x+2=0.$

Factoring, $(x-1)(x-2)=0.$

Hence, $x=1$, and $x=2.$

But $x=1$ reduces two denominators in the original equation to zero. Hence $x=1$ is not a root of the given equation.

$x=2$ is a root, as may be shown by substituting.

Thus, $\dfrac{2}{2-1}+\dfrac{2-1}{2}=2+\frac{1}{2}=\frac{5}{2}.$

WRITTEN EXERCISES

Solve the following equations, making sure that no denominator is reduced to zero by a supposed solution:

1. $\dfrac{3x^3+3}{3x^2-7x+3}=x-7.$

2. $\dfrac{x+2}{x-2}=2x+3.$

3. $\dfrac{3}{2x^2-x-1}+\dfrac{5}{x^2-1}+\dfrac{1}{x+1}=0.$

4. $\dfrac{2x}{2x-1}+\dfrac{x}{x+1}-\dfrac{3x}{x-1}=-1.$

5. $\dfrac{1}{3(x-1)}-\dfrac{1}{x^2-1}=\dfrac{1}{4}.$

6. $\dfrac{2a-1}{a}+\dfrac{1}{2}=\dfrac{3a}{3a-1}.$

7. $\dfrac{2}{x-1}+\dfrac{3}{x-2}=\dfrac{6}{x-3}.$

8. $\dfrac{1}{a-x}-\dfrac{1}{a+x}=-\dfrac{3+x^2}{a^2-x^2}.$

9. $\frac{2}{x-10}+10=x+\frac{2}{10-x}$.   10. $\frac{6-x}{x-4}+\frac{x-4}{6-x}=\frac{5}{2}$.

11. $\frac{a}{2a-1}+\frac{24}{4a^2-1}=\frac{2(a-4)}{2a+1}-\frac{1}{9}$.

12. $\frac{a}{a-1}+\frac{a-1}{a}=\frac{a^2+a-1}{a^2-a}$.

13. $\frac{1}{a^2-4}-\frac{3}{2-a}=1+\frac{1}{3(a+2)}$.

14. $\frac{(a-x)(x-b)}{(a-x)-(x-b)}=x$.

15. $\frac{x+m-2n}{x+m+2n}=\frac{n+2m-2x}{n-2m+2x}$.

16. $\frac{1}{x-2}+\frac{7x}{24(x+2)}=\frac{5}{x^2-4}$.

17. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{2(a^2+1)}{(1+a)(1-a)}$.

18. $\frac{x-m}{x+m}=\frac{n-x}{n+x}$.   19. $\frac{1}{x-a}-\frac{2a}{x^2-a^2}=b$.

20. $\frac{4}{3x+1}+\frac{4(3x-1)}{2x+1}=\frac{2x+1}{3x+1}$.

21. $\frac{2x+3}{2(2x-1)}+\frac{7-3x}{3x-4}+\frac{x-7}{2(x+1)}=0$.

22. $\frac{y^2+2y-2}{y^2+5y+6}+\frac{y}{y+3}=\frac{y}{y+2}$.

23. $\frac{5}{2x+3}+\frac{7}{3x-4}=\frac{8x^2-13x-64}{6x^2+x-12}$.

24. $\frac{27}{x^3+27}-\frac{3}{x^2-3x+9}+\frac{1}{x+3}=0$.

25. $\frac{1}{x^3-1}-\frac{2}{x^2+x+1}=\frac{1}{x-1}$.

26. $\frac{a^2c}{x^3-a^3}-\frac{b}{x^2+xa+a^2}=\frac{c}{x-a}$.

IRRATIONAL EQUATIONS LEADING TO QUADRATICS

235. Many equations containing radicals may be solved by means of *quadratics.* Sometimes they even reduce to ***linear equations,*** as in the first example below.

Example 1. Solve $\frac{\sqrt{4x+1}-\sqrt{3x-2}}{\sqrt{4x+1}+\sqrt{3x-2}}=\frac{1}{5}.$

Solution. Clearing of fractions,

$$5\sqrt{4x+1}-5\sqrt{3x-2}=\sqrt{4x+1}+\sqrt{3x-2}.$$

Transposing, and combining similar radicals,

$$2\sqrt{4x+1}=3\sqrt{3x-2}.$$

Squaring both members, $4(4x+1)=9(3x-2)$.

Transposing and solving, we find $x=2$.

Check. Substituting $x=2$ in the given equation,

$$\frac{\sqrt{9}-\sqrt{4}}{\sqrt{9}+\sqrt{4}}=\frac{3-2}{3+2}=\frac{1}{5}.$$

Example 2. Solve $\frac{6-x}{\sqrt{6-x}}-\sqrt{3}=\frac{x-3}{\sqrt{x-3}}.$

Solution. We see that $6-x=\sqrt{6-x}\times\sqrt{6-x},$

and $x-3=\sqrt{x-3}\times\sqrt{x-3}.$

Hence each fraction can be reduced to lower terms, thus,

$$\frac{\sqrt{6-x}\times\cancel{\sqrt{6-x}}}{\cancel{\sqrt{6-x}}}-\sqrt{3}=\frac{\sqrt{x-3}\times\cancel{\sqrt{x-3}}}{\cancel{\sqrt{x-3}}},$$

or $$\sqrt{6-x}-\sqrt{3}=\sqrt{x-3}.$$

Squaring both sides, $6-x-2\sqrt{3}\sqrt{6-x}+3=x-3.$

Transposing and dividing by 2, $x-6=-\sqrt{3}\,\sqrt{6-x}.$

Squaring again, $x^2-12x+36=3(6-x).$

Transposing, $x^2-9x+18=0.$

Factoring, $(x-6)(x-3)=0.$

Hence, $x=6$, and $x=3$.

But neither of these values will satisfy the given equation, since $x=6$ makes one denominator zero, and $x=3$ makes the other zero. Hence this equation *has no solution.*

Example 3. Solve $\sqrt{x-1}+\sqrt{4-x}=\sqrt{3}$.

Solution. It will be simpler to transpose first, so as to have one radical containing the unknown alone on one side, thus

$$\sqrt{x-1}-\sqrt{3}=-\sqrt{4-x}.$$

Squaring both members, $x-1-2\sqrt{3}\sqrt{x-1}+3=4-x$.

Transposing, $2x-2=2\sqrt{3}\sqrt{x-1}$,

or $x-1=\sqrt{3}\sqrt{x-1}$.

Squaring again and transposing, $x^2-5x+4=0$.

Solving, $x=4$, and $x=1$.

Both of the values satisfy the given equations.

236. The above examples suggest the following **rule** for solving irrational equations:

(1) *If a fraction of the form* $\frac{a-b}{\sqrt{a}-\sqrt{b}}$ *is involved, this should be reduced by dividing numerator and denominator by* $\sqrt{a}-\sqrt{b}$ *before clearing of fractions.*

(2) *If a fraction of the form* $\frac{a+b}{\sqrt{a+b}}$ *is involved, this should be reduced by dividing both numerator and denominator by* $\sqrt{a+b}$, *obtaining* $\sqrt{a+b}$.

(3) *After clearing of fractions, transpose terms so as to leave one radical alone in one member.*

(4) *Square both members, and if the resulting equation still contains radicals, transpose and square as before.*

(5) *In every case verify all results by substituting in the given equation.*

The equation should be looked over with care at the outset to see whether it may be simplified by using (1) or (2) of the above rule.

Thus in Example 3 on the next page, $\frac{by-1}{\sqrt{by}+1}$ may be thus simplified, giving $\sqrt{by}-1$.

Similarly, in Example 11, $\frac{x+5}{\sqrt{x+5}}=\sqrt{x+5}$.

Likewise, note Examples 7, 8, 12, and 13.

EXERCISES

Solve the following equations and check all solutions:

1. $\sqrt{x^2+7x-2}-\sqrt{x^2-3x+6}=2.$

2. $\sqrt{3y}-\sqrt{3y-7}=-\dfrac{5}{\sqrt{3y-7}}.$

3. $\dfrac{by-1}{\sqrt{by}+1}=\dfrac{\sqrt{by}-1}{2}+4.$

4. $\sqrt{5x-19}+\sqrt{3x+4}=9.$

5. $\dfrac{\sqrt{x^2+a^2}-x}{\sqrt{x^2+a^2}+x}=2.$

6. $\sqrt{a+\sqrt{ax+x^2}}=\sqrt{a}-\sqrt{x}.$

7. $\dfrac{y-l}{\sqrt{y}+\sqrt{l}}=\dfrac{\sqrt{y}-\sqrt{l}}{3}+2\sqrt{l}.$

8. $\sqrt{5x}-3=-\dfrac{5x-1}{\sqrt{5x}+1}.$

9. $\dfrac{4}{x}-\dfrac{\sqrt{4-x^2}}{x}=\sqrt{3}.$

10. $\dfrac{4+x+\sqrt{8x+x^2}}{4+x-\sqrt{8x+x^2}}=4.$

11. $\dfrac{5-x}{\sqrt{5-x}}+\dfrac{x+5}{\sqrt{x+5}}=2.$

12. $\dfrac{x-a}{\sqrt{x}-\sqrt{a}}=\dfrac{\sqrt{x}+\sqrt{a}}{2}+2\sqrt{a}.$

13. $\dfrac{m-y}{\sqrt{m-y}}-\sqrt{m-n}=-\dfrac{y-n}{\sqrt{y-n}}.$

14. $2\sqrt{x-a}+3\sqrt{2x}=\dfrac{7a+5x}{\sqrt{x-a}}.$

15. $\sqrt{x-3}+\sqrt{x+9}=\sqrt{x+18}+\sqrt{x-6}.$

16. $\sqrt{x+7}-\sqrt{x-1}=\sqrt{x+2}+\sqrt{x-2}.$

17. $a\sqrt{y+b}-c\sqrt{b-y}=\sqrt{b(a^2+c^2)}.$

18. $y\sqrt{y-c}-\sqrt{y^3+c^3}+c\sqrt{y+c}=0.$

19. $\dfrac{\sqrt{x}}{\sqrt{m}}+\dfrac{\sqrt{m}}{\sqrt{x}}=\dfrac{\sqrt{m}}{\sqrt{n}}+\dfrac{\sqrt{n}}{\sqrt{m}}.$

HIGHER EQUATIONS REDUCIBLE TO QUADRATICS

237. Equations in the Quadratic Form. If an equation contains a certain expression and also the square of this expression, then the equation is a *quadratic in the given expression.*

Thus, $x^4 + 7x^2 = 12$ is a quadratic in x^2.
$2x + 7 + 3\sqrt{2x+7} = 9$ is a quadratic in $\sqrt{2x+7}$.

Such equations may often be solved by quadratic methods, even when of higher degree than the second.

Example 1. Solve $x^4 + 7x^2 = 44$.

This may be written $(x^2)^2 + 7(x^2) = 44$,
which is a quadratic in x^2. Solving for x^2, we find

$$(x^2 - 4)(x^2 + 11) = 0.$$

Hence, $x^2 = 4$ and $x^2 = -11$,
and $x = \pm 2$, and $x = \pm\sqrt{-11}$.

Example 2. Solve $x + 2 + 3\sqrt{x+2} = 18$.

Since $x + 2$ is the square of $\sqrt{x+2}$, this is a quadratic in $\sqrt{x+2}$.
Putting $\sqrt{x+2} = z$ for simplicity in solving, we have

$$z^2 + 3z - 18 = 0.$$

Solving, $z = 3$, and $z = -6$.
Replacing the value of z, we have

$$\sqrt{x+2} = 3, \text{ and } \sqrt{x+2} = -6.$$

Squaring, $x + 2 = 9$, and $x + 2 = 36$.
Hence, $x = 7$, and $x = 34$.
$x = 7$ satisfies the given equation, but $x = 34$ does not.

Example 3. Solve $(2x^2 - 1)^2 - 5(2x^2 - 1) - 14 = 0$.

This is a quadratic in the expression $2x^2 - 1$.
Putting $2x^2 - 1 = z$, we have

$$z^2 - 5z - 14 = 0.$$

Solving for z, $z = -2$, and $z = 7$.
Replacing the values of z,

$$2x^2 - 1 = -2, \text{ and } 2x^2 - 1 = 7.$$

Solving for x, $x = \pm \frac{1}{2}\sqrt{-2}$ and $x = \pm 2$.
All four values of x satisfy the given equation.

Example 4. Solve

$$x^2 - 17x + 40 - 2\sqrt{x^2 - 17x + 69} = -26. \qquad (1)$$

Add 29 to each member, obtaining

$$x^2 - 17x + 69 - 2\sqrt{x^2 - 17x + 69} = 3. \qquad (2)$$

Equation (2) is a quadratic in $\sqrt{x^2 - 17x + 69}$.

Putting $\sqrt{x^2 - 17x + 69} = z$,
we have $z^2 - 2z - 3 = 0. \qquad (3)$
Solving for z, $z = 3$, and $z = -1$.
Replacing the values of z,

$$\sqrt{x^2 - 17x + 69} = 3, \text{ and } \sqrt{x^2 - 7x + 69} = -1.$$

Squaring, $x^2 - 17x + 69 = 9$, and $x^2 - 17x + 69 = 1$.
Transposing, $x^2 - 17x + 60 = 0, \qquad (4)$
and $x^2 - 17x + 68 = 0. \qquad (5)$
From (4), $(x - 5)(x - 12) = 0$ or $x = 5$, $x = 12$.
From (5), $x = \dfrac{17 \pm \sqrt{17}}{2}$, but these values of x do not satisfy (2).

WRITTEN EXERCISES

Solve the following equations:

1. $x^4 - 5x^2 + 6 = 0.$
2. $5x - 4 - 2\sqrt{5x - 4} = 63.$
3. $(2 - x + x^2)^2 + x^2 - x = 18.$
4. $a^2 - 3a + 4 - 3\sqrt{a^2 - 3a + 4} = -2.$
5. $3a^6 - 57a^3 - 648 = 0.$
6. $x^2 - 8x + 16 + 6\sqrt{x^2 - 8x + 16} = 40.$
7. $\left(a + \dfrac{2}{a}\right)^2 + 4\left(a + \dfrac{2}{a}\right) = 21.$
8. $a^8 - 97a^4 + 1296 = 0.$
9. $a^2 - 3a + 4 + \sqrt{a^2 - 3a + 15} = 19.$
10. $(5x - 7 + 3x^2)^2 + 3x^2 + 5x - 247 = 0.$
11. $\sqrt[3]{7x - 6} - 4\sqrt[6]{7x - 6} + 4 = 0.$
12. $x^6 + 19x^3 = 216.$

RELATIONS BETWEEN THE ROOTS AND COEFFICIENTS OF A QUADRATIC

238. The Sum and Product of the Roots. We have seen (§ 233) that in the solution of $ax^2+bx+c=0$ the two roots,

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\ \text{and}\ x_2=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

are such that $x_1+x_2=-\frac{b}{a}$ and $x_1x_2=\frac{c}{a}$.

E.g. in $2x^2-3x-8=0$, $x_1+x_2=-\frac{-3}{2}=\frac{3}{2}$ and $x_1x_2=-\frac{8}{2}=-4$.

239. The Discriminant of the Quadratic Equation. The expression b^2-4ac under the radical sign in the two values of x above is called the *discriminant of the quadratic.* Its value determines the nature of the roots of the equation as follows:

(1) If $b^2>4ac$, that is, if $b^2-4ac>0$, then both x_1 and x_2 are *real* (see § 151), and they are distinct since

$$x_1=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\ \text{and}\ x_2=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

If b^2-4ac is a perfect square, then both x_1 and x_2 are *rational* (see § 15).

(2) If $b^2=4ac$, that is, if $b^2-4ac=0$, then the values of x_1 and x_2 are the same, since $x_1=-\frac{b}{2a}+0$ and $x_2=-\frac{b}{2a}-0$.

(3) If $b^2<4ac$, that is, if $b^2-4ac<0$, then both x_1 and x_2 are *complex numbers,* since $\sqrt{b^2-4ac}$ is *imaginary* (see § 212).

240. Summary. *If x_1 and x_2 are the roots of $ax^2+bx+c=0$, then we have*

$$x_1+x_2=-\frac{b}{a},\ \text{and}\ x_1x_2=\frac{c}{a}, \tag{1}$$

x_1 and x_2 are real and distinct, if $b^2-4ac>0$, (2)

x_1 and x_2 are real and equal, if $b^2-4ac=0$, (3)

x_1 and x_2 are imaginary, if $b^2-4ac<0$. (4)

By means of the results (1) to (4), we may determine the character of the roots of a quadratic without solving it.

Example 1. Determine the character of the roots of

$$8x^2 - 3x - 9 = 0.$$

Since $b^2 - 4ac = 9 - 4 \cdot 8(-9) = 297 > 0$, the roots are *real* and *distinct*. Since $b^2 - 4ac$ is not a perfect square, the roots are *irrational*.

Since $\frac{c}{a} = -\frac{9}{8} = x_1x_2$, the roots have *opposite signs*.

Since $-\frac{b}{a} = \frac{3}{8} = x_1 + x_2$, the *positive root is greater* in absolute value.

Example 2. Examine $3x^2 + 5x + 2 = 0$.

Since $b^2 - 4ac = 25 - 4 \cdot 3 \cdot 2 = 1 > 0$, the roots are *real* and *distinct*. Since $b^2 - 4ac$ is a perfect square, the roots are *rational*.

Since $\frac{c}{a} = \frac{2}{3} = x_1x_2$, the roots have the *same sign*.

Since $-\frac{b}{a} = -\frac{5}{3} = x_1 + x_2$, the roots are *both negative*.

Example 3. Examine $x^2 - 14x + 49 = 0$.

Since $b^2 - 4ac = 196 - 4 \cdot 49 = 0$, the roots are *real* and *equal*.

Example 4. Examine $x^2 - 7x + 15 = 0$.

Since $b^2 - 4ac = 49 - 4 \cdot 15 = -11$, the roots are *imaginary*.

ORAL EXERCISES

1. Find the value of $b^2 - 4ac$ in $2x^2 - 3x + 2 = 0$.

2. Find the sum and also the product of the roots in

$$3x^2 - 4x - 5 = 0.$$

3. Show that the roots of $2x^2 - 5x - 3 = 0$ have *opposite* signs. Which has the *greater* absolute value?

4. Show that the roots of $2x^2 + 8x - 3 = 0$ are *real*.

5. Show that the roots of $2x^2 - 7x + 3 = 0$ are *rational*.

6. Show that the roots of $3x^2 + 11x - 2 = 0$ are *real* but irrational.

7. Show that the roots of $2x^2 - 3x + 2 = 0$ are *imaginary*.

WRITTEN EXERCISES

Without solving, determine the character of the roots in each of the following:

1. $5x^2 - 4x - 5 = 0.$
2. $6x^2 + 4x + 2 = 0.$
3. $x^2 - 4x + 8 = 0.$
4. $2 + 2x^2 = 4x.$
5. $6x + 8x^2 = 9.$
6. $1 - a^2 = 3a.$
7. $6a - 30 = 3a^2.$
8. $6a^2 + 6 = 13a.$
9. $16m^2 + 4 = 16m.$
10. $25a^2 - 10a = 8.$
11. $20 - 13b - 15b^2 = 0.$
12. $10y^2 + 39y + 14 = 0.$
13. $3a^2 + 5a + 22 = 0.$
14. $3a^2 - 22a + 21 = 0.$
15. $5b^2 + 6b = 27.$
16. $6a - 17 = 11a^2.$

FORMATION OF EQUATIONS WHOSE ROOTS ARE GIVEN

241. If the roots of a quadratic equation are given, the equation may be formed by recalling the *solution by factoring.*

Example 1. Form the equation whose roots are 7 and -4.

Solution. Since a quadratic in the form $(x-a)(x-b)=0$ has the roots $x=a$ and $x=b$, we see that $(x-7)(x-(-4))=0$ is an equation whose roots are 7 and -4.

Hence, $(x-7)(x+4)=x^2-3x-28=0$ is the required equation.

Example 2. Form the equation whose roots are 2, 3, and 5.

Solution. The required equation is

$$(x-2)(x-3)(x-5)=x^3-10x^2+31x-30=0.$$

ORAL EXERCISES

Form the equations whose roots are:

1. 1, 2.
2. 1, 3.
3. 2, 3.
4. 1, 4.
5. 2, 4.
6. 2, 6.
7. $-1, -2.$
8. $-1, -3.$
9. $-2, -3.$
10. $-1, -4.$
11. $-2, -4.$
12. $-2, -6.$
13. 3, 4.
14. $-3, -4.$
15. $4, -5.$

WRITTEN EXERCISES

Form the equations whose roots are:

1. $3, -7$.
2. b, c.
3. $a, -b, -c$.
4. $5, -4, -2$.
5. $-5, -6$.
6. $a, -b$.
7. $-b+k, -b-k$.
8. $2, 3, 4, 5$.
9. $1, \frac{1}{2}, \frac{1}{3}, 3$.
10. $\sqrt{5}, -\sqrt{5}$.
11. $\sqrt{-1}, -\sqrt{-1}$
12. $8+\sqrt{3}, 8-\sqrt{3}$.
13. $3+2\sqrt{-1}, 3-2\sqrt{-1}$.
14. $5-\sqrt{-1}, 5+\sqrt{-1}$.
15. $a-\sqrt{3}, a+\sqrt{3}$.
16. $\dfrac{-b+\sqrt{b^2-4ac}}{2a}, \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

242. Factoring by Solving a Quadratic. An expression of the second degree in a single letter may be *resolved into factors*, each of the first degree in that letter, by solving a quadratic equation.

Example 1. Factor $6x^2 - 17x + 5$.

This trinomial may be written, $6(x^2 - \frac{17}{6}x + \frac{5}{6})$.

Solving the equation, $x^2 - \frac{17}{6}x + \frac{5}{6} = 0$, we find $x_1 = \frac{1}{3}$ and $x_2 = \frac{5}{2}$. Hence, by the factor theorem, § 70, $x - \frac{1}{3}$ and $x - \frac{5}{2}$ are factors of $x^2 - \frac{17}{6}x + \frac{5}{6}$. And finally,

$$\begin{aligned}6(x^2 - \tfrac{17}{6}x + \tfrac{5}{6}) &= 6(x - \tfrac{1}{3})(x - \tfrac{5}{2}) = 3(x - \tfrac{1}{3}) \cdot 2(x - \tfrac{5}{2})\\ &= (3x-1)(2x-5).\end{aligned}$$

This process is usually not needed when the factors are *rational.* It is particularly applicable when the factors are *irrational* or *imaginary.*

Example 2. Factor $3x^2 + 8x - 7 = 3(x^2 + \frac{8}{3}x - \frac{7}{3})$.

Solving the equation $x^2 + \frac{8}{3}x - \frac{7}{3} = 0$, we find,

$$x_1 = \frac{-4+\sqrt{37}}{3} \text{ and } x_2 = \frac{-4-\sqrt{37}}{3}.$$

Hence, as above:

$$\begin{aligned}3x^2 + 8x - 7 &= 3\left[x - \frac{-4+\sqrt{37}}{3}\right]\left[x - \frac{-4-\sqrt{37}}{3}\right]\\ &= 3\left[x + \frac{4}{3} - \frac{\sqrt{37}}{3}\right]\left[x + \frac{4}{3} + \frac{\sqrt{37}}{3}\right].\end{aligned}$$

WRITTEN EXERCISES

Using the method of § 242, factor the following:

1. $x^2 - 4x + 8$.
2. $x^2 + 6x - 9$.
3. $a^2 + 3a - 1$.
4. $2x^2 - 4x + 2$.
5. $5x^2 - 4x - 5$.
6. $6x^2 + 4x + 2$.
7. $3a^2 - 6a + 30$.
8. $6a^2 - 13a + 6$.
9. $16m^2 - 32m + 15$.
10. $25a^2 - 10a - 8$.
11. $15b^2 + 13b - 20$.
12. $10y^2 + 39y + 14$.
13. $3a^2 + 5a + 22$.
14. $3a^2 - 22a + 21$.
15. $5b^2 + 6b - 27$.
16. $11a^2 - 6a + 17$.

PROBLEMS IN ONE UNKNOWN LEADING TO QUADRATICS

243. Interpreting Results in Solving Problems. A problem which gives rise to a quadratic equation *may have two solutions,* in which case the two roots of the quadratic are the solutions.

If such a problem has *only one solution,* it is necessary to determine which of the two roots of the quadratic satisfies the conditions of the problem.

Example 1. The dimensions of a picture inside the frame are 10 by 12 inches. The area of the frame is 104 square inches. Find the width of the frame.

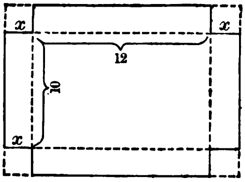
Solution. If x = width of frame, then $4x^2$ is the area of the corners, and $2 \cdot 12x + 2 \cdot 10x = 44x$ is the area of the rest of the frame.

Hence, $4x^2 + 44x = 104,$

or $x^2 + 11x - 26 = 0.$

Hence, $x = 2$, or -13.

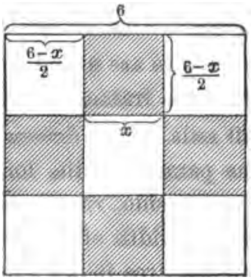
Since the width of a frame cannot be negative, the only solution which satisfies the problem is $x = 2$.

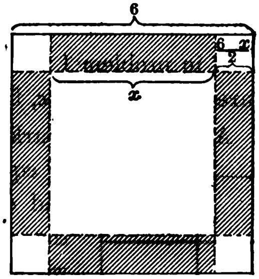


John Napier (1550–1617), Baron of Merchiston, Scotland, inventor of logarithms, was the first Englishman to publish important contributions to mathematics. Most of his life was spent on his family estates, where he took active part in political and religious controversies.

The work containing his treatment of logarithms was published in Latin in 1614, though he had confided the results of his discovery to the astronomer Tycho Brahe as early as 1594.

Example 2. In the accompanying figures, the large squares are 6 inches on each side. The shaded parts occupy $\frac{4}{9}$ of the area of each square. Find the width of the shaded strips.





Solution. Let $x =$ the length of a shaded strip in either figure. Then the total shaded area in each case is $4 \cdot x \cdot \frac{6-x}{2}$.

$$\text{Hence,}\qquad 4x \cdot \frac{6-x}{2} = \frac{4}{9} \cdot 36 = 16,$$

$$\text{or}\qquad x^2 - 6x + 8 = 0,$$

$$\text{or}\qquad (x-2)(x-4) = 0.$$

$$\text{Hence,}\qquad x = 2, \text{ and } x = 4.$$

It is easily seen by studying the figures that this problem has two solutions, and that they are given by the two roots $x = 2$, and $x = 4$.

Example 3. Find three consecutive integers such that the sum of their squares is 110.

Solution. Let $x =$ the smallest integer.

$$\text{Then}\qquad x^2 + (x+1)^2 + (x+2)^2 = 110,$$

$$\text{or}\qquad 3x^2 + 6x - 105 = 0.$$

$$x^2 + 2x - 35 = 0.$$

$$(x+7)(x-5) = 0.$$

$$\text{Hence,}\qquad x = 5, \text{ and } x = -7.$$

The positive integers 5, 6, 7, satisfy the conditions of the problem, as do also the negative integers -7, -6, -5.

If this problem were proposed in arithmetic, where the numbers used are all positive, the value $x = -7$ would not furnish a solution of the problem, whereas in algebra $x = -7$ is a legitimate solution.

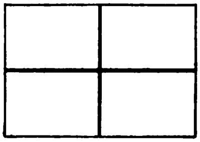
PROBLEMS IN ONE UNKNOWN LEADING TO QUADRATICS

Interpret both solutions of each quadratic:

1. A picture measured inside the frame is 18 by 24 inches. The area of the frame is 288 square inches. Find its width.

2. If in problem 1 the sides of the picture are a and b and the area of the frame c, find the width of the frame.

3. A rectangular park is 80 by 120 rods. Two driveways of equal width, one parallel to the longer and one to the shorter side, run through the park. What is the width of the driveways if their combined area is 591 square rods?



4. If in problem 3 the park is a rods wide and b rods long and the area of the driveways is c square rods, find their width.

5. A and B working together can do a piece of work in 6 days. A can do it alone in 5 days less than B. How long will it require each when working alone?

6. A and B working together can do a piece of work in n days. A does the work in d days less than B. How long will it require each when working alone?

7. The circumference of the rear wheel of a carriage is 4 feet greater than that of the front wheel. In running one mile the front wheel makes 110 revolutions more than the rear wheel. Find the circumference of each wheel.

8. Solve a general problem of which Example 7 is a special case, using b feet instead of one mile, letting the other numbers remain as they are in Example 7.

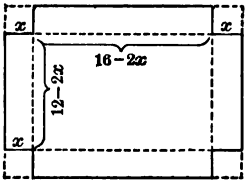
9. On her second westward trip the *Mauretania* traveled 625 knots in a certain time. If her speed had been 5 knots less per hour, it would have required $6\frac{1}{4}$ hours longer to cover the same distance. Find her speed per hour.

10. By increasing the speed a miles per hour, it requires b hours less to go c miles. Find the original speed. Show how problem 9 may be solved by means of the formula thus obtained.

11. A man can row 24 miles downstream and return in 9 hours. If his rate upstream is 4 miles per hour less than downstream, find the rate of the current and the rate of the boat in still water.

12. A man can row a miles downstream and return in b hours. If his rate upstream is c miles per hour less than downstream, find the rate of the current, and the rate of the boat in still water.

13. A rectangular sheet of tin, 12 by 16 inches, is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cubic inches.



The resulting equation is of the third degree. Solve it by factoring. See §§ 70, 228. Obtain three results and determine which are applicable to the problem.

14. A square piece of tin is made into an open box containing a cubic inches, by cutting from each corner a square whose side is b inches and then turning up the sides. Find the side of the original square. Also if $a = 18$, $b = 2$.

15. A rectangular piece of tin is 2 inches longer than it is wide. By cutting from each corner a square whose side is 1 inch and turning up the sides, an open box containing 18 cubic inches is formed. Find the dimensions of the original piece of tin.

16. The difference of the cubes of two consecutive integers is 397. Find the integers.

17. The upper base of a trapezoid is 8 and the lower base is 3 times the altitude. Find the altitude and the lower base if the area is 78.

The area of a trapezoid is equal to one half the sum of the bases multiplied by the altitude. See page 79.

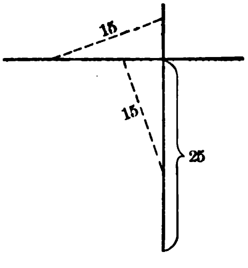
18. The lower base of a trapezoid is 4 greater than twice the altitude, and the upper base is $\frac{1}{2}$ the lower base. Find the two bases and the altitude if the area is $52\frac{1}{2}$.

19. The area of a regular hexagon inscribed in a circle is a. Find the radius of the circle.

20. One edge of a rectangular box is increased 6 inches, another 3 inches, and the third is decreased 4 inches, making a cube whose volume is 864 cubic inches greater than that of the original box. Find its dimensions.

21. Of two trains one runs 12 miles per hour faster than the other, and covers 144 miles in one hour less time. Find the speed of each train.

22. An automobile is running northward on a straight road at the rate of 25 miles per hour. A team is traveling westward on a straight road at the rate of 10 miles per hour. When the team crosses the north and south road the automobile is 25 miles south of the crossing. How long before the machine and the team will be 15 miles apart measured diagonally across the country?



If t is the number of hours required, then

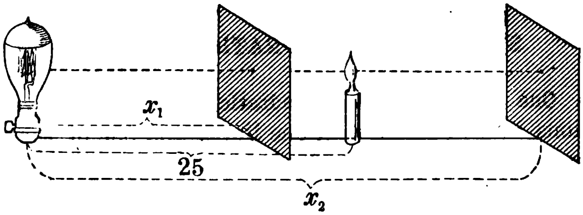
$$(25-25t)^2+(10t)^2=15^2,$$

or

$$25^2(1-t)^2+10^2t^2=15^2.$$

$$5^2(1-t)^2+2^2t^2=3^2.$$

23. An electric light of 32 candle power is 25 feet from a candle of 6 candle power. Where should a card be placed between them so as to receive the same amount of light from each?



Suggestion. The illumination varies inversely as the square of the distance from the source of light, and directly as the intensity of the source of light.

Let x = distance from the candle. Then $\frac{6}{x^2} = \frac{32}{(25-x)^2}$. Compute result accurately to two places of decimals.

24. Where must a card be placed in problem 23 if the candle is between the card and the electric light?

Notice that the roots of the equations in 23 and 24 are the same. Explain what this means.

25. If the distance between the earth and the sun is 93 million miles, and if the mass of the sun is 300,000 times that of the earth, find two positions in which a particle would be equally attracted by the earth and the sun.

The gravitational attraction of one body upon another varies inversely as the square of the distance, and directly as the product of the masses. Represent the mass of the earth by unity.

This problem may be solved more easily if we let d = distance from the earth to the sun, and m = the mass of the sun, the mass of the earth being regarded as unity. Let x = the distance from the earth in millions of miles to the point required. The equation, then, is

$$\frac{1}{x^2} = \frac{m}{(d-x)^2}, \text{ or } \frac{1}{x} = \frac{\sqrt{m}}{d-x}.$$

After solving this equation, substitute the numbers given above.

CHAPTER XV

SYSTEMS OF QUADRATIC EQUATIONS

244. One Linear and One Quadratic Equation. A system of equations in two unknowns, one linear and one of the second degree, may always be solved by a substitution, leading to the solution of an ordinary quadratic.

Example 1. Solve the system $\begin{cases} x - y = 4, & (1) \\ 2x^2 + 3y^2 = 84. & (2) \end{cases}$

Solution. From (1), $x = y + 4.$

Substituting in (2), $2(4 + y)^2 + 3y^2 = 84,$

or $5y^2 + 16y - 52 = 0.$

Factoring, $(5y + 26)(y - 2) = 0.$

Hence, $y = 2$, or $y = -\frac{26}{5}.$

From $y = 2$ and (1), we have $\begin{cases} x = 6, \\ y = 2. \end{cases}$

From $y = -\frac{26}{5}$ and (1), we have $\begin{cases} x = -\frac{6}{5}, \\ y = -\frac{26}{5}. \end{cases}$

Example 2. Solve the system $\begin{cases} 3x - 7y = 1, & (1) \\ x^2 - 2xy - 2x + y = -3. & (2) \end{cases}$

Solution. From (1), $x = \frac{1 + 7y}{3}.$

Substituting in (2),

$$\left(\frac{1 + 7y}{3}\right)^2 - 2 \cdot \frac{1 + 7y}{3} y - 2 \cdot \frac{1 + 7y}{3} + y = -3.$$

Clearing of fractions,

$$1 + 14y + 49y^2 - 6y - 42y^2 - 6 - 42y + 9y = -27,$$

collecting terms $7y^2 - 25y + 22 = 0.$

Factoring, $(y - 2)(7y - 11) = 0$, and $y = 2$, or $\frac{11}{7}.$

From $y = 2$, we have $\begin{cases} x = 5, \\ y = 2. \end{cases}$

From $y = \frac{11}{7}$, we have $\begin{cases} x = 4, \\ y = \frac{11}{7}. \end{cases}$

Example 3. Solve the system $\begin{cases} 2x + 3y = 6, & (1) \\ 4x^2 - 3y^2 - x = 25. & (2) \end{cases}$

From (1), $$x = \frac{6 - 3y}{2}.$$

Substituting in (2),

$$4 \cdot \left(\frac{6 - 3y}{2}\right)^2 - 3y^2 - \frac{6 - 3y}{2} = 25.$$

Hence, $$36 - 36y + 9y^2 - 3y^2 - \frac{6 - 3y}{2} = 25,$$

$$72 - 72y + 18y^2 - 6y^2 - 6 + 3y = 50,$$

$$12y^2 - 69y + 16 = 0.$$

Solving by the formula,

$$y = \frac{69 \pm \sqrt{69^2 - 4 \cdot 12 \cdot 16}}{24} = \frac{69 \pm \sqrt{3993}}{24} = .24 \text{ or } 5.51.$$

Substituting these values of y in (1) and (2), we have

$$\begin{cases} x = 2.64, \\ y = .24, \end{cases} \text{and} \begin{cases} x = -5.27, \\ y = 5.51. \end{cases}$$

WRITTEN EXERCISES

Solve the following systems of equations:

1. $\begin{cases} x^2 + y^2 = 25, \\ x + y = 7. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 10, \\ x + 2y = 5. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 30, \\ x - 2y = 3. \end{cases}$

4. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

5. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2x - y = 4. \end{cases}$

6. $\begin{cases} y = 3x + 3x^2 - 6, \\ 2y + 3x - 3 = 0. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 40, \\ x + 2y = 10. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 29, \\ 3x - 7y = -29. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

10. Solve example 9 if $a = 8$.

11. $\begin{cases} \dfrac{x^2}{16} + \dfrac{y^2}{9} = 1, \\ 3x + 4y = 12. \end{cases}$

12. $\begin{cases} 3x^2 + 2y^2 = 11, \\ x - 3y = 7. \end{cases}$

SPECIAL METHODS OF SOLUTION

245. Certain systems of one linear and one second degree equation may be solved by special methods, as is shown in the following examples:

Example 1. Solve
$$\begin{cases} x^2 + y^2 = 13, & (1)\\ x - y = 1. & (2)\end{cases}$$

Solution. Squaring both members of (2) and subtracting from (1),

$$2xy = 12. \qquad (3)$$

Adding (1) and (3), $$x^2 + 2xy + y^2 = 25. \qquad (4)$$

Taking square roots, $$x + y = \pm 5. \qquad (5)$$

From (2) and (5), $$\begin{cases} x = 3, \\ y = 2, \end{cases} \text{ and } \begin{cases} x = -2, \\ y = -3. \end{cases}$$

Example 2.
$$\begin{cases} x^2 - y^2 = 5, & (1)\\ x - y = 1. & (2)\end{cases}$$

Solution. From (1) $$(x - y)(x + y) = 5. \qquad (3)$$

Dividing the members of (3) by those of (2),

$$x + y = 5. \qquad (4)$$

Then (2) and (4) may be solved in the usual way.

The system $\begin{cases} x^2 - y^2 = 7, \\ x + y = 7, \end{cases}$ may be solved in a similar manner.

Example 3. Solve
$$\begin{cases} x + y = 7, & (1)\\ xy = 12. & (2)\end{cases}$$

Solution. Multiplying (2) by 4, and subtracting from the square of (1), we get

$$x^2 - 2xy + y^2 = 1, \qquad (3)$$

whence, $$x - y = \pm 1. \qquad (4)$$

Then (1) and (4) are linear equations.

The equations $\begin{cases} x - y = 3, \\ xy = 10, \end{cases}$ may be solved in a similar manner.

WRITTEN EXERCISES

Solve the following systems:

1. $\begin{cases} x+y=10, \\ xy=21. \end{cases}$

2. $\begin{cases} x^2+y^2=34, \\ x-y=2. \end{cases}$

3. $\begin{cases} x^2+y^2=32, \\ x+y=8. \end{cases}$

4. $\begin{cases} x^2+y^2=r^2, \\ x+y=a. \end{cases}$

5. $\begin{cases} x+y=a, \\ xy=b. \end{cases}$

6. $\begin{cases} x^2+y^2=r^2, \\ x-y=a. \end{cases}$

7. $\begin{cases} x^2-y^2=a, \\ x+y=b. \end{cases}$

8. $\begin{cases} x^2-y^2=p, \\ x-y=q. \end{cases}$

246. Simultaneous Equations, Each of the Second Degree. We now proceed to study the solution of a pair of equations each of the second degree.

Consider
$$\begin{cases} x^2+y=7, & (1) \\ x+y^2=13. & (2) \end{cases}$$

Solving (1) for y, and substituting in (2), we have
$$x+49-14x^2+x^4=13,$$
which is of the *fourth degree* and cannot be solved by any methods thus far studied. There are, however, special cases in which two equations each of the second degree can be solved by a proper combination of methods already known.

Case I. *When only the squares of the unknowns enter the equations.*

Example. Solve
$$\begin{cases} 2x^2+3y^2=21, & (1) \\ 3x^2-2y^2=25. & (2) \end{cases}$$

Multiplying (1) by 2, and (2) by 3, and adding, we have
$$13x^2=117.$$
$$x^2=9.$$
$$x=\pm 3.$$

Substituting in (1), we have two values of y for each value of x, thus
$$\begin{cases} x=3, \\ y=1; \end{cases} \quad \begin{cases} x=3, \\ y=-1; \end{cases} \quad \begin{cases} x=-3, \\ y=1; \end{cases} \quad \begin{cases} x=-3, \\ y=-1. \end{cases}$$

Case II. *When all terms containing the unknowns are of the second degree in the unknowns.*

Example. Solve
$$\begin{cases} 2x^2 - 3xy + 4y^2 = 3, & (1) \\ 3x^2 - 4xy + 3y^2 = 2. & (2) \end{cases}$$

First Solution. Put $y = vx$ in (1) and (2), obtaining
$$\begin{cases} x^2(2 - 3v + 4v^2) = 3, & (3) \\ x^2(3 - 4v + 3v^2) = 2. & (4) \end{cases}$$

Hence, from (3) and (4),
$$x^2 = \frac{3}{2 - 3v + 4v^2}, \text{ and also } x^2 = \frac{2}{3 - 4v + 3v^2}. \qquad (5)$$

From (5) $$\frac{3}{2 - 3v + 4v^2} = \frac{2}{3 - 4v + 3v^2}, \qquad (6)$$

or $$v^2 - 6v + 5 = 0. \qquad (7)$$

Hence, $$v = 1, \text{ and } v = 5. \qquad (8)$$

From $y = vx$, $$y = x, \text{ and } y = 5x. \qquad (9)$$

If $y = x$, then from (1) and (2),
$$\begin{cases} x = 1, \\ y = 1, \end{cases} \text{ and } \begin{cases} x = -1, \\ y = -1. \end{cases}$$

If $y = 5x$, then from (1) and (2),
$$\begin{cases} x = \dfrac{1}{\sqrt{29}}, \\ y = \dfrac{5}{\sqrt{29}}, \end{cases} \text{ and } \begin{cases} x = -\dfrac{1}{\sqrt{29}}, \\ y = -\dfrac{5}{\sqrt{29}}. \end{cases}$$

Verify each of these four solutions by substituting in equations (1) and (2).

Second Solution. We may eliminate the ***known terms*** by multiplying equation (1) by 2 and equation (2) by 3, and subtracting,

thus, $$5x^2 - 6xy + y^2 = 0.$$

Dividing by x^2,
$$5 - 6\frac{y}{x} + \frac{y^2}{x^2} = 0.$$

Factoring, $$\left(\frac{y}{x} - 5\right)\left(\frac{y}{x} - 1\right) = 0.$$

Hence, $\frac{y}{x} = 5$ or 1, and $y = 5x$, or x.

These are the results reached in equation (9) of the first solution, and the remainder of the work is the same as before.

WRITTEN EXERCISES

Solve the following systems of equations:

1. $\begin{cases} 2x^2 + 4y^2 = 38, \\ 5x^2 - y^2 = -4. \end{cases}$

2. $\begin{cases} 3x^2 - 2y^2 = 97, \\ x^2 + y^2 = 74. \end{cases}$

3. $\begin{cases} x^2 - y^2 = 27, \\ 2x^2 - 3y^2 = 45. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 20, \\ 2x^2 - y^2 = 7. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 10, \\ x^2 + 3y^2 = 34. \end{cases}$

6. $\begin{cases} 3x^2 - 2y^2 = 40, \\ 4x^2 + 3y^2 = 60. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 50, \\ x^2 - y^2 = 14. \end{cases}$

8. $\begin{cases} 3x^2 + y^2 = 14, \\ 2x^2 - y^2 = 6. \end{cases}$

9. $\begin{cases} 5x^2 + y^2 = 42, \\ x^2 + 5y^2 = 18. \end{cases}$

10. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

11. $\begin{cases} 2x^2 + 5y^2 = a, \\ 3x^2 + y^2 = b. \end{cases}$

12. $\begin{cases} 7x^2 - 2y^2 = a, \\ 3x^2 - y^2 = b. \end{cases}$

13. $\begin{cases} x^2 + y^2 = a, \\ x^2 - y^2 = b. \end{cases}$

14. $\begin{cases} 2x^2 + 3y^2 = c, \\ 3x^2 + 2y^2 = d. \end{cases}$

15. $\begin{cases} ax^2 + by^2 = h, \\ cx^2 + dy^2 = k. \end{cases}$

16. $\begin{cases} x^2 + xy + y^2 = 3, \\ x^2 + 3xy - 2y^2 = 2. \end{cases}$

17. $\begin{cases} 2x^2 - xy + y^2 = 7, \\ x^2 + 2xy - y^2 = 7. \end{cases}$

18. $\begin{cases} 3x^2 - 4xy + y^2 = -1, \\ x^2 + 2xy - 3y^2 = -7. \end{cases}$

19. $\begin{cases} 5x^2 - 7xy + 3y^2 = 1, \\ 4x^2 + 5xy - 8y^2 = 1. \end{cases}$

20. $\begin{cases} 3x^2 - 7xy - y^2 = 9, \\ 2x^2 + 4xy - 3y^2 = -5. \end{cases}$

21. $\begin{cases} x^2 - 2xy - y^2 = -14, \\ 2x^2 + 5xy + y^2 = 26. \end{cases}$

22. $\begin{cases} x^2 + y^2 + xy = 3, \\ 2x^2 - y^2 - xy = 9. \end{cases}$

23. $\begin{cases} 4xy + 3x^2 + 2y^2 = 9, \\ 2xy + x^2 + y^2 = 0. \end{cases}$

24. $\begin{cases} 7x^2 + 4xy - y^2 = 8, \\ 3x^2 - 2xy - 4y^2 = 4. \end{cases}$

247. Special Cases. There are many other special cases of simultaneous equations which can be solved by proper combination of the methods thus far used.

The suggestions given in the following examples illustrate the devices in most common use.

The solution should in each case be completed by the student.

Example 1. Solve
$$\begin{cases} x^2 + y^2 = 58, & (1) \\ xy = 21. & (2) \end{cases}$$

Adding twice (2) to (1) and taking square roots, we have

$$x + y = 10, \text{ and } x + y = -10. \qquad (3)$$

Each of the equations (3) may now be solved simultaneously with (2), as in Example 3, page 194.

Example 2. Solve
$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5, & (1) \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13. & (2) \end{cases}$$

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$. Then these equations reduce to

$$\begin{cases} a + b = 5, & (3) \\ a^2 + b^2 = 13. & (4) \end{cases}$$

Solving as in Example 1, page 194, $\begin{cases} a = 3, \\ b = 2, \end{cases}$ $\begin{cases} a = 2, \\ b = 3. \end{cases}$

Hence, $\begin{cases} x = \frac{1}{3}, \\ y = \frac{1}{2}, \end{cases}$ $\begin{cases} x = \frac{1}{2}, \\ y = \frac{1}{3}. \end{cases}$

Example 3. Solve
$$\begin{cases} x^2 + y^2 + x + y = 8, & (1) \\ xy = 2. & (2) \end{cases}$$

Add twice (2) to (1), obtaining

$$x^2 + 2\,xy + y^2 + x + y = 12. \qquad (3)$$

Let $x + y = a$. Then (3) reduces to

$$a^2 + a = 12,$$

or
$$a = 3,\ a = -4. \qquad (4)$$

Hence, $x + y = 3$, and $x + y = -4$. (5)

Now solve each equation in (5) simultaneously with (2).

Example 4. Solve
$$\begin{cases} x^4y^4 + x^2y^2 = 272, & (1) \\ x^2 + y^2 = 10. & (2) \end{cases}$$

In (1) substitute a for x^2y^2. Then

$$a^2 + a = 272, \text{ whence } a = 16, \text{ and } -17.$$

Hence, $$xy = \pm\sqrt{16} = \pm 4, \text{ and } \pm\sqrt{-17}.$$

Each of these four equations may now be solved simultaneously with (2), as in Example 1, page 198.

Example 5. Solve
$$\begin{cases} x^3 - y^3 = 117, & (1) \\ x - y = 3, & (2) \end{cases}$$

By factoring, (1) becomes

$$(x - y)(x^2 + xy + y^2) = 117. \qquad (3)$$

Substituting 3 for $x - y$, we have

$$x^2 + xy + y^2 = 39. \qquad (4)$$

(2) and (4) may now be solved by substitution as in § 244.

Example 6. Solve
$$\begin{cases} x^3 + y^3 = 513, & (1) \\ x + y = 9. & (2) \end{cases}$$

Factor (1) and substitute 9 for $x + y$. Then proceed as in Example 5.

Example 7. Solve
$$\begin{cases} x^2y + xy^2 = 126, & (1) \\ x + y = 9. & (2) \end{cases}$$

Factoring (1) and substituting 9 for $x + y$, we have

$$xy = 14. \qquad (3)$$

(2) and (3) may then be solved as in Example 3, page 194.

Example 8. Solve
$$\begin{cases} x^3 + y^3 = 54\,xy, & (1) \\ x + y = 6. & (2) \end{cases}$$

Factor (1) and substitute 6 for $x + y$, obtaining

$$x^2 - xy + y^2 = 9\,xy. \qquad (3)$$

(2) and (3) may now be solved by substitution, as in § 244.

Example 9. Solve
$$\begin{cases} x^3 + y^3 = 243, & (1) \\ x^2y + xy^2 = 162. & (2) \end{cases}$$

Multiply (2) by 3 and add to (1), obtaining a perfect cube.
Taking cube roots, we have

$$x + y = 9. \qquad (3)$$

(1) and (3) are now solved as in Example 6 above.

WRITTEN EXERCISES

Solve each of the following pairs of equations:

1. $\begin{cases} r^2 + rs + s^2 = 63, \\ r - s = 3. \end{cases}$

2. $\begin{cases} 3x^2 + 2y^2 = 35, \\ 2x^2 - 3y^2 = 6. \end{cases}$

3. $\begin{cases} 3x^2 + 2xy = 16, \\ 4x^2 - 3xy = 10. \end{cases}$

4. $\begin{cases} a^2 + ab + b^2 = 7, \\ a^2 - ab + b^2 = 19. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 5, \\ xy = 2. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 5, \\ x^2 + z^2 = 10, \\ y^2 + z^2 = 13. \end{cases}$

7. $\begin{cases} 3x - 4y = 0, \\ x^2 + y^2 = 4. \end{cases}$

8. $\begin{cases} x^2 + xy = 4, \\ y^2 + xy = 5. \end{cases}$

9. $\begin{cases} x^3 + y^3 = 91, \\ x + y = 7. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 11, \\ x^2 - y^2 = 7. \end{cases}$

11. $\begin{cases} x^2 - 3xy = 0, \\ 5x^2 + 3y^2 = 9. \end{cases}$

12. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 19, \\ \dfrac{1}{x} + \dfrac{1}{y} = 1. \end{cases}$

13. $\begin{cases} 3x - 2y = 6, \\ 3x^2 - 2xy + 4y^2 = 12. \end{cases}$

14. $\begin{cases} a + b + ab = 11, \\ (a + b)^2 + a^2b^2 = 61. \end{cases}$

15. $\begin{cases} \dfrac{1}{a^2} + \dfrac{1}{ab} + \dfrac{1}{b^2} = 49, \\ \dfrac{1}{a} + \dfrac{1}{b} = 8. \end{cases}$

Suggestions. In example 12, substitute $\frac{1}{x} = a, \frac{1}{y} = b$.

In 14, $a + b = x$, $ab = y$.

In 15, $\frac{1}{a} = x$, $\frac{1}{b} = y$.

16. $\begin{cases} 4a^2 - 2ab = b^2 - 16, \\ 5a^2 = 7ab - 36. \end{cases}$

17. $\begin{cases} 3x^2 - 9y^2 = 12, \\ 2x - 3y = 14. \end{cases}$

18. $\begin{cases} x^2 + xy + y^2 = 19, \\ x^2 + y^2 = 13. \end{cases}$

19. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

20. $\begin{cases} x^2 + y^2 + x - y = 36, \\ xy = 15. \end{cases}$

21. $\begin{cases} x^1 - 5xy + y^2 = -2, \\ x^2 + 7xy + y^2 = 22. \end{cases}$

22. $\begin{cases} a^2 + 6ab + b^2 = 124, \\ a + b = 8. \end{cases}$

23. $\begin{cases} a^2 - 3ab + 2b^2 = 0, \\ 2a^2 + ab - b^2 = 9. \end{cases}$

24. $\begin{cases} x^2+y^2+2x+2y=27, \\ xy=-12. \end{cases}$

25. $\begin{cases} x^2+y^2-5x-5y=-4, \\ xy=5. \end{cases}$

26. $\begin{cases} (7+x)(6+y)=80, \\ x+y=5. \end{cases}$

27. $\begin{cases} (x-4)^2+(y+4)^2=100, \\ x+y=14. \end{cases}$

28. $\begin{cases} b+a^2=5(a-b), \\ a+b^2=2(a-b). \end{cases}$

29. $\begin{cases} (13x)^2+2y^2=177, \\ (2y)^2-13x^2=3. \end{cases}$

30. $\begin{cases} x^2+y^2=20, \\ 5x^2-3y^2=28. \end{cases}$

31. $\begin{cases} x^2=-5-3xy, \\ 2xy=y^2-24. \end{cases}$

32. $\begin{cases} x+y+\sqrt{x+y}=12, \\ x^3+y^3=189. \end{cases}$

33. $\begin{cases} x^4+x^2y^2+y^4=133, \\ x^2-xy+y^2=7. \end{cases}$

34. $\begin{cases} x+xy+y=29, \\ x^2+xy+y^2=61. \end{cases}$

35. $\begin{cases} 2x^2-5xy+3x-2y=22, \\ 5xy+7x-8y-2x^2=8. \end{cases}$

36. $\begin{cases} x+y=74, \\ x^2+y^2=3026. \end{cases}$

37. $\begin{cases} x^3+y^3=9, \\ x+y=3. \end{cases}$

38. $\begin{cases} x^3-y^3=37, \\ x-y=1. \end{cases}$

39. $\begin{cases} x^2+y^2-xy=80, \\ x-y-xy=-8. \end{cases}$

40. $\begin{cases} 7y^2-5x^2+20x+13y=29, \\ 5(x-2)^2-7y^2-17y=-17. \end{cases}$

41. $\begin{cases} 8a+8b-ab-a^2=18, \\ 5a+5b-b^2-ab=24. \end{cases}$

42. $\begin{cases} 2(x+4)^2-5(y-7)^2=75, \\ 7(x+4)^2+15(y-7)^2=1075. \end{cases}$

43. $\begin{cases} x^3+y^3=(a+b)(x-y), \\ x^2-xy+y^2=a-b. \end{cases}$

44. $\begin{cases} x+y=4, \\ x^3+x^2y+xy^2+y^3=32. \end{cases}$

45. $\begin{cases} (3x+4y)(7x-2y)+3x+4y=44, \\ (3x+4y)(7x-2y)-7x+2y=30. \end{cases}$

PROBLEMS IN TWO OR MORE UNKNOWNS LEADING TO QUADRATICS

248. Interpretation of Results. In solving a problem leading to a system of quadratic equations, it may be that not all pairs of values resulting from the equations will satisfy the conditions of the problem. See § 243.

In each case it is necessary to determine whether all or any of the results are applicable to the problem, as in the following example.

Example. The sum of the length and width of a rectangle is 18, and its area is 90. Find the dimensions.

Solution. Let the dimensions be x and y.

Then
$$\begin{cases} x + y = 18, \\ xy = 90. \end{cases}$$

Solving,
$$\begin{cases} x = 9 + 3\sqrt{-1}, \\ y = 9 - 3\sqrt{-1}, \end{cases} \quad \begin{cases} x = 9 - 3\sqrt{-1}, \\ y = 9 + 3\sqrt{-1}. \end{cases}$$

The imaginary results show that the problem has no real solution. That is, there is no rectangle having the properties stated in the problem.

PROBLEMS

Determine in each case whether all results are applicable to the problem.

1. The area of a rectangle is 2400 square feet and its perimeter is 200 feet. Find the length of its sides.

2. The area of a rectangle is a square feet and its perimeter is $2\,b$ feet. Find the length of its sides. Solve Example 1 by substituting in the formula thus obtained.

3. The sides a and b of a right triangle are increased by the same amount, thereby increasing the square on the hypotenuse by $2\,k$. Find by how much each side is increased.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

4. The hypotenuse c and one side a of a right triangle are each increased by the same amount, thereby increasing the square on the other side by $2k$. Find how much was added to the hypotenuse.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

5. Find the altitude drawn to the longest side of the triangle whose sides are 6, 7, 8.

6. Find the area of a triangle whose sides are 15, 17, 20.

First find one altitude as in problem 5.

7. Find the area of a triangle whose base is 16 and whose sides are 10 and 14.

8. Find the altitude on a side a of a triangle two of whose sides are a and a third b.

9. The sum of two numbers less 2, divided by their difference is 4, and the sum of their cubes divided by the difference of their squares is $1\frac{2}{5}$ times their sum. Find the numbers.

10. In going one mile the front wheel of a carriage makes 88 revolutions more than the rear wheel. If one foot is added to the circumference of the rear wheel, and 3 feet to that of the front wheel, the latter will make 22 revolutions more than the former. Find the circumference of each wheel.

11. The circumference of the rear wheel of a carriage is a feet greater than that of the front wheel. In running d feet the front wheel makes n more revolutions than the rear wheel. Find the circumference of each wheel.

12. Find a number of two digits whose sum divided by their difference is 4. This number divided by the sum of its digits is equal to twice the digit in units' place plus $\frac{1}{8}$ of the digit in tens' place; the tens' digit being the greater.

13. There is a fraction such that, if 3 is added to each of its terms, the result is $\frac{4}{5}$, and if 3 is subtracted from each of its terms, the result is $\frac{1}{2}$. Find the fraction.

14. Find the fraction such that, if a is added to both numerator and denominator, its value is $\frac{m}{n}$, and if a is subtracted from both numerator and denominator, its value is $\frac{r}{s}$.

15. The diagonal of a rectangle is a and its perimeter $2b$. Find its sides.

Make a problem which is a special case of this and solve it by substituting in the formula just obtained.

16. If the difference between the length and width of a rectangle is b and the diagonal is a, find the sides.

17. The hypotenuse of a right triangle is 20 inches longer than one side and 10 inches longer than the other. Find the dimensions of the triangle.

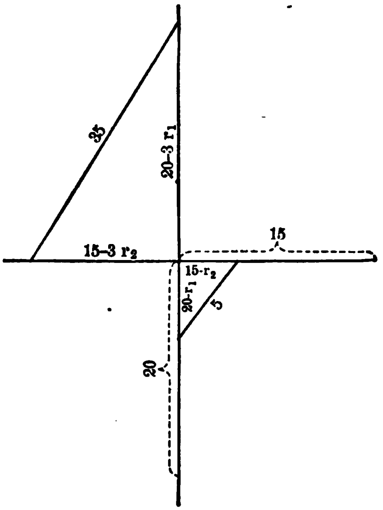
18. If in problem 17 the hypotenuse is a inches longer than one side and b inches longer than the other, find the dimensions of the triangle.

19. The area of a circle exceeds that of a square by 10 square inches, while the perimeter of the circle is 4 less than that of the square. Find the side of the square and the radius of the circle.

Use $3\frac{1}{7}$ as the value of π.

20. The upper base of a trapezoid is equal to the altitude, and the area is 48. If the altitude is decreased by 4, and the upper base by 2, the area is then 14. Find the dimensions of the trapezoid.

21. An automobile running northward at the rate of 15 miles per hour is 20 miles south of the intersection with an east and west road. At the same time another automobile running westward on the cross-road at the rate of 20 miles per hour is 15 miles east of the crossing. How far apart (diagonally) will they be 15 minutes later? One hour later?



22. Under the conditions of problem 21 how long after the first time specified will the automobiles be 10 miles apart? Is there more than one such position?

23. What are the rates of motion of the automobiles in problem 21 if one hour later they are 5 miles apart and 3 hours later they are 35 miles apart? (See the figure.)

If the rates of the automobiles are r_1 and r_2, then after 1 hour we have and after 3 hours we have

$$\begin{cases} (20-r_1)^2+(15-r_2)^2=5^2, & (1)\\ (20-3r_1)^2+(15-3r_2)^2=35^2. & (2)\end{cases}$$

Simplifying (1) and (2), $r_1^2+r_2^2-40r_1-30r_2+600=0$, (3)

and $9r_1^2+9r_2^2-120r_1-90r_2-600=0$. (4)

Multiplying (3) by 9, subtracting from (4), and simplifying,

$$4r_1+3r_2=100. \qquad (5)$$

The solution of this problem may now be completed by solving (5) and (1) simultaneously.

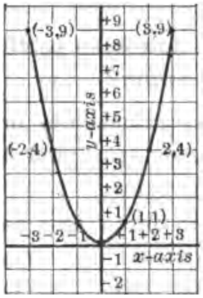
NOTE. — In equation (2), $20-3r_1$ and $15-3r_2$ are both negative numbers. This means that in this problem the distances north and west of the crossing are *negative*, while those south and east are *positive*.

CHAPTER XVI

GRAPHIC REPRESENTATION OF QUADRATICS

249. Graphs of Quadratics. We saw, § 106, that a single equation in two variables is satisfied by indefinitely many pairs of numbers. If such an equation is of the **first degree** in the two variables, the graph is in every case a *straight line.*

We are now to consider graphs of equations of the **second degree** in two variables.



Example 1. Graph the equation $y = x^2$.

By giving various values to x and computing the corresponding value of y, we find pairs of numbers as follows which satisfy this equation:

$$\begin{cases} x = 0, \\ y = 0. \end{cases} \quad \begin{cases} x = 1, \\ y = 1. \end{cases} \quad \begin{cases} x = -1, \\ y = 1. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 4. \end{cases} \quad \begin{cases} x = -2, \\ y = 4. \end{cases} \quad \begin{cases} x = 3, \\ y = 9. \end{cases}$$

$$\begin{cases} x = -3, \\ y = 9. \end{cases} \text{etc.}$$

These pairs of numbers correspond to points which lie on a curve as shown in the figure.

By referring to the graph the curve is seen to be symmetrical with respect to the y-axis. This can be seen directly from the equation itself, since x is involved only as a square, and hence, if $y = x^2$ is satisfied by $x = a$, $y = b$, it must also be satisfied by $x = -a$, $y = b$.

It may easily be verified that no three points of this curve lie on a straight line. The curve is called a **parabola.**

Example 2. Graph the equation $y = x^2 + 2x - 3$.

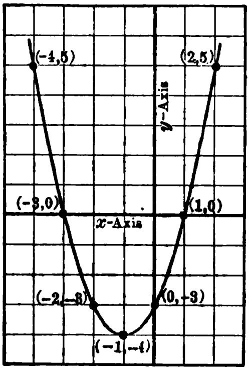
Each of the following pairs of numbers satisfies the equation:

$$\begin{cases} x = 0, \\ y = -3. \end{cases} \quad \begin{cases} x = 1, \\ y = 0. \end{cases} \quad \begin{cases} x = -1, \\ y = -4. \end{cases}$$

$$\begin{cases} x = 2, \\ y = 5. \end{cases} \quad \begin{cases} x = -2, \\ y = -3. \end{cases} \quad \begin{cases} x = -3, \\ y = 0. \end{cases}$$

$$\begin{cases} x = -4, \\ y = 5. \end{cases}$$

Plotting these points and drawing a smooth curve through them, we have the graph of the equation, as shown in the figure.



WRITTEN EXERCISES

In this manner graph each of the following:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 6$.
6. $y = 3x^2 - 7x + 2$.

250. Intersection Points on the x-axis. We now seek to find the points at which each of the above curves cuts the x-axis. The value of y for all points on the x-axis is zero. Hence we put $y = 0$, and solve the resulting equation.

Thus in Example 2 above, if $y = 0$, $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$, which is satisfied by $x = 1$ and $x = -3$. Hence this curve cuts the x-axis in the two points $x = 1$, $y = 0$ and $x = -3$, $y = 0$, as shown in the figure.

Similarly in Example 1 on page 206, if $y = 0$, $x^2 = 0$, and hence $x = 0$. Hence the curve meets the x-axis in the point $x = 0$, $y = 0$, as shown in the figure.

Again, in $y = 2x^2 - x - 6$, if $y = 0$ then $2x^2 - x - 6 = (2x + 3)(x - 2) = 0$, which is satisfied by $x = -\frac{3}{2}$ and $x = 2$. Hence the curve cuts the x-axis in the points $x = -\frac{3}{2}$ and $x = 2$.

WRITTEN EXERCISES

Find the points in which each of the six curves in the preceding list cuts the x-axis.

Notice that in every case the expression to the right of the equality sign can be factored, so that when $y = 0$ the resulting equation in x may be solved by the factoring method.

Example 3. Plot the curve $y = x^2 + 4x + 2$ and find its intersection points with the x-axis.

We are not able to factor $x^2 + 4x + 2$ by inspection. Hence, we solve the equation $x^2 + 4x + 2 = 0$ by completing the square, or by the formula, obtaining $x = -2 + \sqrt{2}$ and $x = -2 - \sqrt{2}$. Hence, the curve cuts the x-axis in points whose abscissas are given by these irrational values of x.

NOTE. — In making this graph, we first plot points corresponding to *integral* values of x, as before; then, in drawing the smooth curve through these, the intersections made with the x-axis are approximately the points on the number scale corresponding to the irrational numbers, $-2 + \sqrt{2}$ and $-2 - \sqrt{2}$.

WRITTEN EXERCISES

In the above manner, find the points at which each of the following curves cuts the x-axis, and plot the curves:

1. $y = x^2 + 5x + 3$.
2. $y = 3x^2 + 8x - 2$.
3. $y = 6x - 4x^2 + 5$.
4. $y = -4 - 2x + 5x^2$.
5. $y = 2x - 5x^2 + 8$.
6. $y = 5 + 8x - 3x^2$.

251. Intersection Points on Lines Parallel to the x-axis. The points in which a line parallel to the x-axis cuts a curve may be found in a manner shown by the following examples.

Example 1. Graph on the same axes the straight line, $y = -2$ and the curve, $y = x^2 + 2x - 3$.

The line $y = -2$ is parallel to the x-axis and two units *below* it. If we substitute -2 for y in the equation

$$y = x^2 + 2x - 3,$$

and then solve for x, we shall find the abscissas of the two points in which the line $y = -2$ cuts this curve.

Solving, $-2 = x^2 + 2x - 3$ for x,

we find $x_1 = -1 + \sqrt{2}$ and $x_2 = -1 - \sqrt{2}$.

Hence, the line $y = -2$ cuts the curve $y = x^2 + 2x - 3$ in the points:

$$\begin{cases} x_1 = -1 + \sqrt{2}, \\ y_1 = -2, \end{cases} \text{ and } \begin{cases} x_2 = -1 - \sqrt{2}, \\ y_2 = -2. \end{cases}$$

Example 2. Graph on the same axes $y = -4$ and $y = x^2 + 2x - 3$.

Substituting and solving as before, we find

$$x_1 = \frac{-2 + \sqrt{4-4}}{2} = \frac{-2+0}{2} = -1;$$

and

$$x_2 = \frac{-2 - \sqrt{4-4}}{2} = \frac{-2-0}{2} = -1.$$

In this case the two values of x are *equal*, and hence there is only *one* point common to the line and the curve, namely, the point $x = -1$, $y = -4$. This may be understood by thinking of the line $y = -2$, in the preceding example, as moving down to the position $y = -4$, whereupon the two values of x which were *distinct* come to *coincide*.

252. Tangent to a Curve. A line which meets a curve in two *coincident points* is said to be *tangent to the curve.*

253. Problem. To find a Line Tangent to a Curve. What is the value of a in $y = a$, if this line is tangent to the curve $y = x^2 + 5x + 8$?

Substituting a for y and solving by means of the formula, we have

$$x = \frac{-5 \pm \sqrt{25 - 4(8-a)}}{2} = \frac{-5 \pm \sqrt{4a-7}}{2}.$$

If the line is to be tangent to the curve, then the expression under the radical sign must be zero. That is, $4a - 7$, or $a = \frac{7}{4}$. Verify by plotting the graphs.

WRITTEN EXERCISES

In each of the following find the value of a in order that $y = a$ may be tangent to the curve:

1. $y = x^2 - 1$.
2. $y = x^2 + 4x$.
3. $y = x^2 + 3x - 4$.
4. $y = x^2 + 5x + 4$.
5. $y = x^2 - 7x + 16$.
6. $y = 3x^2 - 7x + 2$.

254. Problem. To find a Line which does not cut a Given Curve. Find the intersection points of the curve $y = x^2 + 3x + 5$ and the line $y = 2\frac{1}{2}$.

Substituting for y and solving for x, we have

$$x_1 = \frac{-6 + \sqrt{36 - 40}}{4} = \frac{-6 + 2\sqrt{-1}}{4} = \frac{-3 + \sqrt{-1}}{2};$$

$$x_2 = \frac{-6 - \sqrt{36 - 40}}{4} = \frac{-6 - 2\sqrt{-1}}{4} = \frac{-3 - \sqrt{-1}}{2}.$$

The imaginary solutions indicate that the curve and the line have *no point in common,* as is evident on constructing the graphs.

If now we solve the equations $y = a$ and $y = x^2 + 3x + 5$ for x by first substituting a for y, we have

$$x = \frac{-3 \pm \sqrt{4a - 11}}{2}.$$

We now see that

(1) If $a > \frac{11}{4}$, the number under the radical sign is *positive,* and there are *two real and distinct* values of x. Hence the line $y = a$ and the curve meet in two points.

(2) If $a < \frac{11}{4}$, the number under the radical sign is *negative.* Consequently the values of x are *imaginary* and the line $y = a$ and the curve do not meet.

(3) If $a = \frac{11}{6}$, the number under the radical sign is zero, and the two values of x are *equal.* Hence the line is *tangent* to the curve.

255. Nature of Roots Determined by Formula. From the two preceding paragraphs it appears that it is possible to determine the *relative* positions of the line and the curve *without completely solving* the equations. Namely, eliminate y and reduce the resulting equation to the form $ax^2 + bx + c = 0$. Hence we have:

(1) If $b^2 - 4ac > 0$, *i.e. positive,* then the line cuts the curve in two distinct points.

(2) If $b^2 - 4ac = 0$, then the line is tangent to the curve.

(3) If $b^2 - 4ac < 0$, *i.e. negative,* then the line does not cut the curve. See § 240.

Example 1. Does the line $y = 2x - 6$ meet the curve $y = x^2 + 7x - 4$?

Solution. Substituting from the first equation in the second,

$$2x - 6 = x^2 + 7x - 4, \text{ or } x^2 + 5x + 2 = 0.$$

Then $b^2 - 4ac = 25 - 8 = 17$. Hence the line meets the curve in two distinct points.

Example 2. Does the line $x + y = 3$ meet the curve $y = x^2 - 3x + 4$?

Solution. Substituting from the first equation in the second,

$$3 - x = x^2 - 3x + 4 \text{ or } x^2 - 2x + 1 = 0.$$

Then $b^2 - 4ac = 4 - 4 = 0$. Hence the two roots are equal and the line meets the curve in only one point and is tangent to it.

Example 3. Does the line $2x - 3y = 10$ meet the curve $y = x^2 + 4x + 1$?

Solution. Eliminating y,

$$2x - 10 = 3x^2 + 12x + 3 \text{ or } 3x^2 + 10x + 13 = 0.$$

Then $b^2 - 4ac = 100 - 156 = -56$ and the line does not meet the curve.

WRITTEN EXERCISES

In each of the following determine without graphing whether or not the line meets the curve, and in case it does, find the intersection points:

1. $\begin{cases} y = 2x^2 - 3x - 4, \\ y - x = 3. \end{cases}$

2. $\begin{cases} y = 2x^2 + 2x - 1, \\ 2y = x - 1. \end{cases}$

3. $\begin{cases} y = 3x^2 - 2x - 1, \\ 2x - y = 4. \end{cases}$

4. $\begin{cases} y = 4x^2 + 6x + 1, \\ x = y + 5. \end{cases}$

5. $\begin{cases} y = x^2 - 7x + 12, \\ 5x - y = -1. \end{cases}$

6. $\begin{cases} y = 5x^2 + 8x + 3, \\ 2y - 5x - 2 = 0. \end{cases}$

7. $\begin{cases} y = 5x^2 - 8x + 3, \\ 3 - x = 3y. \end{cases}$

8. $\begin{cases} y = -5x^2 + 8x - 3, \\ 2 - 4y - x = 0. \end{cases}$

9. $\begin{cases} y = -5x^2 - 8x - 3, \\ 5y - 3x = 8. \end{cases}$

10. $\begin{cases} y = 3x - 3x^2 + 7, \\ -5 - 3x + 2y = 0. \end{cases}$

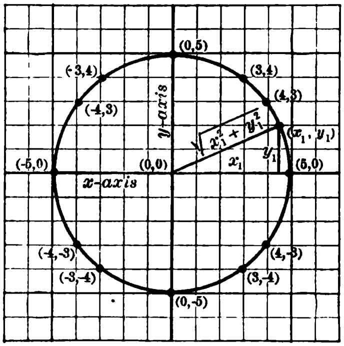
256. The Equation of the Circle. Graph the equation

$$x^2 + y^2 = 25.$$

Writing the equation in the form $y = \pm\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$$\begin{cases} x = 0, \\ y = \pm 5. \end{cases} \quad \begin{cases} x = \pm 5, \\ y = 0. \end{cases} \quad \begin{cases} x = 3, \\ y = \pm 4. \end{cases}$$

$$\begin{cases} x = -3, \\ y = \pm 4. \end{cases} \quad \begin{cases} x = 4, \\ y = \pm 3. \end{cases} \quad \begin{cases} x = -4, \\ y = \pm 3. \end{cases}$$



These points lie on the circumference of a circle whose radius is 5. For, if we consider any point (x_1, y_1) on this circumference, then we have $x_1^2 + y_1^2 = 25$, since the sum of the squares on the sides of a right triangle is equal to the square on the hypotenuse.

$\therefore$ $x^2 + y^2 = 25$ is, therefore, the equation of a circle with radius 5.

257. Problem. To find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $x + y = 7$.

Solving these equations simultaneously we have, $\begin{cases} x_1 = 4, \\ y_1 = 3, \end{cases}$ and $\begin{cases} x_2 = 3, \\ y_2 = 4, \end{cases}$ which represent the required points.

Verify this by graphing the two equations on the same axes.

HISTORICAL NOTE

The **graph** was brought into mathematics by **Descartes**. On the one hand the graph has helped greatly in acquiring a full understanding of the meaning of equations and their solutions; and on the other hand, it has made possible the study of geometric curves by means of the methods of algebra and higher branches of mathematics such as the calculus.

Newton proved that a body moving around the sun under the law of gravitation must always travel in a curve which may be represented by an equation of the second degree. That is, the path is a *circle*, an *ellipse*, a *parabola*, or *hyperbola*. All the planets move in elliptical orbits which are very nearly circular. The orbits of many comets are hyperbolas, some of them being very nearly if not quite parabolic. Mathematical astronomy is based entirely on the algebraic representation of these curves.

258. Problem. To find the points of intersection of the circle $x^2 + y^2 = 25$ and the line $3x + 4y = 25$.

Eliminating y and solving for x, we find $x = \frac{6 \pm 0}{2} = 3$.

Hence, $\begin{cases} x_1 = 3, \\ y_1 = 4, \end{cases}$ and $\begin{cases} x_2 = 3, \\ y_2 = 4. \end{cases}$

Since the two values of x coincide, and likewise the two values of y, the circumference and the line have but *one point* in common. Verify by graphing the line and the circle on the same axes.

259. Problem. To find the points of intersection of the

Circle, $x^2 + y^2 = 25$

and the straight line, $x + y = 10$.

Substituting for y and solving for x, we have

$$x = \frac{20 \pm \sqrt{400 - 600}}{4} = \frac{20 \pm \sqrt{-200}}{4}$$

$$= \frac{20 \pm 10\sqrt{-2}}{4} = \frac{10 \pm 5\sqrt{-2}}{2}.$$

The imaginary values of x indicate that there is no intersection point. Verify by plotting.

EXERCISES

In each of the following determine by solving whether the line and the circumference meet, and in case they do find the points of intersection:

1. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 4. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 36, \\ 4x + y = 6. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + y = -5. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 20, \\ 2x + y = 0. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 7, \\ x + y = 8. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 8, \\ x - y = 4. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 16, \\ 3x - 4y = 12. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 16, \\ 2x + y = 12. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 9, \\ x + y = 8. \end{cases}$

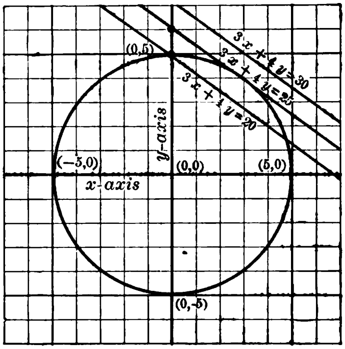
10. $\begin{cases} x^2 + y^2 = 4, \\ x - y = 4. \end{cases}$

11. $\begin{cases} x^2 + y^2 = 6, \\ x - 2y = 6. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 8, \\ 2x - y = 8. \end{cases}$

260. Problem. Graph on the same axes the circle $x^2 + y^2 = 5^2$, and the lines $3x + 4y = 20$, $3x + 4y = 25$, and $3x + 4y = 30$.

The first line *cuts* the circumference in two distinct points, the second seems to be *tangent* to it, and the third does *not meet* it. Observe that the three lines are parallel.



In order to make a general discussion of the relative positions of such straight lines and the circumference of a circle, we solve the following equations simultaneously:

$$x^2 + y^2 = r^2, \qquad (1)$$
$$3x + 4y = c. \qquad (2)$$

Eliminating y by substitution, and solving for x, we find

$$x = \frac{3c \pm 4\sqrt{25r^2 - c^2}}{25}. \qquad (3)$$

The two values of x from (3) are the abscissas of the points of intersection of the circumference (1) and the line (2).

(1) These values of x are *real and distinct*, if $25r^2 - c^2$ is *positive*. That is, if $r = 5$, and $c = 20$, then $25r^2 - c^2 = 25 \cdot 25 - 400 = 225$ and the line cuts the circle in two distinct points.

(2) These values of x are real and coincident if $25r^2 - c^2 = 0$. Thus, if $r = 5$, $c = 25$, then $25r^2 - c^2 = 625 - 625 = 0$ and the line meets the circle in only one point. That is, the line is tangent to the circle.

(3) These values of x are imaginary if $25r^2 - c^2$ is negative. Thus, if $r = 5$, $c = 30$, then $25r^2 - c^2 = 625 - 900 = -275$, and the line fails to meet the circle.

Notice that the three lines are parallel to each other and that their equations differ only in their right members. Hence, if in the equation $3x + 4y = c$, the value of c increases from 20 to 25, the line will move outward while remaining parallel to its original position. The points of intersection will gradually approach each other until they coincide. At the same time the expression $25r^2 - c^2$ will gradually approach zero.

261. Parameters. Letters such as c and r in the above solution to which any arbitrary constant values may be assigned are called **parameters**, while x and y are the **unknowns** of the **equations.**

EXERCISES

Solve each of the following pairs of equations:

Give such values to the parameters involved as will cause the line (*a*) to cut the curve in two distinct points, (*b*) to be tangent to the curve, (*c*) to fail to meet the curve.

1. $\begin{cases} x^2 + y^2 = 4, \\ ax + 3y = 16. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 16, \\ 2x + by = 12. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 2x + 3y = c. \end{cases}$

4. $\begin{cases} y^2 = 8x, \\ 3x + 4y = c. \end{cases}$

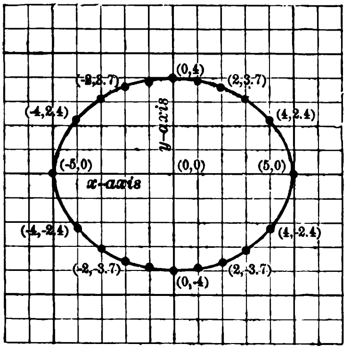
262. Problem. The Equation of the Ellipse. Graph the equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

Writing the equation in the form $y = \pm \frac{4}{5}\sqrt{25 - x^2}$, and assigning values to x, we compute the corresponding values of y as follows:

$\begin{cases} x = 0, \\ y = \pm 4, \end{cases}$ $\begin{cases} x = \pm 5, \\ y = 0, \end{cases}$ $\begin{cases} x = 1, \\ y = \pm 3.9, \end{cases}$

$\begin{cases} x = -1, \\ y = \pm 3.9, \end{cases}$ $\begin{cases} x = 2, \\ y = \pm 3.7, \end{cases}$

$\begin{cases} x = 3, \\ y = \pm 3.2, \end{cases}$ $\begin{cases} x = 4, \\ y = \pm 2.4. \end{cases}$



Evidently if x is greater than 5 in absolute value, the corresponding values of y are imaginary.

Plotting these points, they are found to lie on the curve shown in the figure.

This curve is called an **ellipse**.

EXERCISES

Solve the following pairs of equations:

From the solutions determine whether the straight line and the curve intersect, and in case they do, find the coördinates of the intersection points. Verify each by constructing the graphs.

1. $$\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{9}=1, \\ 3x+4y=12. \end{cases}$$

2. $$\begin{cases} \dfrac{x^2}{49}+\dfrac{y^2}{16}=1, \\ 2x-7y=8. \end{cases}$$

3. $$\begin{cases} x^2+4y^2=25, \\ 2x-y=4. \end{cases}$$

4. $$\begin{cases} 3x^2+2y^2=11, \\ x-3y=7. \end{cases}$$

5. $$\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{9}=1, \\ 2x-y=14. \end{cases}$$

6. $$\begin{cases} y=2x^2-3x+4, \\ y-4x-8=0. \end{cases}$$

When arbitrary constants are introduced in the equations of a straight line and an ellipse, we may determine values for these constants so as to make the line cut the ellipse, touch it, or not cut it, as in the case of the circle, §§ 260, 261.

EXERCISES

Solve each of the following pairs simultaneously:

In each example give such values to the parameter that the line shall (*a*) cut the curve in two distinct points, (*b*) be a tangent to the curve, (*c*) have no point in common with the curve.

1. $$\begin{cases} \dfrac{x^2}{a^2}+\dfrac{y^2}{16}=1, \\ 4x+y=2. \end{cases}$$

2. $$\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{b^2}=1, \\ x+5y=5. \end{cases}$$

3. $$\begin{cases} \dfrac{x^2}{25}+\dfrac{y^2}{16}=1, \\ 4x-5y=c. \end{cases}$$

4. $$\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ 5x-by=20. \end{cases}$$

5. $$\begin{cases} \dfrac{x^2}{16}+\dfrac{y^2}{25}=1, \\ ax+4y=20. \end{cases}$$

6. $$\begin{cases} \dfrac{x^2}{36}+\dfrac{y^2}{16}=1, \\ ax+6y-24=0. \end{cases}$$

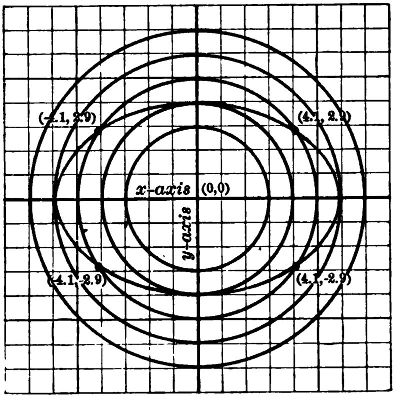
ORAL EXERCISES

In the figure on this page the ellipse is the graph of the equation $\frac{x^2}{36}+\frac{y^2}{16}=1$.

1. What are the radii of the circles? What are their equations?

2. In how many points does each circle meet the ellipse?

3. Is it possible to draw a circle which will meet the ellipse in more than four points?



Example. Solve simultaneously, obtaining results to one decimal place:

$$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, & (1) \\ x^2+y^2=25. & (2) \end{cases}$$

Clear (1) of fractions and proceed as in § 246. Verify the solution by reference to the graph given in the figure.

EXERCISES

Solve simultaneously each of the following pairs of equations and interpret all the solutions in each case from the graph in the above figure:

1. $$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=36. \end{cases}$$

2. $$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=16. \end{cases}$$

3. $$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=49. \end{cases}$$

4. $$\begin{cases} \frac{x^2}{36}+\frac{y^2}{16}=1, \\ x^2+y^2=31. \end{cases}$$

263. Problem. Intersections of a Circle and a Parabola.

Solve simultaneously $\begin{cases} x^2+y^2=25, & (1)\\ x^2=4y+a. & (2)\end{cases}$

Substituting x^2 from (2) in (1),

$$y^2+4y+a-25=0. \qquad (3)$$

Hence, $$y=\frac{-4\pm\sqrt{16-4(a-25)}}{2}=-2\pm\sqrt{29-a}. \qquad (4)$$

Substituting in (2), we have

$$x^2=-8\pm4\sqrt{29-a}+a$$

and $$x=\pm2\sqrt{-2\pm\sqrt{29-a}+a}.$$

Arranging the values in pairs, we have

(1) $\begin{cases} x=\sqrt{-8+4\sqrt{29-a}+a},\\ y=-2+\sqrt{29-a}.\end{cases}$

(2) $\begin{cases} x=\sqrt{-8-4\sqrt{29-a}+a},\\ y=-2-\sqrt{29-a}.\end{cases}$

(3) $\begin{cases} x=-\sqrt{-8+4\sqrt{29-a}+a},\\ y=-2+\sqrt{29-a}.\end{cases}$

(4) $\begin{cases} x=-\sqrt{-8-4\sqrt{29-a}+a},\\ y=-2-\sqrt{29-a}.\end{cases}$

If $a>29$, all four solutions are imaginary.

If $a=29$, (1) and (2) are coincident, and (3) and (4) are also coincident.

If $a=20$, the solutions are:

(1) $\begin{cases} x=\sqrt{24},\\ y=1.\end{cases}$ (2) $\begin{cases} x=0,\\ y=-5.\end{cases}$

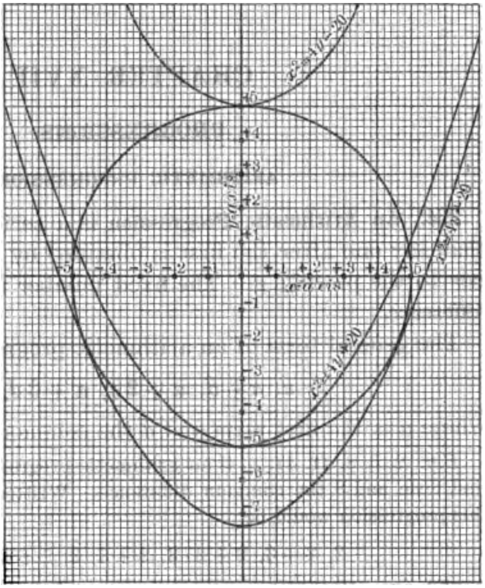
(3) $\begin{cases} x=-\sqrt{24},\\ y=1.\end{cases}$ (4) $\begin{cases} x=0,\\ y=-5.\end{cases}$

That is, (2) and (4) coincide, while (1) and (3) are real and distinct.

If $a=-20$, the solutions are:

(1) $\begin{cases} x=0,\\ y=5.\end{cases}$ (2) $\begin{cases} x=\sqrt{-56},\\ y=-9.\end{cases}$

(3) $\begin{cases} x=0,\\ y=5.\end{cases}$ (4) $\begin{cases} x=-\sqrt{-56},\\ y=-9.\end{cases}$



That is, (1) and (3) are coincident, while (2) and (4) are *imaginary* and *distinct*.

For all values of a between 29 and 20, the solutions are all *real* and *distinct*.

For all values of a between 20 and -20, two solutions are *imaginary* and two are *real* and *distinct*.

For all values of a less than -20, all solutions are *imaginary*.

That is, from the equation $x^2 = 4y + a$, the lowest parabola corresponds to $a = 29$. As a decreases from 29 to 20 the parabola moves up from the lowest position to the second position. As a decreases from 20 to -20, the parabola moves up from the middle position to the upper position, as shown in the figure.

A study of the above graph verifies the correctness of these statements.

WRITTEN EXERCISES

In each of the following, find the values of the parameters for which, (a) the roots are coincident, (b) the roots are real and distinct, (c) the roots are imaginary. (See § 254.)

1. $x^2 + 12x + a = 0$.
2. $x^2 + 4ax + 8 = 0$.
3. $x^2 + ax + a = 0$.
4. $x^2 + 3ax + 2a = 0$.
5. $x^2 + 4x + 2a^2 = 0$.
6. $x^2 + 3ax + a = 0$.
7. $x^2 + 10x - 4a^2 = 0$.
8. $x^2 - 12ax + 20a = 0$.

CHAPTER XVII

PROGRESSIONS

ARITHMETIC PROGRESSIONS

264. An Arithmetic Progression is a series of numbers, such that any one after the first is obtained by adding a fixed number to the preceding. The fixed number is called the **common difference.**

The general form of an arithmetic progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,$$

where a is the first term and d the common difference.

E.g. 2, 5, 8, 11, 14, ⋯ is an arithmetic progression in which 2 is the first term and 3 the common difference. Written in the general form, this progression would be

$$2,\ 2+3,\ 2+2\cdot 3,\ 2+3\cdot 3,\ 2+4\cdot 3,\ \cdots.$$

265. The Last Term. From the general form of the arithmetic progression we see that the third term equals the first plus twice the difference, the fourth term equals the first plus three times the difference, and in general if there are n terms in the progression, then the last term is $a+(n-1)d$. Indicating the last term by l, we have

$$l = a+(n-1)d. \qquad \text{I}$$

An arithmetic progression of n terms would then be written in general form thus,

$$a,\ a+d,\ a+2d,\ \cdots,\ a+(n-2)d,\ a+(n-1)d.$$

ORAL EXERCISES

1. If $a = 2$ and $d = 2$, find the 4th term.
2. If $a = 3$ and $d = 4$, find the 5th term.
3. If $a = 3$ and $d = 6$, find the 6th term.

4. If $a = 1$ and $d = \frac{1}{2}$, find the 10th term.
5. If $a = 2$ and $d = 1$, find the 15th term.
6. If $a = -1$ and $d = 3$, find the 12th term.

WRITTEN EXERCISES

1. Solve I for each letter in terms of all the others.

In each of the following find the value of the letter not given, and write out the progression in each case:

2. $\begin{cases} a = 2, \\ d = 2, \\ n = 7. \end{cases}$

3. $\begin{cases} a = 3, \\ d = 5, \\ l = 43. \end{cases}$

4. $\begin{cases} a = 1, \\ n = 15, \\ l = 15. \end{cases}$

5. $\begin{cases} a = 7, \\ n = 31, \\ l = 91. \end{cases}$

6. $\begin{cases} a = 4, \\ d = -3, \\ n = 18. \end{cases}$

7. $\begin{cases} a = -5, \\ d = 4, \\ n = 7. \end{cases}$

8. $\begin{cases} a = 3, \\ d = -5, \\ l = -32. \end{cases}$

9. $\begin{cases} d = 7, \\ n = 8, \\ l = 24. \end{cases}$

10. $\begin{cases} d = -5, \\ n = 13, \\ l = -63. \end{cases}$

11. $\begin{cases} a = -3, \\ n = 9, \\ l = -27. \end{cases}$

12. $\begin{cases} a = 11, \\ l = -39, \\ d = -5. \end{cases}$

13. $\begin{cases} a = x, \\ l = y, \\ n = z. \end{cases}$

266. The Sum of an Arithmetic Progression of n terms may be obtained as follows:

Let s_n denote the sum of n terms. Call the last term l, then the next preceding is $l - d$, the next $l - 2d$, etc. Hence we may write

$$s_n = a + [a + d] + [a + 2d] + \cdots [l - 2d] + [l - d] + l. \qquad (1)$$

This may also be written, reversing the order of the terms, thus,

$$s_n = l + [l - d] + [l - 2d] + \cdots + [a + 2d] + [a + d] + a. \qquad (2)$$

Adding (1) and (2), we have

$(a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$, since at each step the d's cancel.

Since there are n of these expressions, each equal to $a + l$, we have

$$2s_n = n(a + l).$$

Hence,

$$s_n = \frac{n}{2}(a + l). \qquad \text{II}$$

This formula for the sum of n terms involves a, l, and n, that is, the first term, the last term, and the number of terms.

267. Fundamental Formulas. In the two equations,

$$l = a + (n - 1)d, \qquad \text{I}$$

$$s = \frac{n}{2}(a + l), \qquad \text{II}$$

there are five letters, namely, a, d, l, n, s. If any three of these are given, the equations I and II may be solved simultaneously to find the other two.

Solution of Problems.

Example 1. Given $n = 11$, $l = 23$, $s = 143$. Find a and d.

Substituting the given values in I and II,

$$\begin{cases} 23 = a + (11 - 1)d. & (1) \\ 143 = \frac{11}{2}(a + 23). & (2) \end{cases}$$

From (2), $a = 3$, which in (1) gives $d = 2$.

Example 2. Given $d = 4$, $n = 5$, $s = 75$. Find a and l.

From I and II, $$\begin{cases} l = a + (5 - 1)4, & (1) \\ 75 = \frac{5}{2}(a + l). & (2) \end{cases}$$

Solving (1) and (2) simultaneously, we have $a = 7$, $l = 23$.

Example 3. Given $d = 4$, $l = 35$, $s = 161$. Find a and n.

From I and II, $$\begin{cases} 35 = a + (n - 1)4, & (1) \\ 161 = \frac{n}{2}(a + 35). & (2) \end{cases}$$

Solving (1) and (2) simultaneously, we get

$$\begin{cases} n = \frac{23}{2}, \\ a = -7. \end{cases} \text{ or } \begin{cases} n = 7, \\ a = 11. \end{cases}$$

Since an arithmetic progression must have an *integral* number of terms, only the second values of n and a are applicable to this problem.

Example 4. Given $d = 2$, $l = 11$, $s = 35$. Find a and n.

Substituting in I and II, and solving for a and n, we have

$$a = 3,\ n = 5, \text{ and } a = -1,\ n = 7.$$

Hence, there are two progressions,

$$-1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11,$$

and

$$3,\ 5,\ 7,\ 9,\ 11,$$

each of which satisfies the given conditions.

WRITTEN EXERCISES

In each of the following obtain the values of the two letters not given:

Fractional, negative, or imaginary values of n do not satisfy the conditions of the problems. In each exercise interpret all the values found.

1. $\begin{cases} s = 96, \\ l = 19, \\ d = 2. \end{cases}$

2. $\begin{cases} s = 34, \\ l = 14, \\ d = 3. \end{cases}$

3. $\begin{cases} a = 7, \\ l = 27, \\ s = 187. \end{cases}$

4. $\begin{cases} s = 88, \\ l = -7, \\ d = -3. \end{cases}$

5. $\begin{cases} n = 18, \\ a = 4, \\ l = 13. \end{cases}$

6. $\begin{cases} n = 14, \\ a = 7, \\ s = 14. \end{cases}$

7. $\begin{cases} d = -1, \\ n = 41, \\ l = -35. \end{cases}$

8. $\begin{cases} l = 30, \\ s = 162, \\ n = 9. \end{cases}$

9. $\begin{cases} a = 30, \\ n = 10, \\ s = 120. \end{cases}$

10. $\begin{cases} d = 6, \\ l = 49, \\ s = 232. \end{cases}$

11. $\begin{cases} s = 7, \\ d = 1\frac{1}{2}, \\ l = 7. \end{cases}$

12. $\begin{cases} s = 14, \\ d = 3, \\ l = 4. \end{cases}$

In each of the following call the two letters specified the *unknowns* and solve for their values in terms of the remaining three letters in equations I and II.

13. a, d.
14. a, l.
15. a, n.
16. a, s.
17. d, l.
18. d, n.
19. d, s.
20. l, n.
21. l, s.
22. n, s.

268. Arithmetic Means. The terms between the first and the last terms of an arithmetic progression are called **arithmetic means.**

Thus, in 2, 5, 8, 11, 14, 17, the four arithmetic means between 2 and 17 are 5, 8, 11, 14.

If the first and last terms and the number of arithmetic means between them are given, then these means can be found.

For we have given a, l, and n, and hence d can be found and the whole series constructed.

Example. Insert 7 arithmetic means between 3 and 19.

In this case $a = 3$, $l = 19$, and $n = 9$.

Hence, from $l = a + (n - 1)d$ we find $d = 2$ and the required means are 5, 7, 9, 11, 13, 15, 17.

269. The case of *one* arithmetic mean is important. Let A be the arithmetic mean between a and l. Since a, A, l are in arithmetic progression, we have $A = a + d$, and $l = A + d$. Hence, eliminating d,

$$l - A = A - a$$

or

$$A = \frac{a + l}{2}. \qquad \text{III}$$

EXERCISES AND PROBLEMS

1. Insert 5 arithmetic means between 5 and -7.

2. Insert 3 arithmetic means between -2 and 12.

3. Insert 8 arithmetic means between -3 and -5.

4. Insert 5 arithmetic means between -11 and 40.

5. Insert 15 arithmetic means between 1 and 2.

6. Insert 9 arithmetic means between $2\frac{3}{4}$ and $-1\frac{1}{2}$.

7. Find the arithmetic mean between 3 and 17.

8. Find the arithmetic mean between -4 and 16.

9. Find the tenth and eighteenth terms of the series 4, 7, 10, ….

10. Find the fifteenth and twentieth terms of the series $-8, -4, 0, \cdots$.

11. The fifth term of an arithmetic progression is 13 and the thirtieth term is 49. Find the common difference.

12. Find the sum of all the integers from 1 to 100.

13. Find the sum of all the odd integers between 0 and 200.

14. Find the sum of all integers divisible by 6 between 1 and 500.

15. There are three numbers in arithmetic progression whose sum is 15. The product of the first and last is $3\frac{1}{5}$ times the second. Find the numbers.

16. In a potato race 40 potatoes are placed in a straight line one yard apart, the first potato being two yards from the basket. How far must a contestant travel in bringing them to the basket one at a time?

17. Show that $1 + 3 + 5 + \cdots + n = k^2$, where k is the number of terms.

18. There are four numbers in arithmetic progression whose sum is 20 and the sum of whose squares is 120. Find the numbers.

19. If a body falls from rest 16.08 feet the first second, 48.24 feet the second second, 80.40 the third, etc., how far will it fall in 10 seconds? 15 seconds? t seconds?

20. According to the law indicated in problem 19 in how many seconds will a body fall 1000 feet? s feet?

21. In what time will a body fall 1000 feet if thrown downward with a velocity of 20 feet per second?

If a body is thrown downward with a velocity of v_0 feet per second, then the distance, s, which it falls in t seconds is v_0t feet plus the distance it would fall if starting from rest.

That is, $s = v_0t + \frac{1}{2}gt^2$, where $g = 32.16$.

22. With what velocity must a body be thrown downward in order that it shall fall 360 feet in 3 seconds?

23. A stone is dropped into a well, and the sound of its striking the bottom is heard in 3 seconds. How deep is the well if sound travels 1080 feet per second?

24. A rifle bullet is shot directly upward with a velocity of 2000 feet per second. How high will it rise and how long before it will reach the ground?

A body thrown upward with a certain velocity will rise as far as it would have to fall to acquire this velocity. The velocity (neglecting the resistance of the atmosphere) of a body starting from rest is $v = gt$, where $g = 32.16$ and t is the number of seconds.

25. From a balloon 5800 feet above the earth, a body is thrown downward with a velocity of 40 feet per second. In how many seconds will it reach the ground?

26. If in problem 25 the body is thrown upward at the rate of 40 feet per second, how long before it will reach the ground?

GEOMETRIC PROGRESSIONS

270. A Geometric Progression is a series of numbers in which any term after the first is obtained by multiplying the preceding term by a fixed number, called the **common ratio.**

The general form of a geometric progression is

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},$$

in which a is the first term, r the constant multiplier, or common ratio, and n the number of terms.

E.g. 3, 6, 12, 24, 48, is a geometric progression in which 3 is the first term, 2 is the common ratio, and 5 is the number of terms.

271. The Last Term. If l is the last or nth term of the series, then

$$l = ar^{n-1}. \qquad \text{I}$$

If any three of the four letters in I are given, the remaining one may be found by solving this equation.

WRITTEN EXERCISES

In each of the following find the value of the letter not given:

1. $\begin{cases} l = 162, \\ r = 3, \\ n = 5. \end{cases}$

2. $\begin{cases} a = 1, \\ r = 2, \\ n = 8. \end{cases}$

3. $\begin{cases} a = -4, \\ r = -3, \\ n = 6. \end{cases}$

4. $\begin{cases} a = -1, \\ r = -2, \\ n = 9. \end{cases}$

5. $\begin{cases} l = 1024, \\ r = -2, \\ n = 11. \end{cases}$

6. $\begin{cases} l = 1024, \\ r = 2, \\ n = 11. \end{cases}$

7. $\begin{cases} a = -\frac{1}{2}, \\ r = \frac{3}{2}, \\ n = 6. \end{cases}$

8. $\begin{cases} l = 18, \\ r = \frac{1}{3}, \\ n = 6. \end{cases}$

9. $\begin{cases} l = -16, \\ r = -\frac{3}{4}, \\ n = 5. \end{cases}$

10. $\begin{cases} l = 32, \\ r = -2, \\ n = 6. \end{cases}$

11. $\begin{cases} a = -2, \\ r = -\frac{3}{2}, \\ n = 7. \end{cases}$

12. $\begin{cases} a = 3, \\ r = 2, \\ l = 1536. \end{cases}$

272. The **Sum of n Terms** of a geometric expression may be found as follows:

If s_n denotes the sum of n terms, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \tag{1}$$

Multiplying both members of (1) by r, we have

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{2}$$

Subtracting (1) from (2), and canceling terms, we have

$$rs_n - s_n = ar^n - a. \tag{3}$$

Solving (3) for s_n, we have

$$s_n = \frac{ar^n - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}. \qquad \text{II}$$

This formula for the sum of n terms of a geometric series involves only a, r, and n.

Since $ar^n = r \cdot ar^{n-1} = r \cdot l$, we may also write:

$$s_n = \frac{rl - a}{r - 1} = \frac{a - rl}{1 - r}. \qquad \text{III}$$

This formula involves only s, r, l, and a.

273. Fundamental Formulas. From equations I and II, or I and III, any two of the numbers a, l, r, s, and n can be found when the other three are given, as in the following examples.

Example 1. Given $n = 7$, $r = 2$, $s = 381$. Find a and l.

From I and III, $l = a \cdot 2^6 = 64\,a, \quad (1)$

$$381 = \frac{2\,l - a}{2 - 1} = 2\,l - a. \tag{2}$$

Substituting $l = 64\,a$ in (2), we obtain $a = 3$, and $l = 192$.

Example 2. Given $a = -3$, $l = -243$, $s = -183$. Find r and n.

From I and III, $-243 = (-3)r^{n-1}, \quad (1)$

$$-183 = \frac{-243\,r + 3}{r - 1}. \tag{2}$$

From (2), $r = -3. \quad (3)$

From (1), $81 = (-3)^{n-1}. \quad (4)$

Since $(-3)^4 = 81$, we have $n - 1 = 4$ or $n = 5$.

WRITTEN EXERCISES

1. Solve II for a in terms of the remaining letters.

2. Solve III for each letter in terms of the remaining letters.

In each of the following find the terms represented by the interrogation points:

$$3.\ \left\{\begin{array}{l} a=1, \\ r=3, \\ n=5, \\ s=? \end{array}\right. \quad 4.\ \left\{\begin{array}{l} s=635, \\ r=2, \\ n=7, \\ a=? \end{array}\right. \quad 5.\ \left\{\begin{array}{l} s=13, \\ r=\frac{2}{3}, \\ n=4, \\ a=? \end{array}\right. \quad 6.\ \left\{\begin{array}{l} l=-\frac{16}{81}, \\ s=? \\ n=5, \\ r=\frac{2}{3}. \end{array}\right.$$

$$7.\ \left\{\begin{array}{l} a=1, \\ s=\frac{25}{64}, \\ l=-\frac{27}{64}, \\ r=? \end{array}\right. \quad 8.\ \left\{\begin{array}{l} r=6, \\ n=5, \\ l=1296, \\ a=? \\ s=? \end{array}\right. \quad 9.\ \left\{\begin{array}{l} r=\frac{3}{2}, \\ n=8, \\ s=1050\frac{5}{6}, \\ l=? \\ a=? \end{array}\right. \quad 10.\ \left\{\begin{array}{l} a=\frac{9}{2}, \\ n=7, \\ l=\frac{32}{81}, \\ r=? \\ s=? \end{array}\right.$$

274. Geometric Means. The terms between the first term and the last term of a geometric progression are called **geometric means.**

Thus in 3, 6, 12, 24, 48, the terms 6, 12, 24 are geometric means between 3 and 48.

If the first term, the last term, and the number of geometric means are given, the ratio may be found from I, and then the means may be inserted.

Example. Insert 4 geometric means between 2 and 64.

We have given $a = 2$, $l = 64$, $n = 4 + 2 = 6$, to find r.

From I, $64 = 2 \cdot r^{6-1}$, or $r^5 = 32$, and $r = 2$.

Hence, the required geometric means are, 4, 8, 16, 32.

275. The case of *one* geometric mean is important. If G is the geometric mean between a and b, we have $\frac{G}{a} = \frac{b}{G}$.

Hence, $$G = \sqrt{ab}.$$

That is, G is a mean proportional between a and b.

EXERCISES AND PROBLEMS

1. Insert 5 geometric means between 2 and 128.

2. Insert 7 geometric means between 1 and $\frac{1}{256}$.

3. Find the geometric mean between 8 and 18.

4. Find the geometric mean between $\frac{1}{12}$ and $\frac{1}{4}$.

Given	Find	Given	Find
5. a, r, n	l, s	**9.** a, n, l	s, r
6. a, r, s	l	**10.** r, n, l	s, a
7. r, n, s	l, a	**11.** r, l, s	a
8. a, r, l	s	**12.** a, l, s	r

13. The product of three terms of a geometric progression is 1000. Find the second term.

14. Four numbers are in geometric progression. The sum of the second and third is 18, and the sum of the first and fourth is 27. Find the numbers.

15. Find an arithmetic progression whose first term is 1 and whose first, second, fifth, and fourteenth terms are in geometric progression.

16. Three numbers whose sum is 27 are in arithmetic progression. If 1 is added to the first, 3 to the second, and 11 to the third, the sums will be in geometric progression. Find the numbers.

17. To find the amount at compound interest when the principal, the rate of interest, and the time are given.

Solution. Let p equal the number of dollars invested, r the rate per cent of interest, t the number of years, and a the amount at the end of t years.

Then $a = p\,(1 + r)$ at the end of one year,

$a = p\,(1 + r)(1 + r) = p\,(1 + r)^2$ at the end of two years,

and $a = p\,(1 + r)^t$ at the end of t years. See page 248.

18. Show how to modify the solution given under problem 17 when the interest is compounded semi-annually; quarterly.

276. Infinite Geometric Series. In attempting to reduce $\frac{2}{3}$ to a decimal, we find by division .666 ⋯, the dots indicating that the 6's are repeated indefinitely.

Conversely, we see that $.666 \ldots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$; that is, a geometric progression in which $a = \frac{6}{10}$, $r = \frac{1}{10}$, and n is not fixed but *goes on increasing indefinitely.*

As n grows large, l grows small, and by taking n sufficiently large, *l can be made as small as we please.*

Hence, formula III, § 272, is to be interpreted in this case as follows:

$$s_n = \frac{a - rl}{1 - r} = \frac{\frac{6}{10} - \frac{1}{10} \cdot l}{1 - \frac{1}{10}} = \frac{6 - l}{9},$$

in which *l grows small indefinitely as n increases indefinitely*, so that, by taking n large enough, s_n can be made to *differ as little as we please* from $\frac{6 - 0}{9} = \frac{6}{9} = \frac{2}{3}.$

In this case we say *s_n approaches $\frac{2}{3}$ as a limit as n increases indefinitely.*

Similarly, if we have $.3333 \cdots$ or $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$, then s_n approaches $\frac{1}{3}$ as a limit.

277. Sum of an Infinite Geometric Series. The series $a + ar + ar^2 + \cdots + ar^n + \cdots$, where the number of terms goes on indefinitely, is called an **infinite geometric series.**

The limit of the sum of the *general infinite geometric series* may be found in the above manner, *provided r is a proper fraction.*

The sum s_n of the first n terms is

$$s_n = \frac{a - rl}{1 - r} = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

If r is a *proper fraction*, then *r^n approaches nearer and nearer to zero as n becomes large*, and hence $\frac{ar^n}{1 - r}$ *becomes as small as we please.*

Hence, $\frac{a - ar^n}{1 - r}$ can be made to approach as nearly as we please to $\frac{a}{1 - r}$, by taking n large enough.

Hence, s_n approaches $\frac{a}{1 - r}$ as n increases indefinitely.

Hence, $\frac{a}{1 - r}$ is said to be *the sum of the infinite series,*

$$a + ar + ar^2 + \cdots + ar^n + \cdots,$$

provided r lies between -1 and $+1$.

If r does not lie between -1 and $+1$, then r^n does not approach zero, and hence the sum of n terms of the infinite geometric series does not approach any limit as n increases indefinitely.

Example. Find the fraction which is the limit of .88888 ….

Solution. $a = \frac{8}{10}$, $r = \frac{1}{10}$.

Hence, $s_n = \frac{a - ar^n}{1 - r} = \frac{\frac{8}{10} - \frac{8}{10} \cdot (\frac{1}{10})^n}{1 - \frac{1}{10}} = \frac{\frac{8}{10}}{\frac{9}{10}} - \frac{\frac{8}{10}(\frac{1}{10})^n}{\frac{9}{10}} = \frac{8}{9} - \frac{8}{9} \cdot \left(\frac{1}{10}\right)^n.$

The limit as n increases indefinitely is therefore $\frac{8}{9}$, which is the required fraction.

WRITTEN EXERCISES

1. Find the fraction which is the limit of .333 ….

2. Find the fraction which is the limit of .1666 ….

Suggestion. The series is $\frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots$. Find the sum of the series beginning with $\frac{6}{100}$ and then add $\frac{1}{10}$.

3. Find the fraction which is the limit of .08333 ….

4. Find the fractions which are the limits of .1444 …, .07555, and .19222.

5. Find the fraction which is the limit of .909090.

Suggestion. The series is $\frac{9}{10} + \frac{9}{1000} + \frac{9}{100000} + \cdots$.

6. Find the fraction which is the limit of .141414 ….

7. Find the sum of the infinite series $-\frac{16}{3} + 4 - 3 + \frac{9}{4} \cdots$.

8. Find the sum of the series $-243 + 81 - 27 + \cdots$.

9. Find the sum $\frac{4}{3} + \frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \cdots$.

HARMONIC PROGRESSIONS

278. A **Harmonic Progression** is a series whose terms are the reciprocals of the corresponding terms of an arithmetic progression.

E.g. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9} \cdots$ is a harmonic progression whose terms are the reciprocals of the terms of the arithmetic progression 1, 3, 5, 7, 9 ….

The name *harmonic* is given to such a series because musical strings of uniform size and tension vibrate in harmony if their lengths are proportional to the reciprocals of the positive integers, *i.e.* to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$.

279. The general form of the harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}. \qquad \text{I}$$

It follows that if $a, b, c, d, e, \cdots$ are in harmonic progression, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}, \cdots$ are in arithmetic progression. Hence, all questions pertaining to a harmonic progression may be answered by first converting it into an *arithmetic progression.*

280. Harmonic Means. The terms between the first and the last terms of a harmonic progression are called **harmonic means** between them.

Example. Insert five harmonic means between 30 and 3.

This is done by inserting five arithmetic means between $\frac{1}{30}$ and $\frac{1}{3}$. By the method of § 268 the arithmetic series is found to be $\frac{1}{30}, \frac{1}{12}, \frac{2}{15}, \frac{11}{60}, \frac{7}{30}, \frac{17}{60}, \frac{1}{3}$. Hence, the harmonic series is $30, 12, \frac{15}{2}, \frac{60}{11}, \frac{30}{7}, \frac{60}{17}, 3$.

281. The case of a single harmonic mean is important. Let a, H, l be in harmonic progression. Then $\frac{1}{a}, \frac{1}{H}, \frac{1}{l}$ are in arithmetic progression.

Hence, by § 269, $\frac{1}{H} = \frac{\frac{1}{a}+\frac{1}{l}}{2}$ or $H = \frac{2al}{a+l}$.

282. The *arithmetic, geometric,* and *harmonic means* between a and l are related as follows:

We have seen $A = \frac{a+l}{2}$, $G = \sqrt{al}$, $H = \frac{2al}{a+l}$.

Hence, $\frac{A}{G^2} = \frac{a+l}{2} \div al = \frac{a+l}{2al}$.

Therefore, $\frac{A}{G^2} = \frac{1}{H}$, or $\frac{A}{G} = \frac{G}{H}$.

That is, G is a geometric mean between A and H. See § 275.

283. If a and l are both positive, then A and H are positive, and therefore G is a real number and G lies between A and H.

Moreover, by reducing $\frac{a+l}{2}$ and $\frac{2al}{a+l}$ to a common denominator we get $A = \frac{(a+l)^2}{2(a+l)}$ and $H = \frac{4al}{2(a+l)}$.

Hence we see that $A > H$, since $a^2 + l^2 + 2al - 4al = a^2 - 2al + l^2 = (a-l)^2 > 0$.

Hence, of the three means the arithmetic mean is the greatest and the harmonic is least.

EXERCISES AND PROBLEMS

1. Insert three harmonic means between 22 and 11.

2. Insert six harmonic means between $\frac{1}{8}$ and $\frac{2}{6}\!\!\!\!^{\,3}$.

3. The first term of a harmonic progression is $\frac{1}{2}$ and the tenth term is $\frac{1}{20}$. Find the intervening terms.

4. Two consecutive terms of a harmonic progression are 5 and 6. Find the next two terms and also the two preceding terms.

5. If a, b, c are in harmonic progression, show that $a \div c = (a-b) \div (b-c)$.

6. Find the arithmetic, geometric, and harmonic means between:

(*a*) 16 and 36; (*b*) $m+n$ and $m-n$; (*c*) $\frac{1}{m+n}$ and $\frac{1}{m-n}$.

7. The harmonic mean between two numbers exceeds their arithmetic mean by 7, and one number is three times the other. Find the numbers.

8. If x, y, and z are in arithmetic progression, show that mx, my, and mz are also in arithmetic progression.

9. x, y, and z being in harmonic progression, show that $\frac{x}{x+y+z}$, $\frac{y}{x+y+z}$, and $\frac{z}{x+y+z}$ are in harmonic progression, and also that $\frac{x}{y+z}$, $\frac{y}{x+z}$, and $\frac{z}{x+y}$ are in harmonic progression.

10. The sum of three numbers in harmonic progression is 3, and the third is double the first. Find the numbers.

11. The geometric mean between two numbers is $\frac{1}{4}$ and the harmonic mean is $\frac{1}{5}$. Find the numbers.

12. Insert n harmonic means between the numbers a and b.

13. The first term of a harmonic progression is a and the second is b. Write the 3d and the 4th terms.

14. The arithmetic mean of two numbers is 2 and the harmonic mean is $\frac{3}{2}$. Find the numbers.

15. The first term of a harmonic progression is $\frac{3}{4}$ and the 7th term is $6\frac{3}{4}$. Find the progression.

16. Find two numbers such that their geometric mean is 4 and their harmonic mean $\frac{5}{16}$.

17. The geometric mean of two numbers is 6 and their arithmetic mean is $7\frac{1}{2}$. Find the numbers.

18. a^2, b^2, c^2, ⋯ are in arithmetic progression. Prove that $b+c, c+a, a+b$ are in harmonic progression.

CHAPTER XVIII

THE BINOMIAL FORMULA

284. By actual multiplication the following products are obtained:

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$
$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

In these products the following rule is seen to hold:

1. The number of terms in each product is one greater than the exponent of the binomial.

2. The exponent of a in the *first* term is the same as the exponent of the binomial, and diminishes by unity in each *succeeding* term.

The exponent of b in the *last* term is the same as the exponent of the binomial, and diminishes by unity in each *preceding* term.

3. The sum of the exponents in each term is equal to the exponent of the binomial.

4. The coefficient of the first term is unity; of the second term, the same as the exponent of the binomial; and the coefficient of any other term may be found by multiplying the coefficient of the next preceding term by the exponent of a in that term and dividing this product by a number one greater than the exponent of b in that term.

5. The coefficients of any pair of terms equally distant from the ends are equal.

Let us now find $(a+b)^6$.

$$(a+b)^6 = (a+b)\,(a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,b^4 + b^5)$$

$$= \left\{\begin{array}{llllllll} a^6 & + 5\,a^5b & + 10\,a^4b^2 & + 10\,a^3b^3 & + 5\,a^2b^4 & + ab^5 & & \\ & a^5b & + 5\,a^4b^2 & + 10\,a^3b^3 & + 10\,a^2b^4 & + 5\,ab^5 & + b^6 & \end{array}\right.$$

Hence $(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6$

From this it is seen that the rule holds also for $(a+b)^6$.

PROOF BY MATHEMATICAL INDUCTION

285. A proof that the above rule holds for *all positive integral powers* of a binomial may be made as follows:

First step. Write out the product as it *would be* for the nth power on the *supposition that the rule holds.*

Then the first term would be a^n and the last term b^n. The second terms from the ends would be $na^{n-1}b$ and nab^{n-1}. The third terms from the ends would be $\frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2$ and $\frac{n(n-1)}{1 \cdot 2}a^2b^{n-2}$. The fourth terms from the ends would be

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 \text{ and } \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^3b^{n-3},$$

and so on, giving

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \cdots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

Second step. Multiply this expression by $a + b$ and see if the result, namely, $(a + b)^{r+1}$, can be so arranged as to conform to the same rule.

Multiplying the above expression for $(a + b)^n$ by a and then by b, we have

$$\left\{\begin{array}{llllll} a^{n+1} + na^nb + \frac{n(n-1)}{1 \cdot 2}a^{n-1}b^2 + \cdots & & + na^2b^{n-1} + ab^n & \\ \qquad a^nb \qquad + na^{n-1}b^2 + \cdots & + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-1} & + nab^n + b^{n+1} \end{array}\right.$$

Hence, adding,

$$(a + b)^{n+1} = a^{n+1} + (n + 1)a^nb + \left[\frac{n(n-1)}{1 \cdot 2} + n\right]a^{n-1}b^2 + \cdots$$
$$+ \left[n + \frac{n(n-1)}{1 \cdot 2}\right]a^2b^{n-1} + (n + 1)ab^n + b^{n+1}.$$

Combining the terms in brackets, we have,

$$(a + b)^{n+1} = a^{n+1} + (n + 1)a^nb + \frac{(n+1)n}{1 \cdot 2}a^{n-1}b^2 + \cdots$$
$$+ \frac{(n+1)n}{1 \cdot 2}a^2b^{n-1} + (n + 1)ab^n + b^{n+1}.$$

The last result shows that the rule holds for $(a + b)^{n+1}$ *if it holds for* $(a + b)^n$. That is, *if the rule holds for any positive integral exponent, it holds for the next higher integral exponent.*

Third step. It was found above by *actual multiplication* that the rule does hold up to $(a+b^6)$. Hence by the above argument we know that the rule holds for $(a+b)^7$.

Moreover, since we now know that the rule holds for $(a+b)^7$, we conclude by the same argument that it holds for $(a+b)^8$, and if for $(a+b)^8$, then for $(a+b)$,^{9} and so on.

Since this process of extending to higher powers can be carried on indefinitely, we conclude that the five statements in § 284 *hold for all positive integral powers of a binomial.*

The essential step in this proof consists in showing that the rule does hold for the $(n+1)$th power *if it holds for the nth power.*

286. The General Term. According to the rule now known to hold for any positive integral exponent, we may write as many terms of the expansion of $(a+b)^n$ as may be desired, thus:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$$

$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots \qquad \text{I}$$

From this result, called the **binomial formula**, we see:

(1) *The exponent of b in any term is one less than the number of that term, and the exponent of a is n minus the exponent of b. Hence, the exponent of b in the $(k+1)$st term is k, and the exponent of a is $n-k$.*

(2) *In the coefficient of any term the last factor in the denominator is the same as the exponent of b in that term, and the last factor in the numerator is one greater than the exponent of a.*

Hence the $(k+1)$st term, which is called the **general term**, is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots k}a^{n-k}b^k. \qquad \text{II}$$

E.g. In $(x+y)^{12}$ the 5th term is $\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}a^{12-4}b^4$

$= 11\cdot 5\cdot 9\cdot a^8b^4$.

287. Expansion of a Binomial. The process of writing out a power of a binomial is called **expanding the binomial,** and the result is called an **expansion of the binomial.**

Example 1. Expand $(x-y)^4$.

In this case $a = x$, $b = -y$, $n = 4$.

Hence, substituting in formula I, omitting the factor 1,

$$(x-y)^4 = x^4 + 4x^3(-y) + \frac{4(4-1)}{2}x^2(-y)^2 + \frac{4(4-1)(4-2)}{2\cdot 3}x(-y)^3 + \frac{4(4-1)(4-2)(4-3)}{2\cdot 3\cdot 4}(-y)^4 \qquad (1)$$

$$= x^4 - 4x^3y + \frac{4\cdot 3}{2}x^2y^2 - \frac{4\cdot 3\cdot 2}{2\cdot 3}xy^3 + \frac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}y^4. \qquad (2)$$

Hence, $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$. (3)

Notice that this is precisely the same as the expansion of $(x+y)^4$ except that every other term beginning with the second is *negative*.

Example 2. Expand $(1-2y)^5$.

Here $a = 1$, $b = -2y$, $n = 5$.

Since the coefficients in the expansion of $(a+b)^5$ are 1, 5, 10, 10, 5, 1, we write at once,

$$\begin{aligned}(1-2y)^5 &= 1^5 + 5\cdot 1^4\cdot(-2y) + 10\cdot 1^3\cdot(-2y)^2 \\ &\quad + 10\cdot 1^2\cdot(-2y)^3 + 5\cdot 1\cdot(-2y)^4 + (-2y)^5 \\ &= 1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5.\end{aligned}$$

Example 3. Expand $\left(\frac{1}{x}+\frac{y}{3}\right)^5$.

Remembering the coefficients just given, we write at once,

$$\begin{aligned}\left(\frac{1}{x}+\frac{y}{3}\right)^5 &= \left(\frac{1}{x}\right)^5 + 5\left(\frac{1}{x}\right)^4\left(\frac{y}{3}\right) + 10\left(\frac{1}{x}\right)^3\left(\frac{y}{3}\right)^2 + 10\left(\frac{1}{x}\right)^2\left(\frac{y}{3}\right)^3 \\ &\quad + 5\left(\frac{1}{x}\right)\left(\frac{y}{3}\right)^4 + \left(\frac{y}{3}\right)^5 \\ &= \frac{1}{x^5} + \frac{5y}{3x^4} + \frac{10y^2}{9x^3} + \frac{10y^3}{27x^2} + \frac{5y^4}{81x} + \frac{y^5}{243}.\end{aligned}$$

In a similar manner any positive integral power of a binomial may be written.

Example 4. Write the *sixth term* in the expansion of $(x-2y)^{10}$ without computing any other term.

From II, § 286, we know the $(k+1)$st term for the nth power of $a+b$, namely,

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{2\cdot 3\cdot 4\cdots k}a^{n-k}b^{k}.$$

In this case $a=x$, $b=-2y$, $n=10$, $k+1=6$. Hence, $k=5$.

Substituting these particular values, we have

$$\frac{10(10-1)(10-2)\cdots(10-5+1)}{2\cdot 3\cdot 4\cdot 5}x^{10-5}(-2y)^{5}$$
$$=\frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{2\cdot 3\cdot 4\cdot 5}x^{5}(-2y)^{5}$$
$$=252(-32)x^{5}y^{5}=-8064x^{5}y^{5}.$$

WRITTEN EXERCISES

1. Make a list of the coefficients for each power of a binomial from the 2d to the 10th.

Expand the following:

2. $(x-y)^{3}$.
3. $(2x+3)^{3}$.
4. $(3x+2y)^{4}$.
5. $(3+y)^{5}$.
6. $(x^{3}+y)^{6}$.
7. $(x-y^{3})^{6}$.
8. $(x^{3}-y^{2})^{7}$.
9. $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^{4}$.
10. $(x^{-1}+y^{-2})^{5}$.
11. $(a-b)^{8}$.
12. $(x+y)^{9}$.
13. $(m-n)^{10}$.
14. $(r^{\frac{1}{2}}+s^{2})^{4}$.
15. $(c^{-2}-d^{-\frac{1}{2}})^{5}$.
16. $(\sqrt{a}-\sqrt{b})^{6}$.
17. $\left(\frac{x^{2}}{y}-\frac{y^{2}}{x}\right)^{8}$.
18. $\left(\frac{2x}{y^{2}}-y\sqrt{x}\right)^{8}$.
19. $\left(\frac{\sqrt{m}}{\sqrt[3]{n^{2}}}+\sqrt{\frac{y}{n}}\right)^{4}$.
20. $\left(\frac{c\sqrt[3]{c}}{\sqrt[5]{d^{4}}}-\frac{\sqrt[3]{d}}{c}\right)^{7}$.
21. $(2a^{2}x^{-2}-3by^{-3})^{4}$.
22. $(3xy^{-3}-x^{-3}y)^{8}$.
23. $(a^{3}b^{-1}-a^{2}b^{3})^{5}$.
24. $(a^{\frac{2}{3}}b^{\frac{3}{2}}+a^{\frac{1}{2}}b^{\frac{1}{3}})^{4}$.
25. $(x^{-1}y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{-1})^{6}$.
26. $(3x^{-2}y^{2}+2xy^{-1})^{4}$.
27. $(2x^{-1}y^{-3}-x^{2}y^{2})^{5}$.
28. $(m^{3}n^{-2}+m^{\frac{1}{2}}n^{\frac{1}{4}})^{7}$.

In each of the following find the required term without finding any other term:

29. The 5th term of $(a+b)^{12}$.

30. The 7th term of $(3x-2y)^{11}$.

31. The 6th term of $(\sqrt{x}-\sqrt[3]{y})^{10}$.

32. The 9th term of $(x-y)^{25}$.

33. The 8th term of $(\frac{1}{2}m-\frac{1}{3}n)^{18}$.

34. The 7th term of $(a^2b-ab^2)^{30}$.

35. The 6th term of $(a-a^{-1})^{2k}$.

36. The 7th term of $(x^2y-x^{-2}y^{-1})^{3m}$.

37. The 5th term from each end of the expansion of $(a-b)^{20}$.

38. The 7th term from each end of $(a\sqrt{a}-b\sqrt{b})^{21}$.

39. Which term, counting from the beginning, in the expansion of $(a+b)^{10}$ has the same coefficient as the 7th term? Verify by finding both coefficients. How do the exponents differ in these terms?

40. What other term has the same coefficient as the 19th term of $(a+b)^{24}$? How do the exponents differ? Find in the shortest way the 21st term of $(a+b)^{25}$.

41. Find the 87th term of $(a+b)^{90}$.

42. Find the 53d term of $(a^{\frac{1}{2}}-b^{\frac{1}{3}})^{56}$.

43. What other term has the same coefficient as the 5th term in the expansion of $(x+y)^{19}$?

44. Expand $[(a+b)+c]^3$ by the binomial formula.

45. Expand $[1+(2x+3y)]^4$ by the binomial formula.

46. Expand $(2x-3y+4z)^3$ by the binomial formula.

47. Write the $(k+1)$st term of $(a+b)^n$. Write the $(n+1)$st term of $(a+b)^n$. Show that the next and also all succeeding terms after the $(n+1)$st term have zero coefficients, thus proving that there are exactly $n+1$ terms in the expansion.

CHAPTER XIX

LOGARITHMS

288. Logarithms. The operations of multiplication, division, and finding powers and roots are greatly shortened by the use of *logarithms.*

The logarithm of a number, in the system commonly used, is the *index of that power of* 10 *which equals the given number.*

Thus, 2 is the logarithm of 100 since $10^2 = 100$.

This is written $\log 100 = 2$.

Similarly $\log 1000 = 3$, since $10^3 = 1000$,

and $\log 10000 = 4$, since $10^4 = 10000$.

If the logarithm of a positive number is not an integer it may be written approximately or exactly as a decimal fraction.

Thus, $\log 74 = 1.8692$ since $10^{1.8692} = 74$ approximately.

In higher algebra it is shown that the laws for rational exponents (§ 176) hold also for irrational exponents.

289. Mantissa, Characteristic. The decimal part of a logarithm is called the **mantissa,** and the integral part the **characteristic.**

Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., it follows that for all numbers between 1 and 10 the logarithm lies between 0 and 1, that is, the characteristic is 0. Likewise for numbers between 10 and 100 the characteristic is 1, for numbers between 100 and 1000 it is 2, etc.

290. Tables of logarithms (see p. 244) give the mantissas only, the characteristics being supplied, according to the rule of § 292.

291. An important property of logarithms is illustrated by the following:

From the table of logarithms, page 244, we have:

$$\log 376 = 2.5752, \text{ or } 376 = 10^{2.5752}. \qquad (1)$$

Dividing both members of (1) by 10, we have

$$37.6 = 10^{2.5752} \div 10^1 = 10^{2.5752-1} = 10^{1.5752}.$$

Hence, $\log 37.6 = 1.5752.$

Similarly, $\log 3.76 = 1.5752 - 1 = 0.5752,$

$\log .376 = 0.5752 - 1$, or $\bar{1}.5752$,

$\log .0376 = 0.5752 - 2$, or $\bar{2}.5752$,

where $\bar{1}$ and $\bar{2}$ are written for -1 and -2 to indicate that the characteristics are negative while the mantissas are positive.

Multiplying (1) by 10 gives

$\log 3760 = 2.5752 + 1 = 3.5752,$

and $\log 37600 = 2.5752 + 2 = 4.5752.$

Hence, if the decimal point of a number is moved a certain number of places to the right or to the left, the characteristic of the logarithm is increased or decreased by a corresponding number of units, the mantissa remaining the same.

From the table on pages 244, 245, we may find the mantissas of logarithms for all integral numbers from 1 to 1000. In this table the logarithms are given to four places of decimals, which is sufficiently accurate for most practical purposes.

Example 1. Find log 876.

Solution. Look down the column headed N to 87, then along this line to the column headed 6, where we find the number 9425, which is the mantissa. Hence, $\log 876 = 2.9425$.

Example 2. Find log 3747.

Solution. As above we find $\log 3740 = 3.5729$,
and $\log 3750 = 3.5740$.

The difference between these logarithms is 11, which corresponds to a difference of 10 between the numbers. But 3740 and 3747 differ by 7. Hence, their logarithms should differ by $\frac{7}{10}$ of 11, *i.e.* by 7.7. Adding 8 (the nearest integer to 7.7) to the logarithm of 3740, we have 3.5737. This is the required logarithm.

Example 3. Find the number whose logarithm is 2.3948.

Solution. Looking in the table of mantissas, we find that the nearest *lower* logarithm is 2.3945, which corresponds to the number 248.

The given mantissa is 3 greater than that of 248, while the mantissa of 249 is 17 greater. Hence, the number corresponding to 2.3948 must be 248 plus $\frac{3}{17}$ of 1, that is, $248 + .18 = 248.18$.

Example 4. Find log .043.

Solution. Find log 43 and subtract 3 from the characteristic.

Example 5. Find the number whose logarithm is $\bar{4}.3949$.

Solution. Find the number whose logarithm is 0.3949, and move the decimal point 4 places to the left.

292. Rule for Finding the Characteristic.

(1) *For numbers greater than 1, the characteristic is positive or zero and always one less than the number of digits to the left of the decimal point.*

(2) *For numbers less than 1, the characteristic is negative and numerically one greater than the number of zeros to the right of the decimal point.*

WRITTEN EXERCISES

Find the logarithms of the following numbers:

1. 491.
2. 73.5.
3. 2485.
4. 539.7.
5. 53.27.
6. .541.
7. .051
8. 8104.
9. 70349.
10. 439.26.
11. .006.
12. .1902.
13. .0104.
14. 2.176.
15. 8.094.
16. 79.31.
17. 4.245.
18. .0006.
19. 3.817.
20. .1341.

Find the numbers corresponding to the following logarithms:

21. 1.3179.
22. 3.0146.
23. 0.7145.
24. $\bar{1}.5983$.
25. 2.0013.
26. 2.9900.
27. 0.1731.
28. 0.8974.
29. 0.9171.
30. 3.4015.
31. $\bar{1}.5972$.
32. $\bar{1}.0011$.
33. $\bar{2}.7947$.
34. $\bar{2}.5432$.
35. 0.5987.
36. 0.2468.
37. 0.1357.
38. $\bar{2}.0246$.
39. $\bar{1}.1358$.
40. $\bar{4}.0478$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	7857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8318
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

293. Products and powers may be found by means of logarithms, as shown by the following examples.

Example 1. Find the product $49 \times 134 \times .071$.

Solution. From the table,

$$\log 49 = 1.6902 \text{ or } 49 = 10^{1.6902}$$
$$\log 134 = 2.1271 \text{ or } 134 = 10^{2.1271}$$
$$\log .071 = \bar{2}.8513 \text{ or } .071 = 10^{\bar{2}.8513}$$

Since powers of the same base are multiplied by *adding* exponents, we have $49 \times 134 \times .071 = 10^{2.6686}$.

Hence, $\log(49 \times 134 \times .071) = 2.6686$.

The number corresponding to this logarithm, as found by the method used in Example 3, page 243, is 466.2. This is an error of about 1 in 80,000 and is therefore so small as to be negligible.

Example 2. Find $\log (1.05)^{20}$.

Solution. $(1.05)^{20} = (10^{0.0212})^{20} = 10^{(0.0212)20} = 10^{0.424}$.

Hence, $\log(1.05)^{20} = 0.4240$.

It follows from the laws of exponents that

(*a*) *The logarithm of the product of two or more numbers is the sum of the logarithms of the numbers.*

(*b*) *The logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

That is,

$$\log(a \cdot b \cdot c) = \log a + \log b + \log c \text{ and } \log a^n = n \log a.$$

WRITTEN EXERCISES

By means of logarithms obtain the following products:

1. $243 \times 76 \times .34$.
2. 823.68×3.70.
3. 216.83×2.03.
4. $(57)^2 \times (.71)^2$.
5. $510 \times (9.1)^3$.
6. $43.71 \times (21)^2$.
7. 5.93×10.02.
8. 486×3.45.
9. $(.02)^2 \times 0.8$.
10. $(6.5)^2 \times (.91)^3$.
11. $(8.4)^2 \times (.75)^3$.
12. $(.96)^2 \times (49)^2$.
13. $(49)^3 \times .19 \times .21^2$.
14. $.21084 \times (.53)^2$.
15. $7.865 \times (.013)^2$.
16. $(6.75)^3 \times (7.23)^2$.
17. $(1.46)^2 \times (61.2)^2$.
18. $(3.54)^3 \times (29.3)^2$.
19. $(4.132)^2 \times (5.184)^2$.
20. $1946 \times 3.98 \times .08$.

294. Quotients and roots may be found by means of logarithms as shown by the following examples.

Example 1. Divide 379 by 793.

Solution. From the table,

$$\log 379 = 2.5786 \text{ or } 10^{2.5786} = 379.$$
$$\log 793 = 2.8993 \text{ or } 10^{2.8993} = 793.$$

Hence, by the law of exponents for division, § 176,

$$379 \div 793 = 10^{2.5786-2.8993}.$$

Since in all operations with logarithms the mantissa is positive, write the first exponent 3.5786 – 1 and then subtract 2.8993.

Hence, $\log (379 \div 793) = .6793 - 1 = \bar{1}.6793.$

Hence, the quotient is the number corresponding to this logarithm.

Example 2. By means of logarithms approximate $\sqrt[3]{42^2 \times 37^5}$.

By the methods used above we find

$$\log (42^2 \times 37^5) = 11.0874 \text{ or } 10^{11.0874} = 42^2 \times 37^5.$$

Hence, $\sqrt[3]{42^2 \times 37^5} = (10^{11.0874})^{\frac{1}{3}} = 10^{\frac{11.0874}{3}} = 10^{3.6958}.$

That is, $\log \sqrt[3]{42^2 \times 37^5} = 3.6958.$

Hence, the result sought is the number corresponding to this logarithm.

It follows from the laws of exponents that

(*a*) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

(*b*) *The logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

That is,

$$\log \frac{a}{b} = \log a - \log b \text{ and } \log \sqrt[n]{a} = \frac{\log a}{n}.$$

NOTE. — A formula for finding cube roots of numbers is given on pages 126–131. Still higher roots may also be found by the same process, but the work is laborious and complicated. For practical purposes logarithms should be used for finding cube and higher roots, and even for square roots logarithms are desirable.

EXERCISES

By means of logarithms approximate the following quotients and roots:

1. $\sqrt[3]{275}$.
2. $\sqrt[3]{3412}$.
3. $\sqrt[3]{1.72}$.
4. $\sqrt[3]{.00146}$.
5. $\sqrt[3]{347}$.
6. $\sqrt[4]{72.48}$.
7. $\sqrt[5]{11.56}$.
8. $\sqrt[7]{648}$.
9. $\sqrt[6]{54324}$.
10. $\sqrt[8]{.6843}$.
11. $45.2 \div 8.9$.
12. $231.18 \div 4.2$.
13. $.04905 \div .327$.
14. $\sqrt{196 \times 256}$.
15. $\dfrac{5334 \times .02374}{27.43 \times 3.246}$.
16. $\sqrt[5]{69} \div \sqrt[3]{21}$.
17. $\sqrt[7]{15} \times \sqrt[8]{67}$.
18. $\sqrt[10]{211} \times \sqrt[11]{34.7}$.
19. $(5184)^{\frac{1}{2}} \div (38124)^{\frac{1}{3}}$.
20. $(6.75)^3 \div (2.132)^2$.
21. $\sqrt[9]{105} \div \sqrt[13]{76}$.
22. $(91125)^{\frac{1}{3}} \div (576)^{\frac{1}{3}}$.
23. $(3.04)^3 \div (.65)^3$.
24. $(29.3)^{\frac{1}{3}} \div \sqrt{(3.47)^3}$.
25. $\sqrt[3]{39} \times \sqrt[3]{56} \times \sqrt[4]{87}$.
26. $\sqrt[3]{\dfrac{13^4 \times .31^2 \times 4.31^3}{\sqrt{71} \times \sqrt[3]{41} \times \sqrt{51}}}$.
27. $\sqrt[5]{\dfrac{4^9 \times .57^3 \times 42^3}{\sqrt[3]{5.2} \times \sqrt[5]{.83} \times \sqrt{23}}}$.
28. $\left(\dfrac{\sqrt[3]{54} \times \sqrt[4]{28} \times \sqrt[5]{7}}{\sqrt{4^7} \times \sqrt[3]{7^4} \times (.003)^{\frac{3}{4}}}\right)^{\frac{4}{5}}$.

PROBLEMS

1. Solve $a = p(1 + r)^t$ for p and r.

2. Solve $a = p(1 + r^t)$ for t.

Solution. $\log a = \log p\ (1 + r)^t = \log p + \log\ (1 + r)^t$
$= \log p + t \log\ (1 + r)$. (See § 293.) Hence $t = \dfrac{\log a - \log p}{\log\ (1 + r)}$.

3. At what rate of interest compounded annually will $ 1200 amount to $ 1800 in 12 years?

4. At what rate of interest compounded semi-annually will a sum double itself in 20 years? in 15 years? in 10 years?

5. In what time will $ 8000 amount to $13,500, the rate of interest being $3\frac{1}{2}$ % compounded annually?

6. In what time will a sum double itself at 3 %, 4 %, 5 %, compounded semi-annually?

The present value of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

7. What is the present value of $2500 due in 4 years, money being worth $3\frac{1}{2}$ %, interest compounded semi-annually?

8. A man bequeathed $50,000 to his daughter, payable on her twenty-fifth birthday, with the provision that in case she married before that time, the present worth of the bequest at the time of marriage should then be paid. If she married at 21, how much would she receive, interest being 4 % per annum and compounded quarterly?

9. What is the rate of interest if the present worth of $24,000 due in 7 years is $19,500?

10. In how many years is $5000 due if its present worth is $3500, the rate of interest being $3\frac{3}{4}$ % compounded annually?

HISTORICAL NOTE

Logarithms were given to the world full fledged on a single day when **John Napier's** work on the subject was published in 1614. "It is one of the curiosities of the history of science that Napier constructed logarithms before exponents were used." Napier's development of his idea of logarithms is too complicated for us to follow here, but "it is so unique and so different from all other modes of presenting the subject that there cannot be any doubt that this invention is entirely his own." Napier's purpose was to shorten the labor of multiplication and division which was becoming so serious in the then rapidly developing science of Astronomy; and since his time logarithms have been one of the chief elements in making possible the tremendous amount of calculation which is necessary in modern science and in its application to the industries.

Henry Briggs (1556–1631), professor of mathematics at Oxford, improved Napier's system by making it more directly applicable to the decimal notation. Briggs constructed extensive tables to 14 decimal places.

Tables of logarithms were published in Germany in 1625, in France in 1624, and in Italy in 1626.

HARVARD UNIVERSITY

ELEMENTARY ALGEBRA

TIME: ONE HOUR AND A HALF

1. Solve the simultaneous equations

$$x + y = a + b, \qquad \frac{y+b}{x+a} = \frac{a}{b},$$

and verify your results.

2. Solve the equation $x^2 - 1.6\,x - 0.23 = 0$, obtaining the values of the roots correct to three significant figures.

3. Write out the first four terms of $(a - b)^7$.
Find the fourth term of this expansion when

$$a = \sqrt[3]{x^{-1}y^{\frac{1}{2}}}, \qquad b = \sqrt[6]{9\,xy^{-4}},$$

expressing the result in terms of a single radical, and without fractional or negative exponents.

4. Reduce the following expression to a polynomial in a and b:

$$\frac{6\,a^3 + 7\,ab^2 + 12\,b^3}{3\,a^2 - 5\,ab - 4\,b^2} - \frac{1}{\dfrac{3}{19\,b} - \dfrac{5\,a + 4\,b}{19\,a^2}}.$$

5. The cost of publishing a book consists of two main items: first, the fixed expense of setting up the type; and, second, the running expenses of presswork, binding, etc., which may be assumed to be proportional to the number of copies. A certain book costs 35 cents a copy if 1000 copies are published at one time, but only 19 cents a copy if 5000 copies are published at one time. Find (*a*) the cost of setting up the type for the book, and (*b*) the cost of presswork, binding, etc., per thousand copies.

YALE UNIVERSITY

ALGEBRA A

TIME : ONE HOUR

Omit one question in Group II and one in Group III. Credit will be given for *six* questions only.

Group I

1. Resolve into prime factors: (*a*) $6x^2 - 7x - 20$; (*b*) $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24$; (*c*) $a^4 + 4a^2 + 16$.

2. Simplify $\left(5 - \frac{a^2 - 19x^2}{a^2 - 4x^2}\right) \div \left(3 - \frac{a - 5x}{a - 2x}\right)$.

3. Solve $\frac{2(x-7)}{x^2 + 3x - 28} + \frac{2-x}{4-x} - \frac{x+3}{x+7} = 0$.

Group II

4. Simplify $\frac{\sqrt{2} + 2\sqrt{3}}{\sqrt{2} - \sqrt{12}}$, and compute the value of the fraction to two decimal places.

5. Solve the simultaneous equations $\begin{cases} x^{-\frac{1}{2}} + 2y^{-\frac{1}{2}} = \frac{7}{6}, \\ 2x^{-\frac{1}{2}} - y^{-\frac{1}{2}} = \frac{2}{3}. \end{cases}$

Group III

6. Two numbers are in the ratio of $c : d$. If a be added to the first and subtracted from the second, the results will be in the ratio 3 : 2. Find the numbers.

7. A dealer has two kinds of coffee, worth 30 and 40 cents per pound. How many pounds of each must be taken to make a mixture of 70 pounds, worth 36 cents per pound?

8. A, B, and C can do a piece of work in 30 hours. A can do half as much again as B, and B two thirds as much again as C. How long would each require to do the work alone?

CORNELL UNIVERSITY

ELEMENTARY ALGEBRA

---

1. Find the H. C. F.:

$$x^4 - y^4,$$
$$x^3 - xy^2 + x^2y - y^3,$$
$$x^4 + 2x^2y^2 - 3y^4.$$

2. Solve the following set of equations:

$$x + y = -1,$$
$$x + 3y + 2z = -4,$$
$$x - y + 4z = 5.$$

3. Expand and simplify:

$$\left(2x^3 - \frac{1}{x}\right)^7.$$

4. An automobile goes 80 miles and back in 9 hours. The rate of speed returning was 4 miles per hour faster than the rate going. Find the rate each way.

5. Simplify:

$$\frac{\left(\frac{x+1}{x-1}\right)^2 - 2 + \left(\frac{x-1}{x+1}\right)^2}{\left(\frac{x+1}{x-1}\right)^2 - \left(\frac{x-1}{x+1}\right)^2}.$$

6. Solve for x:

$$\frac{2x+3}{x-1} - 6 = \frac{5}{x^2+2x-3}.$$

7. A, B, and C, all working together, can do a piece of work in $2\frac{2}{3}$ days. A works twice as fast as C, and A and C together could do the work in 4 days. How long would it take each one of the three to do the work alone?

UNIVERSITY OF CHICAGO

ELEMENTARY ALGEBRA

I. Resolve into simplest factors:

1. $a^2 - b^2 - (a - b)^2$.

2. $2a^3 - 3a^2 - 2a + 3$.

3. $x^4 - 14x^2y^2 + y^4$.

4. $a(a + b) - c(c + b)$.

II. Reduce to simplest form:

1. $-10\{x - 6[x - (y - z)]\} + 60\{y - (z + x)\}$.

2. $\dfrac{a}{(b - c)(b - a)} + \dfrac{b}{(c - a)(c - b)} + \dfrac{c}{(b - a)(c - a)}$.

3. $\left(\dfrac{a}{b^2} + \dfrac{b}{a^2}\right) \div \left(\dfrac{1}{a^2} - \dfrac{1}{ab} + \dfrac{1}{b^2}\right)$.

III. 1. Solve for x:

$\frac{1}{2}(x - 5) - \frac{1}{3}(x - 4) = \frac{1}{2}(x - 3) - (x - 2)$.

2. Solve for x and y:

$$\begin{cases} \frac{1}{3}(7 + x) = \frac{1}{5}(9 + y). \\ \frac{1}{7}(11 + x + y) = \frac{1}{5}(9 + y). \end{cases}$$

IV. Three brothers, A, B, C, at a family reunion were discussing their ages. C said to A: "Thirty years ago my age was double yours." Then B said to A: "Twenty-three years ago *my* age was double yours." If C's present age exceeds B's by three years and B's exceeds A's by seven years, find the age of each.

UNIVERSITY OF CALIFORNIA

ELEMENTARY ALGEBRA

1. If $a = 4$, $b = -3$, $c = 2$, and $d = -4$, find the value of:

(a) $ab^3 - 3\,cd^2 + 2(3\,a - b)(c - 2\,d)$.

(b) $2\,a^3 - 3\,b^4 + (4\,c^3 + d^3)(4\,c^2 + d^2)$.

2. Reduce to a mixed number:

$$\frac{3\,a^4 - 4\,a^3 - 10\,a^2 + 41\,a - 28}{a^2 - 3\,a + 4}.$$

Simplify:

3. $\dfrac{a + 2}{a^2 + 3\,a - 40} - \dfrac{b - 2}{ab - 5\,b + 3\,a - 15}.$

4. $\left(1 - \dfrac{2 - 3\,b - 2\,c}{a + 2}\right) \div \dfrac{a^2 - 4\,c^2 + 9\,b^2 + 6\,ab}{2\,a^2 + a - 6}.$

5. A's age 10 years hence will be 4 times what B's age was 11 years ago, and the amount that A's age exceeds B's age is one third of the sum of their ages 8 years ago. Find their present ages.

6. Draw the lines represented by the equations

$$3\,x - 2\,y = 13 \text{ and } 2\,x + 5\,y = -4,$$

and find by algebra the coördinates of the point where they intersect.

7. Solve the equations $\begin{cases} bx - ay = b^2 - ab, \\ y - b = 2(x - 2\,a). \end{cases}$

8. Solve $(2\,x + 1)(3\,x - 2) - (5\,x - 7)(x - 2) = 41.$

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JANUARY, 1912

Answer the first six questions and two of the others.

1. Find the prime factors of *each* of the following: $1 - x^4$; $x^2 - cx - 2\,dx + 2\,cd$; $3\,a^5 + 3\,b^5$; $a^4 + a^2b^2 + b^4$.

2. Divide $4\,x^6 - 9\,x^4 + 25 - 14\,x^3 - x^2$ by $2\,x^3 - x - 5 + 3\,x^2$.

3. Solve $\begin{cases} mx - ny = m^2 + n^2, \\ x - y = 2\,n. \end{cases}$

4. Simplify $\sqrt{3} \times \sqrt{48}$; $(5\sqrt{5} - 4)(5\sqrt{5} + 8)$; $3\sqrt{8} \div 15\sqrt{2}$; $6\sqrt{\tfrac{2}{3}} - 5\sqrt{24} + 12\sqrt{\tfrac{3}{2}}$.

5. Find the square root of the following:

$$25\,x^4 - 30\,ax^3 + 49\,a^2x^2 - 24\,a^3x + 16\,a^4.$$

6. The area of a rectangle is 18 square inches less than twice the area of a square; the width of the rectangle equals the width of the square, but the length of the rectangle exceeds that of the square by 7 inches. Find the side of the square.

7. A, B, and C together have \$1285; A's share is \$25 more than $\frac{5}{6}$ of B's, and C's share is $\frac{4}{15}$ of B's. Find the share of each.

8. Solve $21\,x^2 = 2\,ax + 3\,a^2$.

9. Solve $\begin{cases} x^2 - xy + y^2 = 63, \\ x - y = -3. \end{cases}$

10. Define root of a number, surd, affected quadratic equation.

Expand $(2\,a^2 - 3\,b)^4$.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JUNE, 1912

Answer the first four questions and four of the others.

1. Divide $9x^6 - 10x^3 + 9 - 16x^2 - x^4$ by $4x - x^2 + 3x^3 - 3$. [Credit will not be granted if there is any error in the work.]

2. Factor *each* of the following: $x^4 - x^2 - 12$; $(a+2)^2 - 9x^2$; $x^3 - 27$; $6x^2 - x - 2$; $x^5 + 32$; $a^3 - a^2 - a + 1$.

3. Solve $\begin{cases} 2x + y = a + 1, \\ x - 2y = a - 1. \end{cases}$

4. The length of a rectangular field is twice its width; it costs as much to fence it at 50¢ per yard as it does to sod it at 15¢ a square yard. Find the dimensions.

5. Solve $2 + \sqrt{2x + 8} = 2\sqrt{x + 5}$.

6. Solve $x^2 + ax = 42a^2$.

Expand by the binomial formula $(2a - 3b)^5$, giving all the work.

7. Solve $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

Group the four pairs of roots properly.

8. Two men start at the same time and travel in opposite directions; the ratio of their rates is 2 : 3 and in 5 hours they are 100 miles apart. Find the rate of each.

9. Divide the number c into *two* parts such that a times the larger part shall equal b times the smaller part.

10. If a triangle with equal sides has its sides increased 7 inches, 4 inches and 1 inch respectively, a right triangle is formed; find the sides of the right triangle.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JANUARY, 1913

Answer the first six questions and two of the others.

1. Factor each numerator and denominator in the following expressions; perform the operations indicated and reduce to simplest form:

$$\frac{x^4 - 9x^3 - 36x^2}{x^2a^2 - 10xa^2 + 9a^2} \times \frac{x^4 - 4x^2 + 3}{x^4 - 7x^2 - 18} \div \frac{x^3 + x^2}{a^2x^2 + 2a^2}.$$

2. Solve $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 4, \\ 5x - 3y = 2. \end{cases}$

3. Solve $x + \dfrac{a}{2} = \dfrac{a^2}{2x}$.

4. Simplify *each* of the following:

$$4\sqrt{24} + 2\sqrt{54} - \sqrt{6} + 3\sqrt{96} - 5\sqrt{150}.$$

$$\sqrt[3]{375},\ \sqrt{\frac{2}{3}},\ \sqrt[3]{\frac{1}{4}},\ \frac{1}{\sqrt{3}}.$$

5. Solve $\dfrac{16x + 3}{10} - \dfrac{2x - 5}{5x - 1} = \dfrac{8x - 1}{5}$.

6. Separate 42 into *two* parts such that the greater part divided by the less shall give a quotient of 2 and a remainder of 3.

7. Solve $\begin{cases} 6y^2 - xy = 2x^2, \\ 9y - 12 = -4x. \end{cases}$

8. What must be added to $x + a$ to make $y - b$?

What is the cost of 3 apples if a apples cost c cents?

9. The sum of two numbers is 8 and the sum of their cubes is 152; find the numbers.

10. If 1 is added to the numerator of a fraction the value of the fraction becomes $\frac{1}{2}$; if 1 is added to the denominator of the same fraction the value becomes $\frac{1}{3}$. Find the fraction.

NEW YORK STATE EDUCATION DEPARTMENT

ELEMENTARY ALGEBRA

JUNE, 1918

Answer the first six questions and two of the others.

1. Solve and check or prove $\frac{x}{3} - \frac{x-3}{3} = 12 - \frac{x+4}{2} - x$.

2. Extract the square root of $4c^4 - 4c^3 + 5c^2 - 2c + 1$.

3. Solve $\begin{cases} x + 2y = a, \\ 2x - y = b. \end{cases}$

4. Simplify $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$; $(\sqrt{5} - \sqrt{2})(2\sqrt{5} + 3\sqrt{2})$.

5. In five years A will be twice as old as B; five years ago A was three times as old as B. Find the age of each at the present time.

6. Find the quotient to *three* terms and the remainder when $11a^3 - 5a + 12 - 82a^2 + 30a^4$ is divided by $2a - 4 + 3a^2$.

7. (*a*) What is the dividend which, divided by x, gives a quotient of y and a remainder of z?

(*b*) If a apples are sold for a dime, how many can be bought for c cents?

8. Two men, A and B, can dig a trench in 20 days; it would take A 9 days longer to dig it alone than it would B. How long would it take B alone?

9. Find *three* successive *even* numbers whose sum is $\frac{3}{8}$ of the product of the first two.

10. Simplify $\dfrac{x^2 - \dfrac{1}{x}}{x + \dfrac{1}{x} + 1}$.

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

January, 1910

Answer eight questions, selecting two from each group.

Group I

1. Solve $\frac{a+bx}{3b+2ax}=\frac{a-bx}{b-2ax}$.

2. Find the prime factors of *each* of the following expressions and from the factors determine the highest common factor:

$$27m^5-8m^2,\ 6m^3+8m^2-8m,\ 12m^4-8m^3,\ 27m^3-12m.$$

3. Find the product of $2x+3y-z$ and $x-3y+2z$. Prove by substitution the correctness of the result if $x=1$, $y=2$, and $z=3$.

Group II

4. Reduce *each* of the following to its simplest form:

$$\sqrt{50}-\sqrt{32},\ 2\sqrt{5}\times\sqrt{15},\ 6\sqrt{20}\div2\sqrt{10},\ \sqrt{\tfrac{5}{8}},\ \sqrt{9\sqrt{a^4}}.$$

5. Solve $\begin{cases} 2x+y-z=5, \\ 3x+2y-4z=2, \\ x-2y+3z=3. \end{cases}$

6. Find the square root of the following:

$$49x^6-42x^5-47x^4-4x^3+28x^2+16x+4.$$

Group III

7. If 122 marbles were divided among three boys so that the first had twice as many as the second and the second had 6 more than the third, how many had each? Prove.

8. Find *three* consecutive numbers such that if the first is divided by 5, the second by 7, and the largest by 11, the sum of the three quotients is $\frac{1}{7}$ of the sum of the three numbers. Prove.

9. Solve $\sqrt{2x+1} = 2\sqrt{x} - \sqrt{x-3}$.

Group IV

10. Solve $\begin{cases} 2x + y = 5, \\ 3x^2 - 7y^2 = 5. \end{cases}$

11. Solve $1 - 10ax + 16a^2x^2 = 0$.

12. *A* and *B* can together address 100 envelopes in an hour; when each works alone *A* can address 100 envelopes in 50 minutes less time than *B*. How many can each address in an hour?

THE UNIVERSITY OF THE STATE OF NEW YORK

ELEMENTARY ALGEBRA

JANUARY, 1915

Answer 12 questions, selecting five from group I, two from group II, and five from group III.

Group I

1. Solve $\dfrac{3(x-2)}{x+3} = \dfrac{36-4x}{x^2-9} - \dfrac{2+3x}{3-x}$.

2. Factor *three* of the following:

$$x^4 - 8x^2 - 9,$$
$$m^2 - 6mn - 16x^2y^2 + 9n^2,$$
$$m^2d^2 + 3 - 3m - d^2,$$
$$p^2q^2 - 12pqx + 35x^2.$$

3. Solve $\sqrt{3+x} + \sqrt{x} = \dfrac{5}{\sqrt{x}}$.

4. Solve $3(x-2)(x-4) = (x-5)^2$.

5. (*a*) Simplify $2\sqrt{\tfrac{5}{3}} - \sqrt{60} - 5\sqrt{\tfrac{3}{5}}$.

(*b*) Simplify $\dfrac{a\sqrt{x} - 2b\sqrt{y}}{a\sqrt{x} + b\sqrt{y}}$.

6. Solve $\begin{cases} x - 3y = 1, \\ xy + y^2 = 5. \end{cases}$

Group II

7. A man has \$8000 which he wishes to invest in two enterprises so that his total income will be \$425; if one enterprise pays $5\frac{1}{2}$ % and the other 5 %, how much must he invest in each?

8. Separate a line 20 inches long into two parts such that the product of the whole line and one part shall equal the square of the other part. [Result contains a surd.]

9. A rectangular ceiling has in it two skylights each $2\frac{1}{2}$ feet by 3 feet; the surface of the ceiling, not including the skylights, is 93 feet. If the length of the ceiling is 3 feet more than its width, what are its dimensions?

Group III

10. If one of the factors of $6a^2x^2 - 4a^3x - 4ax^3 + x^4 + a^4$ is $a^2 + x^2 - 2ax$, what is the other factor?

11. Write *three* different expressions of higher degree than the first degree whose H. C. F. is $x - y$. Find the L. C. M. of these expressions.

12. A lady bought 5 dozen buttons at d cents a dozen and 3 yards of cloth at k cents a yard; she gave a two-dollar bill in payment. How many cents should she receive in change?

13. What is the value of $8x^2 - 6ax$ when $x = \frac{a - \sqrt{b}}{2}$?

14. Find, correct to *two* decimal places, the solutions of $2x^2 + 6x - 3 = 0$.

15. Write an expression that can be divided by $a - b$ and also by $2a + b$.

THE UNIVERSITY OF CHICAGO

ELEMENTARY ALGEBRA

I. Remove the symbols of aggregation from:

1. $2x - \{3y - [5z - (x - \overline{2y - 3z})]\}$;
2. $2x + \{5z - [3y + (x - \overline{2y - 3z})]\}$.

II. Arrange the following in groups of equivalent expressions:

$$\frac{x}{y},\ \frac{x+a}{y+a},\ \frac{xa}{ya},\ \frac{x-a}{y-a},\ \frac{x \div a}{y \div a};$$

$$\frac{\frac{x}{a}+1}{\frac{y}{a}+1},\ \frac{1-\frac{a}{x}}{\frac{y}{x}-\frac{a}{x}},\ \frac{\frac{x}{y}+\frac{a}{y}}{1+\frac{a}{y}},\ \frac{\frac{x}{a}-1}{\frac{y}{a}-1}.$$

III. Find the G. C. D. of:

$2ax^3 + 2ax^2 - 4ax$,
$(ax^2 + 2ax)(2x^3 - 2)$,
$2a(x^2 - x)(x^3 + 8)$,
$(x - 1)(6abx^2 + 12abx - 2ax^2y - 4axy)$.

IV. Solve for x:

$$\frac{2x-1}{2x-2} - \frac{23}{10x-10} = \frac{3}{5}\left(\frac{1}{x-1} - \frac{1}{3}\right).$$

V. Two yachts race over a 48-mile course. Owing to difference in measurement, B is given a start of half a mile in the first trial, and is beaten by 6 minutes. In the second trial, the rate of the wind being the same as before, B's start is increased to a mile and a half, and still A wins by 2 minutes. Find the rate in feet per minute of each boat.

INDEX

Abel 394
Abscissa 233, 447
Absolute value 44, 355
Addend 335
Addition 8, 353
  by counting 45
  elimination by . . . 209, 423
  laws of 358
  of complex numbers . 284, 510
  of fractions 168, 397
  of monomials 68, 362
  of polynomials . . . 69, 71, 364
  of radicals 268, 495
  of signed numbers . . 45, 47, 356
  principles of 9, 47
Aggregation, symbols of . 11, 360
Ahmes 41
Algebra (origin of name) . . 32
  and Arithmetic 353
Algebraic, fractions . . 160, 393
  operations 20, 353
  sum 48
Algebraic expressions . . 6, 361
  forming of, 7, 37, 79, 81, 87, 91, 220
Alkarismi 101
Alternation 189, 458
Antecedent 188, 458
Approach as a limit 582
Approximate roots . 251, 253, 483
Arabs 5, 32, 41, 62, 313
Arabic numerals 5
Area problems. See Problems.
Argand 515
Arithmetic, means 575
  progression 572
Associative laws 358
Averaging signed numbers, 49, 64
Axes of coördinates . 232, 233, 446
Axioms 29, 406

Base of a power . . 92, 354, 465
Bhaskara 291, 490
Binomial 67, 361
  expansion of 308, 590
  factors of 126–129, 380
  formula 308, 587
Bombelli 490
Braces, brackets, etc. . . 11, 360
Briggs 601

Cajori 62
Cantor 510
Cardan 515
Cauchy 452
Characteristic of a logarithm, 593
Checking results, 10, 27, 32, 33, 34, 69, 102, 211, 281, 282, 415
Circle, problems on, 261, 317, 339, 347, 348, 349, 350, 564
Clearing of fractions . 182, 415
Coefficients 8, 92, 354
  detached 106, 376
Common, denominator . 164, 394
  difference 572
  factor 8, 155, 390
  multiple 157, 391
Commutative laws . 69, 84, 358
Completing the square, 278, 521
  Hindu method of 280
Complex, fractions . . . 178, 402
  number 509
  plane 517
Composition and **Division**, 190, 458
Compound term 67, 361

Conditional equation 11, 405
Consequent 188, 458
Consistent equations 207, 422
Constant 453
  ratio 453
Contradictory equations, 207, 238, 422, 452
Coördinates 233, 447
Cross products 134
Cube of binomial 116, 308, 309
Cube root 354, 478
  of arithmetic numbers 481
Cubes, the sum or difference of, 118, 380

Data problems, 39, 227, 228, 319, 322, 324, 327, 351, 352
Dedekind 510
Definitions 353
Degree of an equation, 147, 207, 236, 289, 404
Dependent equations, 207, 238, 422, 450
Deriving equivalent equations 27, 28, 29, 256, 276, 407
Descartes, 41, 62, 101, 239, 240, 291, 490, 510, 564
Detached coefficients 106, 376
Determinants 434, 436, 440
  note on 452
  solution by 437, 439
Difference 9, 50
  of two cubes 118, 129, 380
  of two squares 110, 126, 380
Digit problems, 81, 82, 83, 222, 223, 228, 343, 351
Diophantus 41, 62, 229, 291, 510
Directions for written work, 30
Discriminant of the quadratic, 534
Distributive laws 15, 358
Dividend 59, 353
Division 14, 18, 59, 353
  by a monomial 96, 98, 374
  by a polynomial 102, 374
Division — *cont.*
  by detached coefficients 106, 376
  by means of logarithms 599
  by zero, 29, 102, 104, 359, 407, 463
  of complex numbers 512, 513
  of fractions 175, 399
  of imaginaries 285, 573
  of monomials 98, 370
  of polynomials 102, 374
  of a product 18
  of radicals 274, 498
  of signed numbers 59, 357
  of sum or difference 14
  of surds 274
  principles of 15, 18, 59, 97
  synthetic division 377
Divisor 59, 353
Double use of signs 52, 53

Egyptians 41, 151, 161
Elimination, by addition or subtraction 208, 423
  by comparison 424
  by formula 435
  by substitution 211, 423
Ellipse 567
Entire surds 262
Equations 11, 404
  conditional 405
  contradictory, 207, 238, 422, 450, 452
  degree of 147, 207, 236, 289, 404
  dependent 207, 238, 422, 450
  equivalent 25, 406
  forming of 536
  fractional 181, 213, 415, 527
  graphic representation of, 235, 239, 446
  graphic solution of 237, 448, 449
  higher 150, 304, 532
  homogeneous 301, 548
  identical 11, 405
  independent 208, 238, 422
  indeterminate 207, 421, 427, 446
  integral 25, 404

Equations — *cont.*
  interpreting solutions of, 538, 554
  involving radicals . 276, 290, 502, 503
  irrational 276, 529
  linear 236, 404 447
  literal, 1, 2, 3, 4, 188, 190, 191, 196, 198, 199, 201, 203, 204, 205, 206, 217, 218, 219, 229, 283, 425, 438
  quadratic . . 147, 278, 297, 519
  rational 404
  roots of 26, 405, 562
  simultaneous 207, 421
  simultaneous quadratic, 297, 547
  solution of, 26, 27, 28, 29, 147, 181, 207, 208, 211, 224, 237, 256, 278, 289, 407
  solved by factoring, 147, 148, 297, 519
  systems of, 224, 297, 302, 427, 428, 544
  translation of 32
Euclid 510
Euler 490, 515
Exponents 92, 354
  fractional 266, 484
  laws of 465, 484, 487
  negative 100, 370, 485
  zero 100, 370, 485
Expressions, algebraic . . . 6
  integral 167, 379
  mixed 394
  rational, irrational . . 355, 379
Extremes, of a proportion, 189, 458

Factoring 122, 379
  by solving a quadratic . . . 537
  cases of 142, 143
  equations solved by . . 147, 519
Factors 8, 122, 353
  found by the remainder theorem 139, 386
  found by grouping . . 136, 384

Factors — *cont.*
  found by the quadratic formula 537
  highest common . . . 155, 390
  laws of 358
  monomial 122, 379
  of binomials 126, 380
  of polynomials 383
  of trinomials . . . 124, 130, 381
  prime 122, 379
Falling body problems . . 293
First degree equations, graph of 236, 447
  in more than one unknown, 207, 421
  in one unknown 25, 404
Form changes 27, 29
Formulas, 1, 2, 4, 121, 196, 198, 199, 201, 203, 204, 206, 259, 282, 308, 311, 436, 525, 562, 574, 579
Fourth proportional . . 190, 458
Fractional, equations . 181, 415, 527
  exponents 266, 484
Fractions 160, 393
  clearing of 182, 415
  complex 178, 402
  division of 175, 399
  fundamental theorems on, 398, 399, 400
  multiplication of . . . 172, 399
  reduction of . . . 162, 164, 394
  signs of 166
  terms of 160, 393
Functions 314
Fundamental laws . . 15, 69, 358
  operations 361

Gauss 292, 452, 510, 515
Geometric, means 580
  progressions 578
  series, infinite 582
Girard 11
Graphic representation of complex numbers . . . 516

Graphic representation — *cont.*
  of equations . 235, 446, 447, 558
  of real numbers 356, 516
  of signed numbers . . . 44, 356
  of statistics 230
Greeks 62, 161, 206, 229
Grouping terms in factoring . 384

Hamilton 70, 84
Harmonic, means 584
  progressions 584
Harriot . 62, 101, 206, 239, 291, 490
Heine 510
Higher equations . . 150, 304, 532
Highest common factor . 155, 390
Hindu method of completing the square 280, 523
Hindus . . 5, 62, 161, 291, 280, 313
Historical notes, 5, 7, 11, 15, 32, 41, 62, 69, 84, 101, 151, 161, 188, 206, 239, 242, 291, 313, 452, 490, 510, 515, 564, 601
Homogeneous equations, 301, 548

Identities 11, 405
Imaginary, number . 284, 463, 508
  roots of a quadratic . . 287, 534
  unit 463, 508
Inconsistent equations . 238, 450
Independent equations . 208, 238, 422
Indeterminant equations, 207, 421, 427, 446
Induction, mathematical . 310, 588
Infinite geometric series . . 582
Inserting in parentheses . . 77
Integers, even, odd 38
  problems on, 38, 40, 91, 120, 291, 319, 320, 326
Integral, equations 25, 404
  expression 167, 379
  solutions 448
Interest problems, 4, 36, 41, 197, 220, 221, 316, 339, 600
Inversion of a proportion, 189, 458
Irrational, expression 355
  number 508
  root 355

Lagrange 452
Laws, of exponents, 94, 95, 97, 266, 267, 484
  fundamental 358
Leibnitz 452
Leonardo of Pisa 5
Lever problems, 202, 224, 345, 413, 432
Limit, approach as a 582
Linear equations 236, 447
Literal equations, 1, 2, 3, 4, 189, 190, 191, 196, 198, 199, 201, 203, 204, 205, 206, 217, 218, 219, 229, 283, 435, 439, 525, 567, 570
Logarithms 593, 601
  table of 596
Lowest common multiple, 157, 391
Lowest terms, fractions in, 162, 394

Mantissa of a logarithm . . . 593
Mathematical induction, 310, 588
Mean proportional . . 190, 458
Means, arithmetic 575
  geometric 580
  harmonic 584
  of a proportion . . . 188, 458
Minuend 50, 353
Monomial . . 67, 361, 362, 368, 370
  factors 379
Motion problems, 199, 200, 201, 202, 222, 223, 224, 332, 343, 344, 345, 349, 350, 351, 352, 412, 542, 557
Multiple 157, 391
  lowest common . . 157, 158, 391
Multiplication . . . 12, 16, 353
  by means of logarithms, 593, 598
  by zero 29, 407

Multiplication — *cont.*
  laws of 358
  of complex numbers . . . 512
  of fractions 172, 399
  of imaginaries 285
  of monomials 95, 368
  of polynomials . . . 84, 88, 372
  of pure imaginary numbers, 286, 512
  of radicals 496
  of signed numbers . . . 56, 357
  of surds 270

Napier 538, 601
Negative, exponents . 100, 370, 485
  number 43, 44, 62, 355
  products 368
  results, interpretation of . . 63
Newton, 62, 102, 206, 313, 490, 510, 564
Number, absolute value of, 44, 355
  complex 509
  constant, variable 453
  positive and negative, 43, 44, 62, 355
  rational, irrational . . 262, 508
  real, imaginary . . 284, 463, 508
  signed 44, 355
  unknown . 25, 404, 415, 421, 567
Number system, of Algebra, 44, 284, 506
  of Arithmetic 44, 506

Operations, algebraic . . 20, 353
  at sight 31
  fundamental 361
  laws of 358
  order of 20, 360
  symbols of 3, 353
Ordinates 233, 447
Oughtred 7

Parabola 558
Parameters 567
Parentheses . . . 11, 75, 360, 366

Parentheses — *cont.*
  insertion in 77
  removal of 75, 366
Pascal 313, 314
Percentage problems . . 4, 350
Polynomials 67, 361
  addition and subtraction of, 69, 72, 364
  cube root of 478
  division by 102, 372
  division of 374
  factors of 383
  multiplication of . . . 85, 88, 372
  square root of 243, 469
Positive and negative products 368
Positive numbers . . 43, 44, 355
Powers 92, 354, 463
Prime factor 122, 379
Principal root 464
Principles, eighteen fundamental. See *List*, page x.
  concerning addition and subtraction 362–364
  concerning the binomial formula 587
  concerning equivalent equations 406
  concerning factors . . 386, 390
  concerning fractions . . . 399
  concerning fractional and negative exponents . . . 484
  concerning multiplication and division . . . 358, 359
  importance of 22
  involving powers and roots . 465
Problems
  applying theory of radicals, 257, 503
  involving data, 39, 227, 228, 319, 322, 324, 327, 351, 352
  involving the Pythagorean proposition, 152, 153, 257, 258, 259, 260, 292, 293, 306, 307, 341, 342, 502–505, 542, 556–559, 564

Problems — *cont.*
  on areas, 1, 2, 3, 4, 87, 91, 120, 154, 220, 221, 222, 223, 259, 260, 261, 294, 295, 305, 306, 323, 337, 347, 348, 349, 350, 351, 503–505
  on circles, 261, 317, 339, 347, 348, 349, 350, 411, 413, 456, 457, 460, 564
  on digits, 81, 82, 83, 222, 223, 228, 343, 351
  on falling bodies . . . 293, 577
  on integers, 38, 40, 91, 120, 291, 319, 320, 326
  on interest, 4, 36, 41, 197, 220, 221, 316, 339, 581, 600
  on levers, 202, 224, 345, 413, 414, 432
  on motion, 199, 200, 201, 202, 221, 222, 223, 234, 332, 343, 344, 345, 349, 350, 351, 352, 412, 413, 542, 557
  on percentage 4, 350
  on ratios 192, 462
  on rectangles, 1, 3, 4, 38, 39, 40, 87, 91, 120, 154, 220, 221, 222, 223, 292, 294, 295, 305, 306, 319, 338, 341, 347
  on similar triangles . . . 193
  on thermometer readings, 419, 420
  on triangles, 2, 3, 92, 193, 194, 195, 257, 258, 259, 260, 261, 293, 294, 306, 307, 341, 342, 347, 502
  on volumes, 2, 3, 4, 36, 153, 295, 325, 503–505
Product 16, 353
  of binomials 85
  of fractions 172, 399
  positive and negative . . 57, 368
  special 108, 373
Progressions 572
Proof, by induction 588
Proportion 188, 458
Pythagorean theorem . . . 151

Pythagorean theorem — *cont.*
  problems on, see Problems.

Quadrants 233
Quadratic equations, 147, 278, 519
  equations in the form of, 289, 532
  exposition by graphs . . . 558
  homogeneous 301, 548
  interpretation of results . 538, 554
  nature of roots of 562
  solution by completing the square 279, 280, 521
  solution by factoring, 147, 297, 519
  solution by formula . 283, 299, 525
  solution by special methods, 300, 546
  systems of 297, 544
Quadratic surds . . 262, 274, 500
  square root of 501
Quotient 15, 353
  of fractions 175, 399
  of polynomials . . . 102, 374
  special 117, 378

Radical expressions . . 262, 491
Radicals 240, 262, 491
  addition and subtraction of, 268, 495
  and exponents . . . 266, 484
  applications of . . . 257, 502
  division of 274, 498
  equations involving, 276, 290, 502, 503
  multiplication of . . . 270, 496
  of same order 268, 494
  reduction of 263, 491
  sign of 240, 354
  similar 268, 495
  simplification of . . . 252, 495
Radicand 262, 491
Rahn 7
Ratio 188, 192, 453
Rational, equation 404
  expression 355, 379
  numbers 262, 508

Rational roots 262, 355
Rationalizing denominators, 254, 274, 499
Real numbers . . . 284, 463, 508
Reciprocal of a number . . 399
Rectangles, problems on. See Problems.
Reduction, of fractions . 162, 394
  to common denominators, 164, 394
  of surds 263, 491
Remainder theorem . . 138, 386
Reviews, 23, 42, 65, 83, 107, 121, 180, 187, 195, 206, 229, 239, 261, 277, 296, 307, 313, 318–352
Roman numerals 5
Roots, and coefficients . . 282, 534
  and powers 634
  cube 354, 478
  square . . 124, 240, 243, 247, 354, 469
Roots of equations . . . 26, 405
  distinct, coincident, imaginary 287, 534
  rational and irrational . . 355
  sum and product of . . 282, 534

Scale of signed numbers . 45, 356
Servois 15, 84
Signed numbers 44, 355
Signs, double use of 53
  of aggregation 11, 366
  of operation 3, 353
  of quality 44, 355
Similar radicals . . . 268, 495
  terms 68, 361
  triangles 193
Simultaneous equations, 207, 422
  higher 532, 551
  linear 207, 421
  quadratic 297, 547
  quadratic and linear . 297, 544
  three of first degree . 224, 427, 439
  two of first degree . 208, 421, 434
  two quadratic 301, 544
Solution of equations, 25, 27, 29, 31, 54, 147, 181, 207, 208, 211, 224, 237, 256, 278, 289, 406, 407, 415, 421, 434, 437, 519, 544
  by determinants . . . 437, 438
  by factoring . 147, 148, 297, 519
  by formula 283, 299, 525
  by graphs 237, 448, 558
  by special devices . . . 301, 546
Solution of problems, hints on 33, 37
Special products and quotients . . . 108, 117, 373, 378
Square, of binomial . . 108, 373
  of any polynomial . . 115, 373
  of trinomial 114
Square root 124, 240, 354
  applications of . . . 257, 503
  approximate 251, 477
  of arithmetic numbers . . 247, 472
  of fractions 253
  of products 241 491
  of polynomials . . . 243, 469
  of radical expressions . . . 501
Squares, difference of two 126, 380
  trinomial 124, 381
Stevinus 161, 490
Substitution 404
  elimination by . . . 211, 423
Subtraction 8, 50, 353
  elimination by . . . 208, 423
  of fractions 168, 397
  of imaginaries . . . 285, 510
  of monomials 68, 362
  of polynomials 72, 364
  of radicals 268, 495
  of signed numbers . . . 50, 356
Sum 9, 45, 69, 353
  of arithmetic progression . 573
  of geometric progression . . 579
  of two cubes 118, 128
Surds 262–275, 500
Symbols of aggregation . . 11, 360

Symbols — *cont.*
of operation 3, 353
Synthetic division 377
Systems of equations, 207, 224, 297, 302, 427, 428, 544
solved by determinants . . 434

Table of logarithms 596
Tangent to a curve 561
Terms 67, 361
of a fraction 160
of a proportion 188, 458
similar and simple . . . 68, 361
transposition of 31
Third proportional 458
Triangles, similar 193
See Problems.
Trinomial 67, 361
factors of . . 130–134, 381, 382
Trinomial squares . . 124, 381

Unknown numbers 25, 404, 421, 567

Vandermonde 452
Variables 314, 453
Variation 315, 453
direct 315, 454
inverse and other forms, 316, 454, 455
Vieta . . 42, 101, 206, 239, 313, 490
Vinculum 11, 360
Volumes. See Problems.

Wallis 101, 188, 490
Widmann 8
Written work, directions for . 30

Zero, division by, 29, 102, 104, 359, 407, 463
exponents 100, 370, 485
multiplication by . . . 29, 407